Solidary Solutions to Games with Restricted Cooperation

Natalia Naumova

St.Petersburg State University, Faculty of Mathematics and Mechanics
Universitetsky pr. 28, Petrodvorets, St.Petersburg, 198504, Russia
E-mail: Nataliai.Naumova@mail.ru

Abstract In TU–cooperative game with restricted cooperation the values of characteristic function $v(S) > 0$ are defined only for $S \in \mathcal{A}$, where $\mathcal{A}$ is a collection of some nonempty coalitions of players.
We examine generalizations of both the proportional solutions of claim problem (Proportional and Weakly Proportional solutions, the Proportional Nucleolus, and the Weighted Entropy solution) and the uniform losses solution of claim problem (Uniform Losses and Weakly Uniform Losses solutions, the Nucleolus, and the Least Square solution). These generalizations are U–equal sacrifice solution, the U–nucleolus and qU–solutions, where U and q are strictly increasing continuous functions.
We introduce Solidary (Weakly Solidary) solutions, where if a total share of some coalition in $\mathcal{A}$ is less than its claim, then the total shares of all coalitions in $\mathcal{A}$ (that don't intersect this coalition) are less than their claims. The existence conditions on $\mathcal{A}$ for two versions of solidary solution are described. In spite of the fact that the versions of the solidary solution are larger than the corresponding versions of the proportional solution, the necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the U–nucleolus in two versions of the solidary solution coincide with conditions on $\mathcal{A}$ for inclusion of the proportional nucleolus in the corresponding versions of the proportional solution. The necessary and sufficient conditions on $\mathcal{A}$ for inclusion qU–solutions in two versions of the solidary solution coincide with conditions on $\mathcal{A}$ for inclusion of the Weighted Entropy solution in the corresponding versions of the proportional solution.
Moreover, necessary and sufficient conditions on $\mathcal{A}$ for coincidence the U–nucleolus with the U–equal sacrifice solution and conditions on $\mathcal{A}$ for coincidence qU–solutions with the U–equal sacrifice solution are obtained.

Keywords: claim problem; cooperative games; proportional solution; weighted entropy; nucleolus.

1. Introduction

A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of nonempty coalitions of agents, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T\in\mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T. We assume that $\mathcal{A}$ covers N and $N \notin \mathcal{A}$.

A *set of imputations of* $(N, \mathcal{A}, c, v)$ is the set

$$\{\{y_i\}_{i\in N} : \ y_i \geq 0, \ \sum_{i\in N} y_i = c\}.$$

A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of its set of imputations. Then $F(N, \mathcal{A}, c, v)$ is a *solution of* $(N, \mathcal{A}, c, v)$. We denote $y(S) = \sum_{i\in S} y_i$.

If $\mathcal{A} = \{\{i\} : i \in N\}$ then a *claim problem* arises, therefore, a cooperative game with restricted cooperation can be considered as a claim problem with coalition demands.

Solutions of claim problem and their axiomatic justifications are described in surveys (Moulin, 2002) and (Thomson, 2003). For claim problems, the Proportional solution, the Uniform Losses solution and their generalization Equal Sacrifice solution are well known. The papers (Naumova, 2011, 2012) and this paper consider generalizations of these solutions to games with restricted cooperation.

For claim problems, the Proportional solution, the Proportional Nucleolus, and the Weighted Entropy solution give the same results. In the case of generalized claim problems, the Proportional solution is the most natural generalization, but this set can be empty for some games. The larger set is the Weakly Proportional solution, where the ratios of total shares of coalitions to their claims are equal for disjoint coalitions in $\mathcal{A}$. This set can also be empty. The Proportional Nucleolus and the Weighted Entropy solution are always nonempty and define uniquely total shares of coalitions in $\mathcal{A}$. These solutions can give different results.

For claim problems, the Uniform Losses solution, the nucleolus, and the Least Square solution give the same results. For generalized claim problems, the Uniform Losses solution and the Weak Uniform Losses solution are the most natural generalizations but they can be empty. The Nucleolus and the Least Square solution can give different results, but each of them is always nonempty and define uniquely total shares of coalitions in $\mathcal{A}$.

Necessary and sufficient conditions on $\mathcal{A}$ that provide the existence of the Proportional solution (Weakly Proportional solution) are obtained in (Naumova, 2011) and these conditions coincide with conditions that provide the existence of the Uniform Losses solution (Weakly Uniform Losses solution).

Necessary and sufficient conditions on $\mathcal{A}$ that provide inclusion of the Weighted Entropy solution in the Proportional solution are the same as conditions on $\mathcal{A}$ for inclusion of the Least Square solution in the Uniform Losses solution. The same are conditions for inclusion of the Proportional Nucleolus in the Proportional solution and conditions for inclusion of the Nucleolus in the Uniform Losses solution. These conditions were obtained in (Naumova, 2011). That paper also contains necessary and sufficient conditions on $\mathcal{A}$ for coincidence the Weighted Entropy solution and the Weakly Proportional solution.

The paper (Naumova, 2012) considers only generalizations of the Proportional solution of claim problems. Generalizations of the Weighted Entropy solution that are called g–solutions are introduced. Necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the g–solution in the Weakly Proportional solution are the same for all g. These conditions permit to obtain for each g the necessary and sufficient conditions on $\mathcal{A}$ for coincidence the g–solution and the Weakly Proportional solution. The obtained conditions are the same as conditions for coincidence the Weighted Entropy solution and the Weakly Proportional solution. The paper (Naumova, 2012) also contains necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the Proportional Nucleolus in the Weakly Proportional solution. The proofs of that paper are not suitable for obtaining conditions on $\mathcal{A}$ for inclusion of the Nucleolus and the Least Square solution in the Weakly Uniform Losses solution.

In this paper we consider two topics. First, for strictly increasing continuous functions U, we introduce U-equal sacrifice solutions that generalize both the Proportional solution and the Uniform Losses solution, U–nucleolus that generalize both the Proportional Nucleolus and the Nucleolus, and qU–solutions that generalize both q–solutions and the Least Square solution. All results of the paper (Naumova, 2012) concerning the proportional case are generalized. In particular, we obtain conditions on $\mathcal{A}$ that provide inclusion of the Nucleolus in the Weakly Uniform Losses solution and conditions on $\mathcal{A}$ that provide inclusion of the Least Square solution in the Weakly Uniform Losses solution.

Moreover, we obtain the necessary and sufficient conditions on $\mathcal{A}$ that provide coincidence of the U–nucleolus and the Weakly U–equal sacrifice solution.

Second, we introduce new solution concepts of Solidary solution (Weakly Solidary solution) that contain U–equal sacrifice solutions (Weakly U–equal sacrifice solutions).

For almost all solutions of claim problems, if one agent gets less than its claim then each agent gets less than its claim, i.e., the *solidarity property* takes place. Two versions of Solidary solutions are obtained by generalizations of the solidarity property to games with restricted cooperation.

In spite of the fact that the versions of the Solidary solution are larger than the corresponding versions of the Proportional solution, the conditions on $\mathcal{A}$ that ensure existence results for the versions of the Solidary solutions are the same as for the corresponding versions in the proportional case. Moreover, the conditions on $\mathcal{A}$ that provide inclusions of the U–nucleolus in the Solidary solution (Weakly Solidary solution) are the same as conditions on $\mathcal{A}$ that provide inclusions of the Proportional Nucleolus in the Proportional (Weakly Proportional) solution. The conditions on $\mathcal{A}$ that provide inclusion of the qU–solution in the Solidary (Weakly Solidary) solution are the same as conditions on $\mathcal{A}$ for inclusion of the Weighted Entropy solution in the Proportional (Weakly Proportional) solution.

