

Two Approaches for Solving a Group Pursuit Game

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Abstract In this paper we study a game of group pursuit in which players move on a plane with bounded velocities. The game is supposed to be a nonzero-sum simple pursuit game between a pursuer and m evaders acting independently of each other. The case of complete information is considered. Here we assume that the evaders are discriminated. Two different approaches to formalize this pursuit problem are considered: noncooperative and cooperative. In a noncooperative case we construct a Nash equilibrium, and in a cooperative case we construct the core. We proved that the core is not empty for any initial positions of the players.

Keywords: group pursuit game, Nash equilibrium, realizability area, TU-game, core.

1. Introduction

The process of pursuit represents a typical conflict situation. When only two players are involved in the process of pursuit we deal with a classical zero-sum differential pursuit game. These games grew out of the military problems and were developed by Isaaks (1965).

When more than two players participate in a game and the players' objectives are not strictly opposite it is rather reasonable to consider such a game as a nonzero-sum one. This approach for solving a group pursuit problem was introduced in (Petrosjan and Shirjaev, 1981) and further applied in works (Tarashnina, 1998), (Pankratova and Tarashnina, 2004).

It is obvious that players' goals are not always strictly opposed. We want to illustrate how differential games can be used for solving different kind of problems. In this case under "capture" we can understand just meeting of players and delivering some goods or information. In terms, players are not aimed to destruct each other. Moreover, players in a nonzero-sum game may cooperate with each other to get a maximal profit.

We investigate a nonzero-sum group pursuit game using two different approaches. We construct a game in normal form and its TU-cooperative version and find their solutions.

2. Nonzero-sum group pursuit game

In the work we study a game of pursuit in which n players – the pursuer P and evaders E_1, \dots, E_m – move on a plane with constant velocities with the possibility of changing the direction of their motion at each time instant (simple motion). We consider the case of complete information. This means that each player at each time instant $t \geq 0$ knows the moment t and his own as well as all other player's positions. Additionally, we assume that the pursuer uses strategies with discrimination against the evaders. This means that at each instant t the pursuer P knows the vector-speeds chosen by the evaders at that time moment.

The players start their motion at moment $t = 0$ at the initial positions

$$z_P^0 = (x_P^0, y_P^0), \quad z_i^0 = (x_i^0, y_i^0), \quad i = \overline{1, m}.$$

Let α and β_i are velocities of P and E_i ($i = \overline{1, m}$), respectively. Suppose that $\alpha > \max_{i=1, \dots, m} \beta_i$. Denote by $E_i^t = z_i^t = (x_i^t, y_i^t)$ and $P^t = z_P^t = (x_P^t, y_P^t)$ the current positions of evader E_i and pursuer P at the moment $t > 0$, respectively.

The motion of players is described by the following system of differential equations

$$\begin{aligned} \dot{z}_P &= \mathbf{u}_P, \quad \mathbf{u}_P \in U_P, \\ \dot{z}_i &= \mathbf{u}_{E_i}, \quad \mathbf{u}_{E_i} \in U_{E_i}, \quad i = \overline{1, m}, \end{aligned} \quad (1)$$

with initial conditions

$$z_P(0) = z_P^0, \quad z_i(0) = z_i^0, \quad i = \overline{1, m}, \quad (2)$$

where $z_P, z_1, \dots, z_m \in \mathbf{R}^2$. The vectors $\mathbf{u}_P \in U_P$ and $\mathbf{u}_{E_i} \in U_{E_i}$ are control variables of P and E_i ($i = \overline{1, m}$), respectively. The set of control variables U_P, U_{E_i} have the following forms

$$\begin{aligned} U_P &= \{\mathbf{u}_P = (u_P^1, u_P^2) : (u_P^1)^2 + (u_P^2)^2 = \alpha^2\}, \\ U_{E_i} &= \{\mathbf{u}_{E_i} = (u_{E_i}^1, u_{E_i}^2) : (u_{E_i}^1)^2 + (u_{E_i}^2)^2 = \beta_i^2\}, \quad i = \overline{1, m}. \end{aligned}$$

We need to explain how the players choose their control variables throughout the game according to the incoming information. Define a strategy of the evader as a function of time and current positions of the players. A strategy of player E_i is a function $u_{E_i}(t, z_P^t, z_1^t, \dots, z_m^t)$ with values in U_{E_i} . The evaders use piecewise open-loop strategies. Denote by \mathcal{U}_{E_i} the set of admissible strategies of player E_i , $i = \overline{1, m}$.

A strategy of player P is a function of time, players' positions and velocity-vectors of the evaders, i.e.

$$u_P(t, z_P^t, z_1^t, \dots, z_m^t, \mathbf{u}_{E_1}^t, \dots, \mathbf{u}_{E_m}^t).$$

That means, that the class of admissible strategies of the pursuer consists of strategies with discrimination (counterstrategies).

The game is played as follows: at the initial time instant the pursuer dictates to the evaders E_1, \dots, E_m a certain behaviour and chooses some pursuit order. In other words, the pursuer fixes some pursuit order and calculates the total pursuit time taking into account that the evaders use the prescribed behaviour. After that, P consequently pursues the evaders according to the chosen order and changes it

as soon as any of the evader chooses a direction of motion different from the one dictated by the pursuer. So, the pursuer punishes deviated evader, changing pursuit order by starting the pursuit of defected evader. If group of evaders is deviated then the pursuer punishes anyone of this group.

Let $\mathbf{\Pi}$ be the set of all possible orders. Now we define a notion of a punishment strategy of the pursuer.

Definition 1. We say that the triple $u_P^\pi = \langle \pi, u_P, p \rangle$ is a punishment strategy of pursuer P with

- $\pi(z_P^0, z_1^0, \dots, z_m^0, u_{E_1}, \dots, u_{E_m})$ is a pursuit order chosen by the pursuer at the initial instant $t = 0$ for some fixed strategy profile of the evaders u_{E_1}, \dots, u_{E_m} ;
- $u_P(t, z_P^t, z_1^t, \dots, z_m^t, \mathbf{u}_{E_1}^t, \dots, \mathbf{u}_{E_m}^t)$, $t \geq 0$, is a pursuit strategy of P that consists in consequent pursuit of the evaders according to the chosen order;
- $p = p(t, \mathbf{u}_{E_1}^t, \dots, \mathbf{u}_{E_m}^t)$ is an element of punishment that consists in changing the pursuit order at the moment t by starting the pursuit of defected evader in case any of the evaders chooses a direction of motion different from $(\mathbf{u}_{E_1}^t, \dots, \mathbf{u}_{E_m}^t)$ dictated by the pursuer.