The paper is organized as follows. The definitions of U–equal sacrifice solutions, U–nucleolus, qU–solutions, the Solidary solutions and the relations between U–equal sacrifice solutions and the Solidary solutions are given in Section 2. Some properties of qU–solutions that will be used in next sections are obtained in Section 3. Conditions on $\mathcal{A}$ for existence the U–equal sacrifice and the Weakly U–equal sacrifice solutions are described in Section 4. Necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the qU–solution in the U–equal sacrifice solution and in the Solidary solution and for inclusion of the U–nucleolus in the U–equal sacrifice solution and in the Solidary solution are obtained in Section 5. In Section 6 we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the U–nucleolus in the Weakly U—equal sacrifice solution and in the Weakly Solidary solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion of qU–solution in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution. In Section 7 we describe necessary and sufficient conditions on $\mathcal{A}$ for coincidence the qU–solution with the Weakly U–equal sacrifice solution and conditions on $\mathcal{A}$ for coincidence the U–nucleolus with the Weakly U–equal sacrifice solution.

2. Definitions

Definition 1. A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of coalitions of agents, $N \notin \mathcal{A}$, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T\in\mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T.

We assume that $\mathcal{A}$ covers N.

Definition 2. A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of *its set of imputations* $\{\{y_i\}_{i\in N} : \ y_i \geq 0, \ \sum_{i\in N} y_i = c\}$. We denote $y(S) = \sum_{i\in S} y_i$.

Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Denote $U(0) = \lim_{t\to 0} U(t)$.

Definition 3. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *U–equal sacrifice solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$, $y(T) > 0$ implies $U(y(T)) - U(v(T)) \leq U(y(S)) - U(v(S))$.

Definition 4. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Proportional solution* of $(N, \mathcal{A}, c, v)$ iff $y(T)/v(T) = y(S)/v(S)$ for all $S, T \in \mathcal{A}$.

Definition 5. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Uniform Losses solution of* $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$, $y(T) > 0$ implies $y(T) - v(T) \leq y(S) - v(S)$, i.e., y belongs to the U–equal sacrifice solution for $U(t) = t$.

Definition 6. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly U–equal sacrifice solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$,
$y(T) > 0$ implies $U(y(T)) - U(v(T)) \leq U(y(S)) - U(v(S))$.

Definition 7. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Proportional solution* of $(N, \mathcal{A}, c, v)$ iff for $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$, $y(T)/v(T) = y(S)/v(S)$.

Definition 8. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Uniform Losses solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$,
$y(T) > 0$ implies $y(T) - v(T) \leq y(S) - v(S)$.

Remark 1. Let $U(0) = -\infty$. Then for each x in the U–equal sacrifice solution, $U(x(Q)) - U(v(Q)) = U(x(S)) - U(v(S))$ for all $S, Q \in \mathcal{A}$. If x belongs to the Weakly U–equal sacrifice solution, then for each S, Q in the same for each $Q, S \in \mathcal{A}$ with $Q \cap P = \emptyset$, either $x(Q) = x(S) = 0$ or $U(x(Q)) - U(v(Q)) = U(x(S)) - U(v(S))$.

Proof. Let x belong to the U–equal sacrifice solution. Since $\mathcal{A}$ covers N and $x(N) > 0$, there exists $T \in \mathcal{A}$ such that $x(T) > 0$. For each $S \in \mathcal{A}$, we have and $U(x(T)) - U(v(T)) \leq U(x(S)) - U(v(S))$, hence $U(x(S)) > -\infty$ and $x(S) > 0$, then we get the equality.

The case of Weakly U–equal sacrifice solution is considered similarly. □

Therefore, the Proportional solution coincides with the ln–equal sacrifice solution and the Weakly Proportional solution coincides with the Weakly ln–equal sacrifice solution.

Definition 9. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Solidary solution* of $(N, \mathcal{A}, c, v)$ iff $x(Q) < v(Q)$ for some $Q \in \mathcal{A}$ implies $x(T) < v(T)$ for all $T \in \mathcal{A}$.

Definition 10. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Solidary solution* of $(N, \mathcal{A}, c, v)$ iff $x(Q) < v(Q)$ for some $Q \in \mathcal{A}$ implies $x(T) < v(T)$ for all $T \in \mathcal{A}$ with $Q \cap T = \emptyset$.

Proposition 1. *Each U–equal sacrifice solution is contained in the Solidary solution. Each Weakly U–equal sacrifice solution is contained in the Weakly Solidary solution.*

Proof. Let y belong to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and $y(Q) < v(Q)$. Then $U(y(Q)) - U(v(Q)) < 0$. Let $T \in \mathcal{A}$. If $y(T) = 0$, then $y(T) < v(T)$, and if $y(T) > 0$, then $U(y(T)) - U(v(T)) \leq U(y(Q)) - U(v(Q)) < 0$, hence $y(T) < v(T)$.

The case of the Weakly Solidary solution is considered similarly. □

Let U be a strictly increasing continuous function defined on $(0, +\infty)$.

Definition 11. Let $X \subset R^n$, $f_1, \ldots, f_k$ be functions defined on X. For $z \in X$, let π be a permutation of $\{1, \ldots, k\}$ such that $f_{\pi(i)}(z) \leq f_{\pi(i+1)}(z)$, $\theta(z) = \{f_{\pi(i)}(z)\}_{i=1}^k$. Then $y \in X$ belongs to the *nucleolus with respect to* $f_1, \ldots, f_k$ *on* X iff $\theta(y) \geq_{lex} \theta(z)$ for all $z \in X$.

Definition 12. A vector $y = \{y_i\}_{i \in N}$ belongs to the *U– nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ on X, where $f_T(z) = U(z(T)) - U(v(T))$ and X is defined as follows. If $U(0) > -\infty$ then X is the set of imputations of $(N, \mathcal{A}, c, v)$ and if $U(0) = -\infty$ then X is the set of imputations z of $(N, \mathcal{A}, c, v)$ such that $z(T) > 0$.

For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the U–nucleolus of $(N, \mathcal{A}, c, v)$ is nonempty and defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Definition 13. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Proportional nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T)/v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

The Proportional nucleolus coincides with the ln–nucleolus.

Definition 14. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T) - v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

Note that even in the case when $\mathcal{A} = 2^N \setminus \{N, \emptyset\}$, the Nucleolus of $(N, \mathcal{A}, c, v)$ does not coincide with the nucleolus of the corresponding TU game because the set of imputations in our definition does not depend on the values of singletons.

$q - U$–solutions

Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $\mathcal{Q}(U)$ be a class of strictly increasing continuous functions q defined on $(-\infty, +\infty)$ such that $q(0) = 0$ and $\lim\limits_{x \to 0} \int\limits_a^x q(U(t))dt < +\infty$ for each $a > 0$.

Definition 15. A vector $y = \{y_i\}_{i \in N}$ belongs to the *qU–solution* of $(N, \mathcal{A}, c, v)$ iff y minimizes

$\sum\limits_{S \in \mathcal{A}} \int\limits_{v(S)}^{z(S)} q(U(t) - U(v(S)))dt$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

Examples of qU-solutions

1. $U(t) = \ln t$, $q(t) = t$, then

$$\int_{v(S)}^{z(S)} q(U(t) - U(v(S)))dt = z(S)[\ln(z(S)/v(S)) - 1] + v(S)$$

and the qU–solution is the *Weighted Entropy solution* (Naumova, 2000, 2008, 2010).

2. $U(t) = \ln t$, $q(t) = (exp(t))^p - 1$, where $p > 0$, then we obtain the minimization problem for $\sum\limits_{S \in \mathcal{A}} z(S)[\frac{z(S)^p}{(p+1)v(S)^p} - 1]$ that was considered in (Yanovskaya, 2002).

3. $U(t) = t = q(t)$, then we obtain the *Least Square solution* that solves the minimization problem for $\sum\limits_{T \in \mathcal{A}} (z(T) - v(T))^2$ on the set of imputations.

3. Existence rezults

The U–nucleolus and the qU–solution are always nonempty sets. Now we describe conditions on $\mathcal{A}$ which ensure that U–equal sacrifice solutions, Weakly U–equal sacrifice solutions, Solidary solutions, Weakly Solidary solutions are nonempty sets. We found that these conditions are the same for all U and coincide with the corresponding versions for Solidarity solutions.

Theorem 1. *Let U be a strictly increasing continuous function defined on* $(0, +\infty)$. *Then the following 3 statements are equivalent.*
1. The U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ *is nonempty for all* $c > 0$*, all* v *with* $v(T) > 0$.
2. The Solidary solution of $(N, \mathcal{A}, c, v)$ *is nonempty for all* $c > 0$*, all* v *with* $v(T) > 0$.
3. $\mathcal{A}$ is a minimal covering of N.

Proof. Let $\mathcal{A}$ be a minimal covering of N. Then for each $S \in \mathcal{A}$ there exists $j(S) \in S \setminus \cup_{Q \in \mathcal{A} \setminus \{S\}} Q$. Denote $J = \{j(S) : S \in \mathcal{A}\}$. For $(N, \mathcal{A}, c, v)$, take $y = \{y_i\}_{i \in N}$ such that $y_i = 0$ for all $i \in N \setminus J$, $\sum_{i \in N} y_i = c$, and $\{y_{j(S)}\}_{S \in \mathcal{A}}$ is the U–equal sacrifice solution of the claim problem $(J, c, \{v(S)\}_{S \in \mathcal{A}})$. Then y belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and by Proposition 1, y belongs to the Solidary solution of $(N, \mathcal{A}, c, v)$.

Let the Solidary solution of $(N, \mathcal{A}, c, v)$ be nonempty for all $c > 0$, all v with $v(T) > 0$. Suppose that $\mathcal{A}$ is not a minimal covering of N, then there exists $S \in \mathcal{A}$ such that $\mathcal{A} \setminus \{S\}$ covers N. Take $c > 0$, $v(S) > c$, $v(Q) = \epsilon$, where $0 < \epsilon < c/|\mathcal{A}|$ for all $Q \in \mathcal{A} \setminus \{S\}$. Let y belong to the Solidary solution of $(N, \mathcal{A}, c, v)$. Then $y(S) \le c < v(S)$ and for each $Q \in \mathcal{A} \setminus \{S\}$, $y(Q) < \epsilon$, hence $\sum_{i \in N} y_i \le |\mathcal{A}|\epsilon < c$, but this contradicts to $\sum_{i \in N} y_i = c$. □

Now we describe conditions on $\mathcal{A}$ that ensure existence of Weakly U–equal sacrifice solutions and Weakly Solidary solutions. The following result of the author will be used.

Theorem 2 (Naumova, 1978, Theorem 2 or 2008 Corollary 1). *Let $c > 0$, $I(c) = \{x \in R^{|N|} : x_i \geq 0,\ x(N) = c\}$, Gr be an undirected graph with the set of nodes $\mathcal{A}$, $\{\succ_x\}_{x \in I(c)}$ be a family of relations on $\mathcal{A}$, and for each $K \in \mathcal{A}$*

$$F^K = \{x \in I(c) : L \not\succ_x K \text{ for all } L \in \mathcal{A}\}.$$

Let $\{\succ_x\}_{x \in I(c)}$ satisfy the following 5 conditions.
1. $\succ_x$ is acyclic on $\mathcal{A}$.
2. If $K \in \mathcal{A}$ and $x_i = 0$ for all $i \in K$, then $x \in F^K$.
3. The set F^K is closed for each $K \in \mathcal{A}$.
4. If $K \succ_x L$, then K and L are adjacent in the graph Gr.
5. If a single node is taken out from each component of Gr, then the remaining elements of $\mathcal{A}$ do not cover N.

Then there exists $x^0 \in I(c)$ such that $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$.

Theorem 3. *Let $G(\mathcal{A})$ be the undirected graph, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$. Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Then the following 3 statements are equivalent.*
1. The Weakly solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$.
2. The Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$.
3. $\mathcal{A}$ satisfies the following condition.
C0. If a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

Proof. Suppose that $\mathcal{A}$ satisfies C0. Fix $(N, \mathcal{A}, c, v)$. For each imputation x, consider the following relation on $\mathcal{A}$: $P \succ_x Q$ iff $P \cap Q = \emptyset$, $x(Q) > 0$, and $U(x(P)) - U(v(P)) < U(x(Q)) - U(v(Q))$. Then x^0 belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ iff $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$. This family of relations and the graph $G(\mathcal{A})$ satisfy all conditions of Theorem 2, hence the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ is a nonempty set. In view of Proposition 1, this implies that the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set.

Now suppose that the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$ and let us prove that C0 is satisfied. Suppose that $\mathcal{A}$ does not satisfy the condition C0. Let m be the number of components of $G(\mathcal{A})$, $S_1, \dots, S_m$ be the nodes taken out from each component of $G(\mathcal{A})$ such that $\mathcal{A} \setminus \{S_1, \dots, S_m\}$ cover N.

Let us take $c > 0$, $v(S_i) = c$ for all $i = 1, \dots, m$, $v(Q) = \epsilon$ for remaining $Q \in \mathcal{A}$, where $\epsilon|\mathcal{A}| < c$. Let y belong to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$. If $Q \cap S_i = \emptyset$, then $y(S_i) > 0$ implies $y(S_i) < v(S_i)$, therefore $y(Q) < \epsilon$ for $Q \neq S_i$, and as such Q cover N, we get $y(N) \leq |\mathcal{A}|\epsilon < c = y(N)$. This contradiction completes the proof. □

4. Properties of qU–solutions

Property 1. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $U(t) \to -\infty$ as $t \to 0$, $q \in \mathcal{Q}(U)$, $q \to \infty$ as $t \to \infty$, and x belong to the qU–solution of $(N, \mathcal{A}, c, v)$. Then $x(S) > 0$ for all $S \in \mathcal{A}$.

Proof. Suppose that there exist $(N, \mathcal{A}, c, v)$, $S \in \mathcal{A}$, and x in qU–solution of $(N, \mathcal{A}, c, v)$ such that $x(S) = 0$. Let $0 < \epsilon < \min\{x_k : x_k > 0\}$. Let

$$M = \max_{T: T \in \mathcal{A}, x(T) > 0} \max_{t \in [x(T)-\epsilon, x(T)+\epsilon]} |q(U(t) - U(v(T)))|.$$

Fix $\delta > 0$ such that $\delta < \min\{\epsilon, \min_{T \in \mathcal{A}} v(T)\}$ and $|q(U(\delta) - U(v(S)))| > 2^{|N|}M$. Let $i \in S$, $j \in N$, $x_j > 0$.

Take $z \in R^{|N|}$ such that $z_i = x_i + \delta$, $z_j = x_j - \delta$, $z_k = x_k$ for $k \neq i, j$. Then $\sum_{T \in \mathcal{A}} \int_{v(T)}^{z(T)} q(U(t) - U(v(T)))dt - \sum_{T \in \mathcal{A}} \int_{v(T)}^{x(T)} q(U(t) - U(v(T)))dt =$
$\sum_{T \in \mathcal{A}: i \in T, j \notin T} \int_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt - \sum_{T \in \mathcal{A}: i \notin T, j \in T} \int_{x(T)-\delta}^{x(T)} q(U(t) - U(v(T)))dt.$

If $i \notin T$, $j \in T$ then $|\int_{x(T)-\delta}^{x(T)} q(U(t) - U(v(T)))dt| \leq \delta M$.

If $T = S$ then $\int_{x(S)}^{x(S)+\delta} q(U(t) - U(v(S)))dt = \int_0^{\delta} q(U(t) - U(v(S)))dt < -2^{|N|}M\delta$.