Denote by $\mathcal{U}_P = \{u_P^\pi\}_{\pi \in \mathbf{\Pi}}$ the set of punishment strategies of the pursuer.

Evader E_i is considered caught if the positions of P and E_i coincide at some time instant. We say that the game is over if the pursuer captures all the evaders.

Let $\pi = \{1, \dots, i, \dots, m\}$ be a pursuit order chosen by pursuer P .

Denoting by K_P the payoff function of P , and by K_{E_i} the payoff function of evader E_i , $i = \overline{1, m}$, we have

$$K_{E_i}(u_P^\pi, u_{E_1}, \dots, u_{E_i}, \dots, u_{E_m}) = \sum_{k \leq i, k = \overline{1, m}} T_k^\pi, \quad (3)$$

where T_k^π is the time spent by the pursuer for capture of the evader E_k ($k = \overline{1, m}$) minus time according to the pursuit order $\pi \in \mathbf{\Pi}$. Here i is a number of the evader E_i in the pursuit order $\pi = \{1, \dots, i, \dots, m\}$ and k ($k \leq i$) is a number of the evader which is pursued before E_i inclusively.

The payoff of P is defined as the negative value of the payoff of evader E_i that is caught last. Thus,

$$K_P(u_P^\pi, u_{E_1}, \dots, u_{E_m}) = -T^\pi, \quad (4)$$

where $T^\pi = \sum_{k=1}^m T_k^\pi$ is the total pursuit time, and π is the chosen pursuit order.

So, we define the nonzero-sum pursuit game as a normal form game as follows

$$\Gamma(z_P^0, z_1^0, \dots, z_m^0) = \langle N, \{\mathcal{U}_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle, \quad (5)$$

where $N = \{P, E_1, \dots, E_m\}$ is the set of players, \mathcal{U}_i is the set of admissible strategies of player i and K_i is a payoff function of player i ($i \in N$), defined by (3) and (4). Each game depends on a choice of the initial positions of the players. Let us fix the players' initial positions and consider the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$.

3. Nash equilibrium in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$

In nonzero-sum games there is a number of solution concepts that are based on some additional assumptions for players' behaviour and structure of the game. One of them is the well-known concept of Nash equilibrium. In considered game there exists a whole family of Nash equilibria that includes some which are extremely adverse to the evaders' interests, and some which are favorable for them, as well as all intermediate equilibria. Different kind of Nash equilibria in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$ were constructed in (Petrosjan and Shirjaev, 1981), (Tarashnina, 1998), (Pankratova and Tarashnina, 2004). In this game we consider the extremely odd Nash equilibrium that is the most disadvantageous for the evaders among all the equilibria.

The strategy set of pursuer P consists of u_P^π corresponding to the pursuit order $\pi \in \Pi$. The pursuer aims to minimize the total pursuit time and each evader wants to avoid his own capture as long as possible and does not care about the other evaders. Denote by π^* the pursuit order which minimizes the total pursuit time and by $u_P^{\pi^*}$ the corresponding strategy of the pursuer.

Let $E_i^{j'}$ be the evader who is currently pursued, $j' \in \{1, \dots, m\}$. E_i^j is the j -th in the line of pursuit evader among the ones not yet caught, $j \in \{1, \dots, m\}$, $j > j'$.

Now let us describe two types of behaviours of evader E_i ($i = \overline{1, m}$):

- $E_i^{j'}$, $j' \in \{1, \dots, m\}$, uses behaviour $[u_{E_i}^{j'}]$ that prescribes to move along the straight line connecting his own and the pursuer's current positions in the direction from P (to the current capture point $N^{j'}$).
- E_i^j , $j \in \{1, \dots, m\}$, uses behaviour $[u_{E_i}^j]$ that prescribes to move along the straight line to the capture point of the currently pursued evader $E_i^{j'}$, $j > j'$, namely, to the current capture point $N^{j'}$, where $N^{j'} = P^{T_{j'}}$.

It is obvious that throughout the game at some moment $t_{E_i} > 0$ each evader E_i changes its type from E_i^j into $E_i^{j'}$. So, the strategy $u_{E_i}^*(t, \cdot)$ of evader E_i ($i = \overline{2, m}$) can be describe as

$$u_{E_i}^*(t, \cdot) = \begin{cases} [u_{E_i}^j], & 0 \leq t < t_{E_i}, \\ [u_{E_i}^{j'}], & t \geq t_{E_i}. \end{cases}$$

During the game each evaders, accept E_1 , consequently applies both types of behaviours. The player E_1 uses just type $[u_{E_1}^{j'}]$, i.e. his strategy is $u_{E_1}^*(t, \cdot) = [u_{E_1}^{j'}]$, $t \geq 0$.

Suppose that $T^0 = 0$, $N^0 = P^0 = (0, 0)$.

The following theorem (Tarashnina, 1998) defines the conditions that support the described Nash equilibrium $(u_P^{\pi^*}, u_{E_1}^*, \dots, u_{E_m}^*)$ in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$.

Theorem 1. *In the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$ in case the conditons*

$$T^{\pi^*} = \min_{\pi \in \Pi} T^\pi, \tag{6}$$

and

$$\frac{\alpha - \beta_i}{\alpha - \beta_{i-1}} \left| N^{i-2} E_{i-1}^{T_{i-2}} \right| + \left| N^{i-1} E_i^{T_{i-1}} \right| > \left| N^{i-2} E_i^{T_{i-2}} \right|, \quad i = \overline{2, m}. \tag{7}$$

hold for all $i = \overline{1, m}$ there exists a Nash equilibrium that is constructed as follows:

1. E_i ($i = \overline{1, m}$) chooses the strategy $u_{E_i}^*$ that dictates to him
 - according to the behaviour $[u_{E_i}^{j'}]$ to move along the straight line connecting current positions $E_i^{j'}$ and P at the moment T_{i-1} in the direction from P if $i = j'$, where $j' \in \{1, \dots, m\}$;
 - according to the behaviour $[u_{E_i}^j]$ to move along the straight line to the capture point of the currently pursued evader E_i^j , $j' \in \{1, \dots, m\}$, $j > j'$, i.e. to point $N^{j'}$, where $N^{j'} = P^{T_{j'}}$, if $i = j$, where $j > j'$.
2. P chooses the strategy u_P^* that minimizes the total pursuit time if each E_i ($i = \overline{1, m}$) adheres to the strategy $u_{E_i}^*$, and P changes the pursuit order as soon as any of the evaders E_i^j ($i > j'$) that are not yet caught deviates from the strategy $u_{E_i}^*$ and pursuits the deviated evader the first.

Now we introduce the notion of a realizability area. For this purpose we associate with each evader E_i an area Ω_i of initial positions of evaders that support the Nash equilibrium and refer to it as the realizability area of the punishment strategy of the pursuer with respect to evader E_i , ($i = \overline{1, m}$).

In words, area Ω_i is the set of all E_i 's initial positions such that, when there, evader E_i has to adhere the strategy $u_{E_i}^*$ dictated to him by the pursuer.

Definition 2. The punishment strategy of pursuer P is called realizable with respect to evader E_i , if the life time of evader E_i , $i \in \{2, \dots, m\}$, in case E_i adheres to the strategy $u_{E_i}^*$ is larger then if E_i deviates, i. e. inequality (7) holds for fixed i .

Definition 3. The punishment strategy of pursuer P is called realizable in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$ if inequality (7) holds for all $i = \overline{2, m}$.

In (Pankratova and Tarashnina, 2004) some illustrative examples for constructing of realizability areas of the punishment strategy are have been presented.

4. Cooperative pursuit game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$

Let us suppose that the players in the game can form a coalitions. Construct a cooperative game between pursuer P and evaders E_1, \dots, E_m in assumption the players use the strategies described in the previous paragraph without the threat of punishment.

Assume that utility of any player is transferable.

Let $2^{\mathbf{N}}$ be the set of all subsets of \mathbf{N} . The function $v: 2^{\mathbf{N}} \rightarrow \mathbf{R}^1$ with the following two properties

1. $v(\emptyset) = 0$, where \emptyset is an empty set,
2. $v(\mathbf{S} \cup \mathbf{R}) \geq v(\mathbf{S}) + v(\mathbf{R})$ for all $\mathbf{R}, \mathbf{S} \subset \mathbf{N}$ with $\mathbf{S} \cap \mathbf{R} = \emptyset$,

is called *the characteristic function* of the game . Condition 2 is a superadditivity property.

For any coalition $\mathbf{S} \subset \mathbf{N}$ we define the characteristic function as follows

$$v(\mathbf{S}) = \max_{u_{\mathbf{S}}} \min_{u_{\mathbf{N} \setminus \mathbf{S}}} \sum_{i \in \mathbf{S}} K_i(u_{\mathbf{S}}, u_{\mathbf{N} \setminus \mathbf{S}}),$$

where $u_{\mathbf{S}}$ and $u_{\mathbf{N} \setminus \mathbf{S}}$ are vectors of admissible strategies of the coalitions \mathbf{S} and $\mathbf{N} \setminus \mathbf{S}$, respectively. Using this approach, we construct the characteristic function v for the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$.

Consider an arbitrary permutation π of the ordered set of indexes $M = \{1, 2, \dots, m\}$. With this permutation we associate a substitution k_π , i.e. $k_\pi : M \rightarrow M$. This means that $k \in M$ goes to $k_\pi \in M$ in permutation π .

The characteristic function of the game has the form

$$v(\{E_{i_1}\}; z_P^0, z_1^0, \dots, z_m^0) = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\}, \quad i_1 = \overline{1, m}.$$

$$v(\{E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \dots, z_m^0) = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\}, \quad \text{where } i_1, i_2 = \overline{1, m}, \\ i_1 \neq i_2.$$

$$v(\{E_{i_1}, E_{i_2}, E_{i_3}\}; z_P^0, z_1^0, \dots, z_m^0) = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\}, \quad \text{where} \\ i_1, i_2, i_3 = \overline{1, m}, \quad i_1 \neq i_2 \neq i_3.$$

...

$$v(\{E_1, \dots, E_m\}; z_P^0, z_1^0, \dots, z_m^0) = \min_{\pi \in \Pi} \left\{ T_{i_1}^\pi + \sum_{k_\pi=i_1}^{i_2} T_{k_\pi}^\pi + \dots + \sum_{k_\pi=i_1}^{i_{m-1}} T_{k_\pi}^\pi + \sum_{k_\pi=i_1}^{i_m} T_{k_\pi}^\pi \right\}.$$

$$v(\{P\}; z_P^0, z_1^0, \dots, z_m^0) = \max_{\pi \in \Pi} \{-T^\pi\}.$$

$$v(\{P, E_{i_1}\}; z_P^0, z_1^0, \dots, z_m^0) = \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\} = 0, \quad \text{where } i_1 = \overline{1, m}.$$

$$v(\{P, E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \dots, z_m^0) = \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\}, \quad \text{where } i_1, i_2 = \\ \overline{1, m}, \quad i_1 \neq i_2.$$

...

$$v(\{P, E_1, \dots, E_m\}; z_P^0, z_1^0, \dots, z_m^0) =$$

$$= \max_{\pi \in \Pi} \left\{ -T^\pi + T_{i_1}^\pi + \sum_{k=i_1}^{i_2} T_k^\pi + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi + \sum_{k=i_1}^{i_m} T_k^\pi \right\} =$$

$$= \max_{\pi \in \Pi} \left\{ T_{i_1}^\pi + \sum_{k=i_1}^{i_2} T_k^\pi + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi \right\}.$$

For simplicity denote by

$$\begin{aligned}
 \hat{T}_{i_1} &= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\}, \\
 \hat{T}_{i_1 i_2} &= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\}, \\
 \hat{T}_{i_1 i_2 i_3} &= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\}, \\
 &\vdots \\
 \hat{T}_{i_1 i_2 \dots i_m} &= \min_{\pi \in \Pi} \left\{ T_{i_1}^\pi + \sum_{k_\pi = i_1}^{i_2} T_{k_\pi}^\pi + \dots + \sum_{k_\pi = i_1}^{i_{m-1}} T_{k_\pi}^\pi + \sum_{k_\pi = i_1}^{i_m} T_{k_\pi}^\pi \right\}, \\
 \tilde{T} &= \max_{\pi \in \Pi} \{-T^\pi\}, \\
 \tilde{T}_{i_1} &= \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\}, \\
 \tilde{T}_{i_1 i_2} &= \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\}, \\
 \tilde{T}_{i_1 i_2 i_3} &= \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\}, \\
 &\vdots \\
 T^* &= \max_{\pi \in \Pi} \left\{ T_{i_1}^\pi + \sum_{k_\pi = i_1}^{i_2} T_{k_\pi}^\pi + \dots + \sum_{k_\pi = i_1}^{i_{m-1}} T_{k_\pi}^\pi \right\}.
 \end{aligned} \tag{8}$$