If $i \in T$, $j \notin T$, $x(T) = 0$, then $\int_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt < 0$ since $\delta < v(T)$.

If $i \in T$, $j \notin T$, $x(T) > 0$, then $|q(U(t) - U(v(T)))| \leq M$ as $t \in [x(T), x(T) + \delta]$, hence $|\int_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt| \leq \delta M$.

Thus,
$\sum_{T \in \mathcal{A}} \int_{v(T)}^{z(T)} q(U(t) - U(v(T)))dt - \sum_{T \in \mathcal{A}} \int_{v(T)}^{x(T)} q(U(t) - U(v(T)))dt <$
$(|\mathcal{A}| - 1)\delta M - 2^{|N|}M\delta < 0$ and x is not in the qU–solution of $(N, \mathcal{A}, c, v)$. □

Property 2. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$, then $f(z) = \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{z(Q)} q(U(t) - U(v(Q)))dt$ is a continuous convex function of z defined on the set of imputations of $(N, \mathcal{A}, c, v)$ and for all $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the qU–solution of $(N, \mathcal{A}, c, v)$ defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Proof. Let $a > 0$, $\psi(r) = \int_a^r q(U(t))dt$ for $r \geq 0$. If $\lim_{t \to 0} q(U(t)) > -\infty$, then $\psi(r)$ is a strictly convex function on $[0, +\infty)$. If $\lim_{t \to 0} q(U(t)) = -\infty$, then $\psi(r)$ is a convex function on $[0, +\infty)$ and a strictly convex function on $(0, +\infty)$. Therefore $f(z)$ is a convex function of z and in view of Property 1, if y and z belong to qU–solution of $(N, \mathcal{A}, c, v)$, then $y(T) = z(T)$ for all $T \in \mathcal{A}$. □

Property 3. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$. Then for each x in the qU–solution of $(N, \mathcal{A}, c, v)$, $x_i > 0$ implies

$$\sum_{T \in \mathcal{A}: i \in T} q(U(x(T)) - q(U(v(T))) \leq \sum_{T \in \mathcal{A}: j \in T} q(U(x(T)) - U(v(T)))$$

for all $j \in N$.

Proof. Let x belong to the qU–solution of $(N,\mathcal{A},c,v)$. Note that in view of Property 1, $q(U(x(Q))-U(v(Q)))$ are well defined for all $Q\in\mathcal{A}$. Let $x_i>0$. Suppose that there exists $j\in N$ such that

$$\sum_{T\in\mathcal{A}:j\in T} q(U(x(T))-U(v(T)))<\sum_{T\in\mathcal{A}:i\in T} q(U(x(T))-U(v(T))).$$

Consider $\epsilon\geq 0$ and $y(\epsilon)\in R^{|N|}$ such that $\epsilon<x_i$, $y(\epsilon)_i=x_i-\epsilon$, $y(\epsilon)_j=x_j+\epsilon$, $y(\epsilon)_k=x_k$ for $k\neq i,j$. Let

$$F(\epsilon)=\sum_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{y(\epsilon)(Q)} q(U(t)-U(v(Q)))dt-\sum_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{x(Q)} q(U(t)-U(v(Q)))dt,$$

then

$$F(\epsilon)=\sum_{Q\in\mathcal{A}:i\in Q,j\notin Q}\int\limits_{x(Q)}^{x(Q)-\epsilon} q(U(t)-U(v(Q)))dt+$$
$$\sum_{Q\in\mathcal{A}:i\notin Q,j\in Q}\int\limits_{x(Q)}^{x(Q)+\epsilon} q(U(t)-U(v(Q)))dt,$$
$$F'(0)=-\sum_{Q\in\mathcal{A}:i\in Q,j\notin Q} q(U(x(Q))-U(v(Q)))+$$
$$\sum_{Q\in\mathcal{A}:i\notin Q,j\in Q} q(U(x(Q))-U(v(Q)))<0.$$

Hence, $F(\epsilon)<0$ for some $\epsilon>0$ and x does not belong to the qU–solution of $(N,\mathcal{A},c,v)$. □

Property 4. Let U be a strictly increasing continuous function defined on $(0,+\infty)$, $q\in\mathcal{Q}(U)$, and x be an imputation of $(N,\mathcal{A},c,v)$ such that $x_i>0$ implies

$$\sum_{T\in\mathcal{A}:i\in T} q(U(x(T))-q(U(v(T)))\leq\sum_{T\in\mathcal{A}:j\in T} q(U(x(T))-U(v(T)))$$

for all $j\in N$.

Then x belongs to the qU–solution of $(N,\mathcal{A},c,v)$.

Proof. For each imputation z of $(N,\mathcal{A},c,v)$, let $f(z)=\sum\limits_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{z(Q)} q(U(t)-U(v(Q)))dt$.

If $z_j>0$ for all $j\in N$ then f is differentiable at z and

$$\frac{\partial}{\partial z_j}f(z)=\sum_{T\in\mathcal{A}:\,T\ni j} q(U(z(T))-U(v(T))). \qquad (1)$$

If z and w are imputations of $(N,\mathcal{A},c,v)$ such that $z_j,w_j>0$ for all $j\in N$, then, in view of Property 2,

$$f(w)-f(z)\geq\sum_{j\in N}\frac{\partial f(z)}{\partial z_j}(w_j-z_j). \qquad (2)$$

Note that if $x_i>0$ then for all $Q\ni i$, $x(Q)>0$ and $q(U(x(Q))-U(v(Q)))$ are well defined. Hence for all $j\in N$, $\sum\limits_{T\in\mathcal{A}:\,T\ni j} q(U(x(T))-U(v(T)))$ are well defined.

Let y be an imputation of $(N, \mathcal{A}, c, v)$. There exist imputations z^k and w^k with positive coordinates such that $\lim_{k\to+\infty} z^k = x$, $\lim_{k\to+\infty} w^k = y$, then it follows from (2) and (1) that

$$f(y) - f(x) \geq \sum_{j\in N}(y_j - x_j) \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))). \tag{3}$$

Let $x_i > 0$, then (1) implies

$$\sum_{j\in N} x_j \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))) = c \sum_{T\in\mathcal{A}:\, T\ni i} q(U(x(T)) - U(v(T))), \tag{4}$$

$$\sum_{j\in N} y_j \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))) \geq c \sum_{T\in\mathcal{A}:\, T\ni i} q(U(x(T)) - U(v(T))). \tag{5}$$

It follows from (3),(4), (5) that $f(y) - f(x) \geq 0$, i.e., x belongs to the qU–solution of $(N, \mathcal{A}, c, v)$. □

5. When generalized solutions satisfy solidarity properties?

We describe conditions on the collection of coalitions $\mathcal{A}$ that ensure the inclusion of the U–nucleolus (qU–solution) in the U–equal sacrifice solution and in the Solidary solution. We prove that these conditions depend neither on U nor on q and are the same.

Theorem 4. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Then the following 3 statements are equivalent.*
1. $\mathcal{A}$ is a partition of N.
2. The U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.
3. The U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.

Proof. Let $\mathcal{A}$ be a partition of N, then the U–nucleolus always coincides with the U-equal sacrifice solution, and by Proposition 1, it is contained in the Solidary solution.

Let the U–nucleolus be always contained in the Solidary solution. Suppose that there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) > 1$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/(4|N|)$.