The characteristic function v can be described in the following form

$$\begin{aligned}
 v(\{P\}; z_P^0, z_1^0, \dots, z_m^0) &= \tilde{T}, \\
 v(\{E_{i_1}\}; z_P^0, z_1^0, \dots, z_m^0) &= \hat{T}_{i_1}, \quad i_1 = \overline{1, m}, \\
 v(\{P, E_{i_1}\}; z_P^0, z_1^0, \dots, z_m^0) &= \tilde{T}_{i_1} = 0, \quad i_1 = \overline{1, m}, \\
 v(\{E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \dots, z_m^0) &= \hat{T}_{i_1 i_2}, \quad i_1, i_2 = \overline{1, m}, \quad i_1 \neq i_2, \\
 v(\{P, E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \dots, z_m^0) &= \tilde{T}_{i_1 i_2}, \quad i_1, i_2 = \overline{1, m}, \quad i_1 \neq i_2, \\
 v(\{E_{i_1}, E_{i_2}, E_{i_3}\}; z_P^0, z_1^0, \dots, z_m^0) &= \hat{T}_{i_1 i_2 i_3}, \quad i_1, i_2, i_3 = \overline{1, m}, \quad i_1 \neq i_2 \neq i_3, \\
 v(\{P, E_{i_1}, E_{i_2}, E_{i_3}\}; z_P^0, z_1^0, \dots, z_m^0) &= \tilde{T}_{i_1 i_2 i_3}, \quad i_1, i_2, i_3 = \overline{1, m}, \quad i_1 \neq i_2 \neq i_3, \\
 &\vdots \\
 v(\{E_1, \dots, E_m\}; z_P^0, z_1^0, \dots, z_m^0) &= \hat{T}_{1 \dots m}, \\
 v(\{P, E_1, \dots, E_m\}; z_P^0, z_1^0, \dots, z_m^0) &= T^*.
 \end{aligned} \tag{9}$$

Here and then we will use following designation $v(\{E_{i_1}, \dots, E_{i_k}\}; z_P^0, z_1^0, \dots, z_m^0) = v(E_{i_1}, \dots, E_{i_k})$ and $v(\{P, E_{i_1}, \dots, E_{i_k}\}; z_P^0, z_1^0, \dots, z_m^0) = v(P, E_{i_1}, \dots, E_{i_k})$

Definition 4. The pair $\langle N, v(S; z_P^0, z_1^0, \dots, z_m^0), S \subset N \rangle$, where N is the set of players, and v is the characteristic function defined by (8)-(9) is called a cooperative pursuit game in characteristic function form and denoted by $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$.

Example 1. let us construct the characteristic function for a pursuit game with a pursuer and three evaders according to formulas (8) and (9). Let $\alpha = 1$ and $\beta_i = \frac{1}{2}$, $i = 1, 2, 3$.

Fix the initial positions of the players: $P^0 = (0, 0)$, $E_1^0 = (1, 0)$, $E_2^0 = (-2, 4)$, $E_3^0 = (5, 7)$. Note that for the chosen initial positions of the players the punishment strategy of P is realizable. First of all, we compose a table with the players' payoffs for all pursuit orders $\pi_i \in \{1, \dots, 6\}$.

Table 1: The players' payoffs for different pursuit orders.

Payoff	$\pi_1 = \{1, 2, 3\}$	$\pi_2 = \{1, 3, 2\}$
K_{E_1}	$T_1^{\pi_1} = 2$	$T_1^{\pi_2} = 2$
K_{E_2}	$T_{12}^{\pi_1} = 2 + 9, 31 = 11, 31$	$T_{132}^{\pi_2} = 2 + 13, 23 + 11, 34 = 26, 57$
K_{E_3}	$T_{123}^{\pi_1} = 2 + 9, 31 + 9, 09 = 20, 4$	$T_{13}^{\pi_2} = 2 + 13, 23 = 15, 23$
K_P	$T_P^{\pi_1} = 20, 4$	$T_P^{\pi_2} = 26, 57$
	$\pi_3 = \{2, 3, 1\}$	$\pi_4 = \{2, 3, 1\}$
K_{E_1}	$T_{21}^{\pi_3} = 8, 94 + 9, 92 = 18, 86$	$T_{231}^{\pi_4} = 8, 94 + 9, 7 + 9, 73 = 28, 37$
K_{E_2}	$T_2^{\pi_3} = 8, 94$	$T_2^{\pi_4} = 8, 94$
K_{E_3}	$T_{213}^{\pi_3} = 8, 94 + 9, 92 + 26, 03 = 44, 89$	$T_{23}^{\pi_4} = 8, 94 + 9, 7 = 18, 64$
K_P	$T_P^{\pi_3} = 44, 89$	$T_P^{\pi_4} = 28, 37$
	$\pi_5 = \{3, 1, 2\}$	$\pi_6 = \{3, 2, 1\}$
K_{E_1}	$T_{31}^{\pi_5} = 17, 2 + 16, 08 = 33, 28$	$T_{321}^{\pi_6} = 17, 2 + 14, 04 + 27, 23 = 58, 47$
K_{E_2}	$T_{312}^{\pi_5} = 17, 2 + 16, 08 + 33, 52 = 66, 8$	$T_{31}^{\pi_6} = 17, 2 + 14, 04 = 31, 24$
K_{E_3}	$T_3^{\pi_5} = 17, 2$	$T_3^{\pi_6} = 17, 2$
K_P	$T_P^{\pi_5} = 66, 8$	$T_P^{\pi_6} = 58, 47$

The characteristic function, according to formulas (8) and (9), has the following form