Let x belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$, then $x(P) < v(P)$ and due to the solidarity property this implies $x(T) < \epsilon$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i < \epsilon$ for all $i \in N \setminus P$. As long as $\mathcal{A}$ covers N, $x(P) > 3/4$. Since x belongs to the U–nucleolus and $\mathcal{A}_P = \{T \in \mathcal{A} \setminus \{P\} : T \cap P \neq \emptyset\} \neq \emptyset$, we have $x_i = 0$ for all $i \in P \setminus \cup_{T\in\mathcal{A}_P} T$. Then $x(S) \geq x(P)/|P|$ for some $S \in \mathcal{A}_P$. Therefore,

$$x(S) \geq 3/(4|N|) > \epsilon,$$

but this contradicts to $x(S) < \epsilon$. □

Theorem 5. *Let U be a strictly increasing continuous function defined on $(0,+\infty)$, $q \in \mathcal{Q}(U)$. Then the following 3 statements are equivalent.*
1. $\mathcal{A}$ is a partition of N.
2. The qU–solution of $(N,\mathcal{A},c,v)$ is contained in the U–equal sacrifice solution of $(N,\mathcal{A},c,v)$ for all $c>0$, all v with $v(T)>0$.
3. The qU–solution of $(N,\mathcal{A},c,v)$ is contained in the Solidary solution of $(N,\mathcal{A},c,v)$ for all $c>0$, all v with $v(T)>0$.

Proof. Let $\mathcal{A}$ be a partition of N. Then for each imputation x of $(N,\mathcal{A},c,v)$,

$$\sum_{T\in\mathcal{A}:\,T\ni i} q(U(x(T))-U(v(T))) = q(U(x(S))-U(v(S))) \quad \text{for all} \quad S\in\mathcal{A},\ i\in S.$$

Let x belong to the qU–solution of $(N,\mathcal{A},c,v)$, then by Property 3, $x(S)>0$ for some $S\in\mathcal{A}$ implies $q(U(x(S))-U(v(S))) \le q(U(x(T))-U(v(T)))$ for all $T\in\mathcal{A}$. As q is a strictly increasing function, this implies $U(x(S))-U(v(S)) \le U(x(T))-U(v(T))$. Thus, x belongs to the U–equal sacrifice solution of $(N,\mathcal{A},c,v)$. Then, by Proposition 1, x belongs to the Solidary solution of $(N,\mathcal{A},c,v)$.

Let the qU–solution be always contained in the Solidary solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P,Q\in\mathcal{A}$ such that $P\cap Q\neq\emptyset$. We take the following v: $v(P)=2$, $v(T)=\epsilon$ otherwise, where $\epsilon<1/|N|$.

Let x belong to the qU–solution of $(N,\mathcal{A},1,v)$. Then $x(P)<v(P)$ and it follows from the solidarity property that $x(T)<\epsilon$ for all $T\in\mathcal{A}\setminus\{P\}$, hence $x_i<\epsilon$ for all $i\in N\setminus P$. If $x_i\le\epsilon$ for all $i\in P$, then $x(N)\le\epsilon|N|<1$, hence there exists $j_0\in P\setminus\cup_{T\in\mathcal{A}\setminus\{P\}}T$ such that $x_{j_0}>\epsilon$. Let $i_0\in P\cap Q$. By Property 3,
$q(U(x(P))-U(v(P))) = \sum_{T\in\mathcal{A}:\,T\ni j_0} q(U(x(T))-U(v(T))) \le$
$\sum_{T\in\mathcal{A}:\,T\ni i_0} q(U(x(T))-U(v(T)))$.

Since $x(T)<v(T)$ for all $T\in\mathcal{A}$ and $q(0)=0$, this implies

$$0 \le \sum_{T\in\mathcal{A}:\,T\ni i_0\,T\neq P} q(U(x(T))-U(v(T))) \le q(U(x(Q))-U(v(Q))<0.$$

In view of this contradiction, $\mathcal{A}$ is a partition of N. □

6. When generalized solutions satisfy weak solidarity properties?

In this section we obtain conditions on the collection of coalitions $\mathcal{A}$ that ensure the inclusion of the U–nucleolus in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution. We prove that these conditions coincide. We also obtain necessary and sufficient conditions on $\mathcal{A}$ that ensure the inclusion of qU–solutions in the Weakly U–equal sacrifice solution. These conditions depend neither on U nor on q and coincide with the conditions that ensure the inclusion of the qU–solutions in the Weakly Solidary solution.

For $i\in N$, denote $\mathcal{A}_i=\{T\in\mathcal{A}: i\in T\}$.

Definition 16. A collection of coalitions $\mathcal{A}$ is *weakly mixed at N* if $\mathcal{A}=\cup_{i=1}^k\mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;

C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C3) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Remark 2. If $k \leq 2$ then C3 follows from C1 and C2.

Remark 3. If $\mathcal{A}$ is a weakly mixed collection of coalitions, then it satisfies the condition C0 of Theorem 3.

Proof. Let $\mathcal{A}$ be weakly mixed at N. Take $j_0 \in N$ such that $|\mathcal{A}_{j_0}| \geq |\mathcal{A}_i|$ for all $i \in N$. Let $\mathcal{A}_{j_0} = \{Q_t\}_{t \in M}$, where $Q_t \in \mathcal{B}^t$, $M \subset \{1, \ldots, k\}$.

Let $S_t \in \mathcal{B}^t$ for all $t \leq k$. Since $\mathcal{A}$ is weakly mixed, there exists $i_o \in \bigcap_{t \in M} S_t$. In view of definition of j_0, $\mathcal{A}_{i_0} = \{S_t : t \in M\}$. Therefore, if for each $t \in \{1, \ldots, k\}$, S_t is taken out from $\mathcal{A}$, then the remaining elements of $\mathcal{A}$ do not cover i_0. □

Example 1. Let $N = \{1, 2, \ldots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N.

Example 2. $N = \{1, 2, \ldots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12, 13\}\}$.
Then $\mathcal{A}$ is weakly mixed at N.

Example 3. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
$\mathcal{B}^3 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$,
then $\mathcal{C}$ satisfies C0, C1, and C2, but does not satisfy C3 (for $i = 1$ and $Q = \{1, 2\}$), hence $\mathcal{C}$ is not weaky mixed at N.

Proposition 2. *Let U be a strictly increasing continuous function defined on* $(0, +\infty)$ *and the U–nucleolus of $(N, \mathcal{A}, t, v)$ be contained in the Weakly Solidary solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$.*

Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let us take the following v: $v(S) = v(P) = 1$, $v(T) = \epsilon$ for all $T \in \mathcal{A} \setminus \{S, P\}$, where $0 < \epsilon < 1/2|N|$. Let $j \in P \cap Q$.

Let x belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$. First, we prove that $x(Q) \geq x(P)/|P|$. Assume the contrary, then $x(P \cap Q) < x(P)/|P|$, hence there exists $i_0 \in P \setminus Q$ such that $x_{i_0} > x(P)/|P|$.

Let $i_0 \notin T$ for all $T \in \mathcal{A} \setminus \{P\}$ then we take $y \in R^{|N|}$: $y_{i_0} = 0$, $y_j = x_j + x_{i_0}$, $y_i = x_i$ otherwise. Then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A}$, hence x does not belong to the U–nucleolus.

Let $i_0 \in T$ for some $T \in \mathcal{A} \setminus \{P\}$, then $T \neq S$ and $x(T) > x(P)/|P| > x(Q)$. This implies

$$U(x(T)) - U(v(T)) = U(x(T)) - U(\epsilon) > U(x(Q)) - U(v(Q)).$$

Let $z = z(\delta) \in R^{|N|}$, $z_{i_0} = x_{i_0} - \delta$, $z_j = x_j + \delta$, $z_i = x_i$ otherwise. If $\delta > 0$ and δ is sufficiently small, then for $T \in \mathcal{A}_{i_0} \setminus \{P\}$,

$$U(z(T)) - U(v(T)) > U(z(Q)) - U(v(Q)) > U(x(Q)) - U(v(Q)),$$

otherwise $z(T) \geq x(T)$, hence $\theta(z(\delta)) >_{lex} \theta(x)$. Thus

$$x(Q) \geq x(P)/|P|.$$

Weak solidarity condition for Q and S implies $x(Q) < \epsilon$, hence $x(P) < \epsilon|P|$. We consider 4 cases.