$$\begin{aligned}
 v(P) &= \max\{-20, 4; -26, 57; -44, 89; -28, 37; -66, 8; -58, 47\} = -20, 4, \\
 v(E_1) &= \min\{2; 2; 18, 86; 28, 37; 33, 28; 58, 47\} = 2, \\
 v(E_2) &= \min\{11, 31; 26, 57; 8, 94; 8, 94; 66, 8; 31, 24\} = 8, 94, \\
 v(E_3) &= \min\{20, 4; 15, 23; 44, 89; 18, 64; 17, 2; 17, 2\} = 15, 23, \\
 v(P, E_1) &= \max\{-20, 4 + 2; -26, 57 + 2; -44, 89 + 18, 86; \\
 &\quad -28, 37 + 28, 37; -66, 8 + 33, 28; -58, 47 + 58, 47\} = 0, \\
 v(P, E_2) &= \max\{-20, 4 + 11, 31; -26, 57 + 26, 57; -44, 89 + 8, 94; \\
 &\quad -28, 37 + 8, 94; -66, 8 + 66, 8; -58, 47 + 31, 24\} = 0, \\
 v(P, E_3) &= \max\{-20, 4 + 20, 4; -26, 57 + 15, 23; -44, 89 + 44, 89; \\
 &\quad -28, 37 + 18, 64; -66, 8 + 17, 2; -58, 47 + 17, 2\} = 0, \\
 v(E_1, E_2) &= \min\{2 + 11, 31; 2 + 26, 57; 18, 86 + 8, 94; \\
 &\quad 28, 37 + 8, 94; 33, 28 + 66, 8; 58, 47 + 31, 24\} = 13, 31, \\
 v(E_1, E_3) &= \min\{2 + 20, 4; 2 + 15, 23; 18, 86 + 44, 89; \\
 &\quad 28, 37 + 18, 64; 33, 28 + 17, 2; 58, 47 + 17, 2\} = 17, 23, \\
 v(E_2, E_3) &= \min\{11, 31 + 20, 4; 26, 57 + 15, 23; 8, 94 + 18, 64; \\
 &\quad 8, 94 + 44, 89; 66, 8 + 17, 2; 31, 24 + 17, 2\} = 27, 58, \\
 v(P, E_1, E_2) &= \max\{-20, 4 + 2 + 11, 31; -26, 57 + 2 + 26, 57; \\
 &\quad -44, 89 + 18, 86 + 8, 94; -28, 37 + 28, 37 + 8, 94; \\
 &\quad -66, 8 + 33, 28 + 66, 8; -58, 47 + 58, 47 + 31, 24\} = 33, 28, \\
 v(P, E_1, E_3) &= \max\{-20, 4 + 2 + 20, 4; -26, 57 + 2 + 15, 23; \\
 &\quad -44, 89 + 18, 86 + 44, 89; -28, 37 + 28, 37 + 18, 64; \\
 &\quad -66, 8 + 33, 28 + 17, 2; -58, 47 + 58, 47 + 17, 2\} = 18, 86,
 \end{aligned}$$

$$\begin{aligned}
v(P, E_2, E_3) &= \max\{-20, 4 + 11, 31 + 20, 4; -26, 57 + 26, 57 + 15, 23; \\
&\quad -44, 89 + 8, 94 + 44, 89; -28, 37 + 8, 94 + 18, 64; \\
&\quad -66, 8 + 66, 8 + 17, 2; -58, 47 + 31, 24 + 17, 2\} = 17, 2, \\
v(E_1, E_2, E_3) &= \min\{2 + 11, 31 + 20, 4; 2 + 26, 57 + 15, 23; \\
&\quad 18, 86 + 8, 94 + 44, 89; 28, 37 + 8, 94 + 18, 64; \\
&\quad 33, 28 + 66, 8 + 17, 2; 58, 47 + 31, 24 + 17, 2\} = 33, 71, \\
v(P, E_1, E_2, E_3) &= \max\{-20, 4 + 2 + 11, 31 + 20, 4; \\
&\quad -26, 57 + 2 + 26, 57 + 15, 23; -44, 89 + 18, 86 + 8, 94 + 44, 89; \\
&\quad -28, 37 + 28, 37 + 8, 94 + 18, 64; -66, 8 + 33, 28 + 66, 8 + 17, 2; \\
&\quad -58, 47 + 58, 47 + 31, 24 + 17, 2\} = 50, 48.
\end{aligned}$$

Finally, we construct a cooperative pursuit game in the characteristic function form.

That is

$$\begin{aligned}
v(P) &= -20, 4, \\
v(E_1) &= 2, \\
v(E_2) &= 8, 94, \\
v(E_3) &= 15, 23, \\
v(P, E_1) &= 0, \\
v(P, E_2) &= 0, \\
v(P, E_3) &= 0, \\
v(E_1, E_2) &= 13, 31, \\
v(E_1, E_3) &= 17, 23, \\
v(E_2, E_3) &= 27, 58, \\
v(P, E_1, E_2) &= 33, 28, \\
v(P, E_1, E_3) &= 18, 86, \\
v(P, E_2, E_3) &= 17, 2, \\
v(E_1, E_2, E_3) &= 33, 71, \\
v(P, E_1, E_2, E_3) &= 50, 48.
\end{aligned}$$

On this example we can see that the characteristic function of the game is super-additive.

Further we proof that it is true for any number of evaders and any initial positions of the players.

Theorem 2. *In the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$ the characteristic function v that is constructed by formulas (8) and (9) is superadditive.*

Proof. In order to prove the theorem we have to show that inequality

$$v(S) + v(T) \leq v(S \cup T)$$

holds for all coalitions $S, T \subset N, S \cap T = \emptyset$.

In fact, the following inequalities are fulfilled.

$$\begin{aligned}
\hat{T}_{i_1} &= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\} \leq \hat{T}_{i_1 i_2} = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} \leq \hat{T}_{i_1 i_2 i_3} = \\
&= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\} \leq \dots \leq \hat{T}_{i_1 \dots i_{m-1}} = \\
&= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_{m-1}} T_{k_\pi}^\pi \right\} \leq \\
&\leq \hat{T}_{i_1 \dots i_m} = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_m} T_{k_\pi}^\pi \right\}.
\end{aligned}$$

It can be easily shown that

$$\begin{aligned} -\tilde{T} &= \max_{\pi \in \Pi} \{-T^\pi\} \leq \tilde{T}_{i_1 i_2} = \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} \leq \dots \leq \\ &\leq \tilde{T}_{i_1 \dots i_{m-1}} = \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_{m-1}} T_{k_\pi}^\pi \right\} \leq \\ &\leq T^* = \max_{\pi \in \Pi} \left\{ -T^\pi + T_{i_1}^\pi + \sum_{k=i_1}^{i_2} T_k^\pi + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi + \sum_{k=i_1}^{i_m} T_k^\pi \right\}. \end{aligned}$$

So, we have

$$\hat{T}_{i_1} \leq \hat{T}_{i_1 i_2} \leq \hat{T}_{i_1 i_2 i_3} \leq \dots \leq \hat{T}_{i_1 \dots i_{m-1}} \leq \hat{T}_{i_1 \dots i_m}, \tag{10}$$

$$-\tilde{T} \leq \tilde{T}_{i_1 i_2} \leq \tilde{T}_{i_1 i_2 i_3} \leq \dots \leq \tilde{T}_{i_1 i_2 \dots i_{m-1}} \leq T^*. \tag{11}$$

Let $S = \{P\}$ and $T = \{E_{i_1}\}$. Since $\tilde{T} = \min_{\pi \in \Pi} \{T^\pi\} \geq \hat{T}_{i_1}$, we have $v(P) + v(E_{i_1}) = -\tilde{T} + \hat{T}_{i_1} \leq 0 = v(P, E_{i_1})$, $i_1 = \overline{1, m}$.

For $S = \{E_{i_1}\}$ and $T = \{E_{i_2}\}$ we have $v(E_{i_1}) + v(E_{i_2}) = \hat{T}_{i_1} + \hat{T}_{i_2} \leq \hat{T}_{i_1 i_2} = v(E_{i_1}, E_{i_2})$, $i_1, i_2 = \overline{1, m}$, $i_1 \neq i_2$. This follows from (10).

For $S = \{P\}$ and $T = \{E_{i_1}, E_{i_2}\}$ we have $v(P) + v(E_{i_1}, E_{i_2}) = -\tilde{T} + \hat{T}_{i_1 i_2} =$

$$\begin{aligned} &= \max_{\pi \in \Pi} \{-T^\pi\} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} = \\ &= -T^{\pi^*} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} \leq -T^{\pi^*} + \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} + \sum_{k_{\pi^*} \leq i_2} T_{k_{\pi^*}}^{\pi^*} \leq \\ &\leq \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} = \tilde{T}_{i_1 i_2} = v(P, E_{i_1}, E_{i_2}), \end{aligned}$$

$i_1, i_2 = \overline{1, m}$, $i_1 \neq i_2$.

Now consider $S = \{E_{i_1}\}$ and $T = \{P, E_{i_2}\}$. Then

$$\begin{aligned} v(E_{i_1}) + v(P, E_{i_2}) &= \hat{T}_{i_1} + \tilde{T}_{i_2} = \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\} + \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} = \\ &= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi \right\} - T^{\pi^*} + \sum_{k_{\pi^*} \leq i_2} T_{k_{\pi^*}}^{\pi^*} \leq \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} - T^{\pi^*} + \sum_{k_{\pi^*} \leq i_2} T_{k_{\pi^*}}^{\pi^*} \leq \\ &\leq \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi \right\} = \tilde{T}_{i_1 i_2}, \quad i_1, i_2 = \overline{1, m}, \quad i_1 \neq i_2. \end{aligned}$$

For $S = \{P, E_{i_1}\}$ and $T = \{E_{i_2}, E_{i_3}\}$ we have

$$\begin{aligned} v(P, E_{i_1}) + v(E_{i_2}, E_{i_3}) &= \hat{T}_{i_1} + \hat{T}_{i_2 i_3} = \max_{\pi \in \Pi} \{-T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi\} + \\ &+ \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\} = -T^{\pi^*} + \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\} \leq \\ &\leq -T^{\pi^*} + \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} + \sum_{k_{\pi^*} \leq i_2} T_{k_{\pi^*}}^{\pi^*} + \sum_{k_{\pi^*} \leq i_3} T_{k_{\pi^*}}^{\pi^*} \leq \end{aligned}$$

$$\leq \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_2} T_{k_\pi}^\pi + \sum_{k_\pi \leq i_3} T_{k_\pi}^\pi \right\} =$$

$$= \tilde{T}_{i_1 i_2 i_3} = v(P, E_{i_1}, E_{i_2}, E_{i_3}), \quad i_1, i_2, i_3 = \overline{1, m}, \quad i_1 \neq i_2 \neq i_3.$$

Now consider two coalitions each of which includes only evaders: $\mathbf{E}_l = \{E_{i_1}, \dots, E_{i_l}\}$ and $\mathbf{E}_s = \{E_{j_1}, \dots, E_{j_s}\}$, $\mathbf{E}_l \cap \mathbf{E}_s = \emptyset$, $i_k, j_q = \overline{1, m}$, $i_k \neq j_q$, $i_1 \neq \dots \neq i_l$, $j_1 \neq \dots \neq j_s$, $k = \overline{1, l}$ è $q = \overline{1, s}$. For this coalitions we get

$$v(E_{i_1}, \dots, E_{i_l}) + v(E_{j_1}, \dots, E_{j_s}) = \hat{T}_{i_1 \dots i_l} + \hat{T}_{j_1 \dots j_s} =$$

$$= \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_l} T_{k_\pi}^\pi \right\} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq j_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq j_s} T_{k_\pi}^\pi \right\} \leq$$

$$\leq \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_l} T_{k_\pi}^\pi + \sum_{k_\pi \leq j_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq j_s} T_{k_\pi}^\pi \right\} =$$

$$= \hat{T}_{i_1 \dots i_l j_1 \dots j_s} = v(E_{i_1}, \dots, E_{i_l}, E_{j_1}, \dots, E_{j_s}).$$

It remains to consider the coalitions $S = \{P, E_{i_1}, \dots, E_{i_l}\}$ and $T = \{E_{j_1}, \dots, E_{j_s}\}$, $i_k, j_q = \overline{1, m}$, $i_k \neq j_q$, $i_1 \neq \dots \neq i_l$, $j_1 \neq \dots \neq j_s$, $k = \overline{1, l}$ and $q = \overline{1, s}$.

Then

$$v(P, E_{i_1}, \dots, E_{i_l}) + v(E_{j_1}, \dots, E_{j_s}) = \tilde{T}_{i_1 \dots i_l} + \hat{T}_{j_1 \dots j_s} =$$

$$= \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_l} T_{k_\pi}^\pi \right\} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq j_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq j_s} T_{k_\pi}^\pi \right\} =$$

$$= -T^{\pi^*} + \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} + \dots + \sum_{k_{\pi^*} \leq i_l} T_{k_{\pi^*}}^{\pi^*} + \min_{\pi \in \Pi} \left\{ \sum_{k_\pi \leq j_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq j_s} T_{k_\pi}^\pi \right\} \leq$$

$$\leq -T^{\pi^*} + \sum_{k_{\pi^*} \leq i_1} T_{k_{\pi^*}}^{\pi^*} + \dots + \sum_{k_{\pi^*} \leq i_l} T_{k_{\pi^*}}^{\pi^*} + \sum_{k_{\pi^*} \leq j_1} T_{k_{\pi^*}}^{\pi^*} + \dots + \sum_{k_{\pi^*} \leq j_s} T_{k_{\pi^*}}^{\pi^*} \leq$$

$$\leq \max_{\pi \in \Pi} \left\{ -T^\pi + \sum_{k_\pi \leq i_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq i_l} T_{k_\pi}^\pi + \sum_{k_\pi \leq j_1} T_{k_\pi}^\pi + \dots + \sum_{k_\pi \leq j_s} T_{k_\pi}^\pi \right\} =$$

$$= \tilde{T}_{i_1 \dots i_l j_1 \dots j_s} = v(P, E_{i_1}, \dots, E_{i_l}, E_{j_1}, \dots, E_{j_s}).$$