Case 1. There exists $j_0 \notin P \cup Q \cup S$ such that $x_{j_0} > \epsilon$. Then for all $T \ni j_0$, $x(T) > v(T)$. Let $w = w(\delta) \in R^{|N|}$, $w_{j_0} = x_{j_0} - \delta$, $w_j = x_j + \delta$, $w_i = x_i$ otherwise. Then for $\delta > 0$, $w(Q) > x(Q)$ and for sufficiently small δ, $w(Q) < v(Q)$, and $w(T) > v(T)$ for all $T \ni j_0$, hence we get $\theta(w(\delta)) >_{lex} \theta(x)$, and the Case 1 is impossible.

Case 2. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$ and $x(S) \leq x(P)$. Then

$$x(N) \leq x(Q) + 2\epsilon|P| + \epsilon|N \setminus (P \cup S \cup Q)| \leq 2\epsilon|N| < 1$$

and this contradicts $x(N) = 1$.

Case 3. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$ and $x_i \leq \epsilon$ for all $i \in S$. This implies $1 = x(N) \leq \epsilon|N| < 1$, hence this case is impossible.

Case 4. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$, $x(S) > x(P)$ and $x_{i_0} > \epsilon$ for some $i_0 \in S$. Then $x(T) > v(T)$ for $T \neq S, T \ni i_0$. Let $y = y(\delta) \in R^{|N|}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_i = x_i$ otherwise. Then for $\delta > 0$, $y(Q) > x(Q)$, $y(P) > x(P)$ and for sufficiently small $\delta > 0$, we get $\theta(y(\delta)) >_{lex} \theta(x)$. This contradiction completes the proof. □

Theorem 6. *Let U be a strictly increasing continuous function defined on* $(0, +\infty)$.

If $\mathcal{A}$ is a weakly mixed collection of coalitions at N then for all $c > 0$, all v with $v(T) > 0$, the U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let, moreover, either U be a convex function or $U(t) \to +\infty$ as $t \to +\infty$. Let the U–nucleolus of $(N, \mathcal{A}, c, v)$ be contained in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. Then $\mathcal{A}$ is a weakly mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be weakly mixed at N and x belong to the U–nucleolus of $(N, \mathcal{A}, c, v)$. We prove that x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Suppose the contrary, i.e., there exist $S, Q \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $U(x(Q)) - U(v(Q)) < U(x(S)) - U(v(S))$ and $x(S) > 0$. Take $i_0 \in S$ such that $x_{i_0} > 0$. Since $\mathcal{A}$ is weakly mixed, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Take $\delta > 0$ such that

$$U(x(Q) + \delta)U(v(Q)) < U(x(S) - \delta) - U(v(S))$$

and $\delta < x_{i_0}$. Let $y = \{y_i\}_{i\in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Then $y(P) < x(P)$ only for $P = S$ and $y(Q) > x(Q)$. Since $U(y(Q)) - U(v(Q)) < U(y(S)) - U(v(S))$, this contradicts the definition of the U-nucleolus. Therefore, x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and by Proposition 1, x belongs to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let either U be is a convex function or $U(t) \to +\infty$ as $t \to +\infty$ and the U–nucleolus be always contained in the Weakly Solidary solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 3. Then $\mathcal{A}$ satisfies C2 by the definition of $G(\mathcal{A})$ and satisfies C1 in view of Proposition 2. Suppose that $\mathcal{A}$ is not weakly mixed. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ for all $j \in N$. Let $0 < \epsilon < 1/|N|$. We can take v with the following properties:
$v(S) = 1$,
$U(v(P)) > 2U(1) - U(1/|N|)$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.
Let x belong to the U–nucleolus and to the Weakly Solidary solution of $(N, \mathcal{A}, 1, v)$. Since $S \cap Q = \emptyset$, $x(N) = 1$, and $v(S) + v(Q) > 1$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Take $\delta > 0$ such that $\delta < 1/|N|$ and for each $T, P \in \mathcal{A}$,

$$U(x(T)) - U(v(T)) < U(x(P)) - U(v(P))$$

implies

$$U(x(T) + \delta) - U(v(T)) < U(x(P) - \delta) - U(v(P)).$$

Let $y = \{y_i\}_{i\in N}$, $y_{i_0} = x_{i_0} + \delta$, $y_{j_0} = x_{j_0} - \delta$, $y_i = x_i$ otherwise.

We prove that there exists $P \in \mathcal{A}$ such that $y(P) > x(P)$ and $U(x(P)) - U(v(P)) < U(x(T)) - U(v(T))$ for all $T \in \mathcal{A}$ with $y(T) < x(T)$ and this would imply that x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$. Consider 2 cases.

Case 1. $j_0 \notin S$. Let $y(T) < x(T)$, then $T \ni j_0$ and $v(T) = \epsilon$, hence

$$U(x(T)) - U(v(T)) \geq U(x_{j_0}) - U\epsilon > 0.$$

Since $U(x(Q)) - U(v(Q)) < 0$ and $y(Q) > x(Q)$, we can take $P = Q$, hence x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$ in this case.

Case 2. $j_0 \in S$. Then there exists $P \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, where $y(P) > x(P)$. Let us check that

$$y(T) < x(T) \quad \text{implies} \quad U(x(P)) - U(v(P)) < U(x(T)) - U(v(T)).$$

If $T = S$ then

$$U(x(S)) - U(v(S)) \geq U(1/|N|) - U(1) > U(1) - U(v(P)) \geq U(x(P)) - U(v(P)).$$

If $T \neq S$ then $v(T) = \epsilon$ and $U(x(T)) - U(v(T)) \geq U(x_{j_0}) - U\epsilon > 0$. Since U is strictly increasing, $U(v(P)) > U(1)$, hence $U(x(P)) - U(v(P)) < 0$ and $U(x(P)) - U(v(P)) < U(x(T)) - U(v(T))$. Thus, x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$ in this case. □

Corollary 1. *The Proportional Nucleolus of* $(N, \mathcal{A}, c, v)$ *is contained in the Weakly Proportional solution and in the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$, v *with* $v(T) > 0$ *if and only if* $\mathcal{A}$ *is a weakly mixed collection of coalitions at* N.

Corollary 2. *The Nucleolus of* $(N, \mathcal{A}, c, v)$ *is contained in the Weakly Uniform Losses solution and in the the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$, v *with* $v(T) > 0$ *if and only if* $\mathcal{A}$ *is a weakly mixed collection of coalitions at* N.

Definition 17. A collection of coalitions $\mathcal{A}$ is *mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C4) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Note that if $\mathcal{A}$ is mixed at N then $\mathcal{A}$ is weakly mixed at N.

Example 4. If $\mathcal{A}$ is weakly mixed at N and all $i \in N$ belong to the same number of coalitions, then $\mathcal{A}$ is mixed at N.

Example 5. Let $N = \{1, 2, \dots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N.

Example 6. Let $N = \{1, 2, \dots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N but not mixed at N. (For $i = 3$, the condition C4 is not realized.)

Proposition 3. *Let the* qU*–solution of* $(N, \mathcal{A}, c, v)$ *be contained in the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$, *all* v *with* $v(T) > 0$. *Then the case* $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ *is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to the qU–solution of $(N, \mathcal{A}, 1, v)$. Since x satisfies the weakly solidarity property, $v(P) + v(S) > 1$, and $S \cap P = \emptyset$, we have $x(S) < v(S)$. Since $Q \cap S = \emptyset$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P\}$, hence $v(T) = \epsilon$ and $x(T)/v(T) > 1$. Thus,

$$\sum_{T \in \mathcal{A}\colon T \ni j_0, T \not\ni i_0} q(U(x(T)) - U(v(T))) \geq 0. \tag{6}$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and it follows from the weak solidarity property that $x(T) < v(T)$. If $v(T) = 1$, then $v(T) \geq x(T)$. Therefore

$$\sum_{T \in \mathcal{A}\colon T \not\ni j_0, T \ni i_0} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q))) < 0. \tag{7}$$

It follows from (6) and (7) that

$$\sum_{T \in \mathcal{A}\colon T \ni j_0} q(U(x(T)) - U(v(T))) > \sum_{T \in \mathcal{A}\colon T \ni i_0} q(U(x(T)) - U(v(T))),$$

but this contradicts Property 3. □

Theorem 7. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$.*

The qU–solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be a mixed collection of coalitions. Let x belong to the qU–solution of $(N, \mathcal{A}, c, v)$. We prove that x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Suppose that there exist $Q, S \in \mathcal{A}$ such that $Q \cap S = \emptyset$, $x(Q) > 0$, and $U(x(Q)) - U(v(Q)) > U(x(S)) - U(v(S))$. There exists $i_0 \in Q$ with $x_{i_0} > 0$. Since $\mathcal{A}$ is mixed, there exists $j_0 \in N$ such that $\mathcal{A}_{j_0} = \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Then

$$\sum_{T \in \mathcal{A}\colon T \not\ni j_0, T \ni i_0} q(U(x(T)) - U(v(T))) = q(U(x(Q)) - U(v(Q))),$$

$$\sum_{T \in \mathcal{A}\colon T \ni j_0, T \not\ni i_0} q(U(x(T)) - U(v(T))) = q(U(x(S)) - U(v(S))),$$

hence

$$\sum_{T \in \mathcal{A}\colon T \ni i_0} q(U(x(T)) - U(v(T))) > \sum_{T \in \mathcal{A}\colon T \ni j_0} q(U(x(T)) - U(v(T))),$$

but this contradicts Property 3. Thus, x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, and due to Proposition 1, x belongs to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let the qU–solution be always contained in the Weakly Solidary solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 3. By the definition of $G(\mathcal{A})$, $\mathcal{A}$ satisfies C2. In view of Proposition 3, $\mathcal{A}$ satisfies C1. Suppose that $\mathcal{A}$ is not mixed at N. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ with $S \cap Q = \emptyset$ such that for each $j \in N$, $\mathcal{A}_j \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$.

Let $L > 1$. Since q and U are strictly increasing continuous functions, there exists $\epsilon > 0$ such that
$\epsilon < 1/|N|$,
$U(1) - U(1 - \epsilon|N|) \leq U(L) - U(1)$,
$q(U(1 - \epsilon|N|) - U(1)) + q(U(1/|N|) - U(\epsilon)) \geq 0$.

We take the following v:
$v(S) = 1$,
$U(v(P)) = L$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to the qU–solution and to the Weakly Solidary solution of $(N, \mathcal{A}, 1, v)$. Since $v(S) + v(Q) > 1$ and $x(N) = 1$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$. We shall prove that there exists $j_1 \in N$ such that $x_{j_1} > 0$ and

$$\sum_{T\in\mathcal{A}:T\ni i_0} q(U(x(T)) - U(v(T))) < \sum_{T\in\mathcal{A}:T\ni j_1} q(U(x(T)) - U(v(T))), \qquad (8)$$

and this would contradict Property 3 of qU–solutions.

The following 3 cases are possible.
1. There exists $j_1 \notin S$ such that $x_{j_1} \geq \epsilon$.
2. $x_j < \epsilon$ for all $j \notin S$ and $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \neq \{Q\}$.
3. $x_j < \epsilon$ for all $j \notin S$ and $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} = \{Q\}$.

Case 1. Since $v(P) > 1$ for all $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,

$$\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_1}} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q))) < 0.$$

Since $j_1 \notin S$, $x(T) \geq v(T) = \epsilon$ for all $T \in \mathcal{A}_{j_1} \setminus \mathcal{A}_{i_0}$, therefore,

$$\sum_{T\in\mathcal{A}_{j_1}\setminus\mathcal{A}_{i_0}} q(U(x(T)) - U(v(T))) \geq 0,$$

this implies (8).

Case 2. We have $x(S) \geq 1 - \epsilon|N|$ and $j_0 \in S$. There exists $P^0 \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, then $\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0}} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q)))+$
$q(U(x(P^0)) - (v(P^0))) < q(U(x(P^0)) - U(v(P^0)))$.

If $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ then either $T = S$ or $x(T) > v(T) = \epsilon$, therefore

$$\sum_{T\in\mathcal{A}_{j_0}\setminus\mathcal{A}_{i_0}} q(U(x(T)) - U(v(T))) \geq q(U(x(S)) - U(v(S))).$$

Since $U(1 - \epsilon|N|) - U(1) \geq U(1) - U(L)$, we get

$$q(U(x(S)) - U(S)) \geq q(U(1 - \epsilon|N|) - U(1)) \geq q(U(1) - U(L)) \geq q(U(x(P^0)) - U(v(P^0)))$$

and this implies (8) for $j_1 = j_0$.

Case 3. Since $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} = \{Q\}$ and $\mathcal{A}_{j_0} \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$, there exists $T_0 \in \mathcal{A} \setminus \mathcal{A}_{i_0}$ such that $j_0 \in T_0$ and $T_0 \neq S$. Then

$$\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0}} q(U(x(T)) - U(v(T))) = q(U(x(Q)) - U(v(Q))) < 0,$$

In view of $x(S) \geq 1-\epsilon|N|$, $x(T_0) \geq 1/|N|$, $v(T_0)=\epsilon$, and restrictions on ϵ, we have
$\sum_{T\in\mathcal{A}_{j_0}\setminus\mathcal{A}_{i_0}} q(U(x(T))-U(v(T))) \geq q(U(x(S))-U(v(S)))+$
$q(U(x(T_0))-U(v(T_0))) \geq q(U(1-\epsilon|N|)-U(1))+q(U(1/|N|)-U(\epsilon)) \geq 0.$
Thus, we obtain (8) for $j_1=j_0$. □

Corollary 3. *The Weighted Entropy solution of* $(N,\mathcal{A},c,v)$ *is contained in the Weakly Proportional solution and in the Weakly Solidary solution of* $(N,\mathcal{A},c,v)$ *for all* $c>0$*, all* v *with* $v(T)>0$ *if and only if* $\mathcal{A}$ *is a mixed collection of coalitions at* N*.*

Corollary 4. *The Least Square solution of* $(N,\mathcal{A},c,v)$ *is contained in the Weakly Uniform Losses solution and in the Weakly Solidary solution of* $(N,\mathcal{A},c,v)$ *for all* $c>0$*, all* v *with* $v(T)>0$ *if and only if* $\mathcal{A}$ *is a mixed collection of coalitions at* N*.*

7. When different U–generalizations give the same result?

In this section necessary and sufficient conditions on $\mathcal{A}$ that provide the coincidence of the U–nucleolus with the Weakly U–equal sacrifice solution and conditions on $\mathcal{A}$ that provide the coincidence of the qU–solution with the Weakly U–equal sacrifice solution. These conditions are the same for all U and $q\in\mathcal{Q}(U)$. The result concerning qU–solutions is a generalization of the corresponding results concerning the Weighted Entropy solution (Naumova, 2011, Theorem 4) and g–solutions (Naumova, 2012, Theorem 4), but the proof of this paper also permits to solve the problem of coincidence the Least Square solution with the Uniform Losses solution. The result concerning the U–nucleolus is completely new.