Finally, we consider $S = \{P\}$ and $T = \{E_{i_1}, E_{i_2}, \dots, E_{i_m}\}$. Hence,

$$v(P) + v(E_{i_1}, E_{i_2}, \dots, E_{i_m}) = -\tilde{T} + \hat{T}_{i_1 i_2 \dots i_m} =$$

$$\leq \max_{\pi \in \Pi} \{-T^\pi\} + \min_{\pi \in \Pi} \left\{ T_{i_1}^\pi + \sum_{k=i_2}^{i_2} T_k^\pi + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi + \sum_{k=i_1}^{i_m} T_k^\pi \right\} =$$

$$\begin{aligned}
 &\leq -T^{\pi^*} + \min_{\pi \in \Pi} \left\{ T_{i_1}^{\pi} + \sum_{k=i_1}^{i_2} T_k^{\pi} + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^{\pi} + \sum_{k=i_1}^{i_m} T_k^{\pi} \right\} \leq \\
 &\leq -T^{\pi^*} + T_{i_1}^{\pi^*} + \sum_{k=i_1}^{i_2} T_k^{\pi^*} + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^{\pi^*} + \sum_{k=i_1}^{i_m} T_k^{\pi^*} \leq \\
 &\leq \max_{\pi \in \Pi} \left\{ -T^{\pi} + T_{i_1}^{\pi} + \sum_{k=i_1}^{i_2} T_k^{\pi} + \dots + \sum_{k=i_1}^{i_{m-1}} T_k^{\pi} + \sum_{k=i_1}^{i_m} T_k^{\pi} \right\} = \\
 &= T^* = v(P, E_1, \dots, E_m).
 \end{aligned}$$

This completes the proof.

It follows from the superadditivity of v that it is profitable for the players to form the maximal coalition N and obtain the maximal total payoff that is possible in the game.

There exist various methods for distribution of the total payoff between the players in a cooperative TU-game. In our paper we consider the core as a solution concept of the game.

5. The core in the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$

Let us describe the imputation set in the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$. Denote by $\xi = (\xi_P, \xi_{E_1}, \dots, \xi_{E_m})$ an imputation in the game. The imputation set is defined as follows

$$E_v(z_P^0, z_1^0, \dots, z_m^0) = \left\{ \xi : \xi_{E_i} \geq \hat{T}_i, i \in \overline{1, m}, \xi_P \geq -\tilde{T}; \sum_{i \in N} \xi_i = T^* \right\}. \tag{12}$$

From (Bondareva, 1963) and (Shapley, 1967) follows the result. For an imputation ξ to belong to the core of the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$ it is necessary and sufficient that the following system of inequalities holds

$$\left\{ \begin{array}{ll}
 \xi_P \geq \tilde{T}, & \\
 \xi_{E_{i_1}} \geq \hat{T}_{i_1}, & i_1 = \overline{1, m}, \\
 \xi_P + \xi_{E_{i_1}} \geq \tilde{T}_{i_1}, & i_1 = \overline{1, m}, \\
 \xi_{E_{i_1}} + \xi_{E_{i_2}} \geq \hat{T}_{i_1 i_2}, & i_1, i_2 = \overline{1, m}, i_1 \neq i_2, \\
 \xi_P + \xi_{E_{i_1}} + \xi_{E_{i_2}} \geq \tilde{T}_{i_1 i_2}, & i_1, i_2 = \overline{1, m}, i_1 \neq i_2, \\
 \xi_{E_{i_1}} + \xi_{E_{i_2}} + \xi_{E_{i_3}} \geq \hat{T}_{i_1 i_2 i_3}, & i_1, i_2, i_3 = \overline{1, m}, i_1 \neq i_2 \neq i_3, \\
 \xi_P + \xi_{E_{i_1}} + \xi_{E_{i_2}} + \xi_{E_{i_3}} \geq \tilde{T}_{i_1 i_2 i_3}, & i_1, i_2, i_3 = \overline{1, m}, i_1 \neq i_2 \neq i_3, \\
 \dots & \\
 \xi_P + \xi_{E_{i_1}} + \dots + \xi_{E_{i_{m-1}}} \geq \tilde{T}_{i_1 \dots i_{m-1}}, & i_1, \dots, i_{m-1} = \overline{1, m}, i_1 \neq \dots \neq i_{m-1}, \\
 \xi_{E_{i_1}} + \dots + \xi_{E_{i_m}} \geq \hat{T}_{i_1 \dots i_m}. &
 \end{array} \right. \tag{13}$$

Denote by $C_v(z_P^0, z_1^0, \dots, z_m^0)$ the core of the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$.

The following theorem holds.

Theorem 3. *In the cooperative pursuit game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$ there exists the non-empty core for any initial positions of the players.*

Proof. First of all, we show that any imputation from the core satisfies the system (13). Suppose that imputation $\xi = (\xi_P, \xi_{E_1}, \dots, \xi_{E_m})$ belongs to the core $C_v(z_P^0, z_1^0, \dots, z_m^0)$. We have to show that system (13) is combined.

Summing the inequalities of system (13), we obtain

$$\begin{aligned}
 & (1 + C_m^1 + C_m^2 + \dots + C_m^{m-1}) \cdot (\xi_P + \xi_{E_1} + \dots + \xi_{E_m}) \geq \tilde{T} + \sum_{k=1}^m \hat{T}_k + \\
 & + \sum_{k=1}^m \tilde{T}_k + \underbrace{(\hat{T}_{12} + \hat{T}_{13} + \dots + \hat{T}_{m-1,m})}_{C_m^2} + \underbrace{(\tilde{T}_{12} + \tilde{T}_{13} + \dots + \tilde{T}_{m-1,m})}_{C_m^2} + \\
 & + \underbrace{(\hat{T}_{123} + \hat{T}_{124} + \dots)}_{C_m^3} + \underbrace{(\tilde{T}_{123} + \tilde{T}_{134} + \dots)}_{C_m^3} + \\
 & + \underbrace{(\hat{T}_{1234} + \hat{T}_{1235} + \dots)}_{C_m^4} + \underbrace{(\tilde{T}_{1234} + \tilde{T}_{1234} + \dots)}_{C_m^4} + \\
 & + \dots + \underbrace{(\hat{T}_{12\dots m-1} + \hat{T}_{12\dots m-2m} + \dots)}_{C_m^{m-1}} + \underbrace{(\tilde{T}_{12\dots m-1} + \tilde{T}_{12\dots m-2m} + \dots)}_{C_m^{m-1}} + \\
 & + \underbrace{\hat{T}_{12\dots m}}_{C_m^m}.
 \end{aligned} \tag{14}$$

Let us consider of the left part of inequality (14). Taking into account $v(N) = T^*$ and $1 + C_m^1 + \dots + C_m^{m-1} = 2^m - 1$, we have

$$\begin{aligned}
 & (2^m - 1) \cdot T^* \geq \sum_{k=1}^m \hat{T}_k + \sum_{k=1}^m \tilde{T}_k \\
 & + \underbrace{(\hat{T}_{12} + \hat{T}_{13} + \dots + \hat{T}_{m-1,m})}_{C_m^2} + \underbrace{(\tilde{T}_{12} + \tilde{T}_{13} + \dots + \tilde{T}_{m-1,m})}_{C_m^2} \\
 & + \underbrace{(\hat{T}_{123} + \hat{T}_{124} + \dots)}_{C_m^3} + \underbrace{(\tilde{T}_{123} + \tilde{T}_{134} + \dots)}_{C_m^3} + \\
 & + \underbrace{(\hat{T}_{1234} + \hat{T}_{1235} + \dots)}_{C_m^4} + \underbrace{(\tilde{T}_{1234} + \tilde{T}_{1234} + \dots)}_{C_m^4} + \\
 & + \dots + \underbrace{(\hat{T}_{12\dots m-1} + \hat{T}_{12\dots m-2m} + \dots)}_{C_m^{m-1}} + \underbrace{(\tilde{T}_{12\dots m-1} + \tilde{T}_{12\dots m-2m} + \dots)}_{C_m^{m-1}} - \\
 & + \tilde{T} + \hat{T}_{1\dots m}.
 \end{aligned} \tag{15}$$

Let us consider the following of values

- \hat{T}_{j_i} and $\tilde{T}_{j_1\dots j_{i-1}j_{i+1}\dots j_m}$ ($m = C_m^1$ pairs);
- \tilde{T}_{j_i} and $\hat{T}_{j_1\dots j_{i-1}j_{i+1}\dots j_m}$ ($m = C_m^{m-1}$ pairs);
- $\hat{T}_{j_i j_k}$ and $\tilde{T}_{j_1\dots j_{i-1}j_{i+1}\dots j_{k-1}j_{k+1}\dots j_m}$ (C_m^2 pairs);
- $\tilde{T}_{j_i j_k}$ and $\hat{T}_{j_1\dots j_{i-1}j_{i+1}\dots j_{k-1}j_{k+1}\dots j_m}$ (C_m^{m-2} pairs);
- ...
- \tilde{T} and $\hat{T}_{1\dots m}$ (1 pair).

By superadditivity of the game, we get

$$\begin{aligned} \hat{T}_{j_i} + \tilde{T}_{j_1 \dots j_{i-1} j_{i+1} \dots j_m} &\leq T^* \\ \tilde{T}_{j_i} + \hat{T}_{j_1 \dots j_{i-1} j_{i+1} \dots j_m} &\leq T^* \\ \hat{T}_{j_i j_k} + \tilde{T}_{j_1 \dots j_{i-1} j_{i+1} \dots j_{k-1} j_{k+1} \dots j_m} &\leq T^* \\ \tilde{T}_{j_i j_k} + \hat{T}_{j_1 \dots j_{i-1} j_{i+1} \dots j_{k-1} j_{k+1} \dots j_m} &\leq T^* \\ \dots & \\ \tilde{T} + \hat{T}_{1 \dots m} &\leq T^*. \end{aligned}$$

The right side of (15) can be estimated as follows:

$$\begin{aligned} & \underbrace{((\hat{T}_1 + \tilde{T}_{23 \dots m}) + \dots + (\hat{T}_m + \tilde{T}_{12 \dots m-1}))}_{C_m^1} + \\ & + \underbrace{((\hat{T}_{12} + \tilde{T}_{34 \dots m}) + \dots + (\hat{T}_{m-1, m} + \tilde{T}_{12 \dots m-2}))}_{C_m^2} + \\ & + \underbrace{((\hat{T}_{123} + \tilde{T}_{4 \dots m}) + \dots + (\hat{T}_{m-1 m-2 m} + \tilde{T}_{1 \dots m-3}))}_{C_m^3} + \dots + \\ & + \underbrace{((\tilde{T}_{123} + \hat{T}_{4 \dots m}) + \dots + (\tilde{T}_{m-2 m-1 m} + \hat{T}_{1 \dots m-3}))}_{C_m^{m-3}} + \\ & + \underbrace{((\tilde{T}_{12} + \hat{T}_{34 \dots m}) + \dots + (\tilde{T}_{m-1, m} + \hat{T}_{12 \dots m-2}))}_{C_m^{m-2}} + \\ & + \underbrace{(\tilde{T}_1 + \hat{T}_{23 \dots m} + \dots + \hat{T}_m + \tilde{T}_{12 \dots m-1})}_{C_m^{m-1}} + \\ & \leq (C_m^1 + C_m^2 + C_m^3 + \dots + C_m^{m-1} + C_m^m) \cdot T^* + (\tilde{T} + \hat{T}_{1,2, \dots, m}) \leq \end{aligned} \tag{16}$$

It remains to show that the following inequality holds

$$(2^m - 1) \cdot T^* \geq (C_m^1 + C_m^2 + C_m^3 + \dots + C_m^{m-1} + C_m^m) \cdot T^*.$$

If the last inequality is true then (16) is fulfilled. Taking into account that $C_m^1 + C_m^2 + C_m^3 + \dots + C_m^{m-1} + C_m^m = 2^m - 1$, we get

$$(2^m - 1) \cdot T^* \geq (2^m - 1) \cdot T^*.$$

It is obvious that the last inequality holds for any m . So, inequality (15) is satisfied. This means that inequality (14) is also satisfied for any initial positions of the players. Hence, system (13) is combined.

It remains to show that any vector satisfying the system (13) is an imputation of the game $\Gamma_v(z_P^0, z_1^0, \dots, z_m^0)$. Indeed, it can be easily checked that the vector $\eta^0 = (T^* - \sum_{i=1}^m \hat{T}_i, \hat{T}_1, \hat{T}_2, \dots, \hat{T}_{m-1}, \hat{T}_m)$ satisfies system (13) and is an imputation of this game. This completes the proof.

6. Conclusion

The considered cooperative and noncooperative approaches to investigation of group pursuit games with one pursuer and m -evaders give us various interesting solutions and allow to look at the same problem from different points of view. This paper extends an application area of group pursuit games.

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