Definition 18. A collection of coalitions $\mathcal{A}$ is *totally mixed at* N if $\mathcal{A}=\cup_{i=1}^{k}\mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N and for each collection $\{S_i\}_{i=1}^{k}$ ($S_i\in\mathcal{P}^i$), we have $\cap_{i=1}^{k}S_i\neq\emptyset$.

Example 7. Let $N=\{1,2,3,4\}$, $\mathcal{C}=\mathcal{B}^1\cup\mathcal{B}^2$, where
$\mathcal{B}^1=\{\{1,2\},\{3,4\}\}$,
$\mathcal{B}^2=\{\{1,3\},\{2,4\}\}$,
then $\mathcal{C}$ is a totally mixed collection of coalitions at N.

Theorem 8. *Let* U *be a continuous strictly increasing function defined on* $(0,+\infty)$ *and either* U *is a convex function or* $U(t)\to+\infty$ *as* $t\to+\infty$*. Then the* U*–nucleolus of* $(N,\mathcal{A},c,v)$ *coincides with the Weakly* U*–equal sacrifice solution of* $(N,\mathcal{A},c,v)$ *for all* $c>0$*, all* v *with* $v(T)>0$ *if and only if* $\mathcal{A}$ *is a totally mixed collection of coalitions at* N*.*

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is weakly mixed at N and it follows from Theorem 4 that the U–nucleolus of $(N,\mathcal{A},c,v)$ is always contained in the Weakly U–equal sacrifice solution of $(N,\mathcal{A},c,v)$. Since in this case for all x in the Weakly U–equal sacrifice solution of $(N,\mathcal{A},c,v)$, $x(S)$ are uniquely defined, this implies coincidence of the U–nucleolus and the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$.

Now suppose that the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ coincides with the U–nucleolus of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. Note that for each x in the U–nucleolus of $(N, \mathcal{A}, c, v)$,

$$x_i > 0 \quad \text{and} \quad \mathcal{A}_j \supset \mathcal{A}_i \quad \text{imply} \quad \mathcal{A}_j = \mathcal{A}_i. \tag{9}$$

By Proposition 2, $\mathcal{A} = \bigcup_{i=1}^k \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 2, for each collection $\{S_i\}_{i=1}^k$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^k S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $q < k$ such that $\cup_{i=1}^q \mathcal{B}^i$ does not cover N and $\cup_{i=1}^q \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > q$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^q \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = q+1, \ldots, k$, there exists $S_j \in \mathcal{B}^j$, such that if S_j is taken out from $\mathcal{B}^j$ for each $j = q+1, \ldots, k$, then the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $(N \setminus N^0)$.

We prove that for each y in the U–nucleolus of $(N, \mathcal{A}, t, v)$, $y(N \setminus N^0) = 0$. Suppose that there exist x in the U–nucleolus of $(N, \mathcal{A}, t, v)$ and $j_0 \in N \setminus N^0$ such that $x_{j_0} > 0$. Let $\mathcal{A}_{j_0} = \{Q_i\}_{i \in M}$, then $Q_i \in \mathcal{B}^i$, $i \in \{q+1, \ldots, k\}$. Since $\mathcal{A}$ is weakly mixed by Theorem 4, there exists $j_1 \in N$ such that $\mathcal{A}_{j_1} \supset \{S_i\}_{i \in M}$.

If $\mathcal{A}_{j_1} = \{S_i\}_{i \in M}$, then $j_1 \in N \setminus N^0$ by the definition of N^0, hence the Case 1 is impossible.

Let $\mathcal{A}_{j_1} \neq \{S_i\}_{i \in M}$. Since $\mathcal{A}$ is weakly mixed, there exists $j_2 \in N$ such that $\mathcal{A}_{j_2} \supset \mathcal{A}_{j_1} \cup \{Q_i\}_{i \in M} \setminus \{S_i\}_{i \in M}$. Then $\mathcal{A}_{j_2} \supset\neq \mathcal{A}_{j_0}$, but this contradicts (9).

Take $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$, then $\tilde{x}$ belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, 1, \tilde{v})$ Proportional solution of $(N, \mathcal{A}, 1, \tilde{v})$ and $\tilde{x}(N \setminus N^0) > 0$. By the proved above, $\tilde{x}$ does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, \tilde{v})$, hence Case 1 is impossible.

Case 2. If $S_j \in \mathcal{B}^j$ is taken out from $\mathcal{B}^j$, for all $j = q+1, \ldots, k$, then the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ do not cover $N \setminus N^0$.

For each $j = q+1, \ldots, k$, $S_j \in \mathcal{B}^j$, we have $S_j \cap (N \setminus N^0) \neq \emptyset$. Indeed, suppose that $S_{j_0} \subset N^0$ for some $j_0 > q$. Then if we take S_{j_0} and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$, $j \neq j_0$ out from $\cup_{j=q+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ as if $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C} = \{(N \setminus N^0) \cap S : S \in \mathcal{B}^j, |\mathcal{B}^j| > 1, j > q\}.$$

Note that $P, S \in \cup_{j=q+1}^k \mathcal{B}^j$, $P \neq S$, $P \cap (N \setminus N^0) \in \mathcal{C}$ imply $P \cap (N \setminus N^0) \neq S \cap (N \setminus N^0)$.

Indeed, suppose that $P \cap (N \setminus N^0) = S \cap (N \setminus N^0)$. There exists $P^1 \in \mathcal{A}$ such that $P \cap P^1 = \emptyset$. If we take S, P^1 and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$ with $P \notin \mathcal{B}^j$ out from $\cup_{j=q+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ because $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0} \ni S$, but this is impossible in the considered case.

For arbitrary problem $(N, \mathcal{A}, c, v)$, where $\mathcal{A}$ is under the Case 2, consider the problem $(N \setminus N^0, \mathcal{C}, c, w)$, where $w(T) = v(S)$ for $T = S \cap (N \setminus N^0) \in \mathcal{C}$. As was proved above, w is well defined. Under the Case 2, due to Theorem 2, there exists y in the Weakly U–equal sacrifice solution of $(N \setminus N^0, \mathcal{C}, c, w)$. Let $x \in R^{|N|}$, $x_i = 0$

for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then x belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and $x(T) = 0$ for all $T \in \bigcup_{i=1}^{q} \mathcal{B}^i$.

Take $c = 1$ and $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$. As was proved above, there exists z in the Weakly U–equal sacrifice solution of of $(N, \mathcal{A}, 1, \tilde{v})$ with $x(T) = 0$ for some $T \in \mathcal{A}$. If $U(0) = -\infty$, z does not belong to the U–nucleolus by the definition of the U–nucleolus. Let $U(0) > -\infty$. We have $U(z(T)) - U(\tilde{v}(T)) < 0$ and $U(\tilde{x}(S)) - U(\tilde{v}(S)) = 0$ for all $S \in \mathcal{A}$, hence z does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, \tilde{v})$. Thus, Case 2 is impossible. □

Corollary 5. *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v with $v(T) > 0$ if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.*

Corollary 6. *The Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with the Weakly Uniform Losses solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v with $v(T) > 0$ if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.*

Theorem 9. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$.*

The qU–solution of $(N, \mathcal{A}, c, v)$ coincides with the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is mixed at N and it follows from Theorem 7 that the qU–solution of $(N, \mathcal{A}, c, v)$ is always contained in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Since $x(S)$ are uniquely defined for all x in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, this implies coincidence of the qU–solution and the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$.

Now suppose that the Wealky U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ coincides with the qU–solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. By Proposition 3, $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 3, for each collection $\{S_i\}_{i=1}^{k}$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^{k} S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $p < k$ such that $\cup_{i=1}^{p} \mathcal{B}^i$ does not cover N and $\cup_{i=1}^{p} \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > p$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^{p} \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = p + 1, \ldots, k$, there exists $S_j \in \mathcal{B}^j$, such that if S_j is taken out from $\mathcal{B}^j$ for all $j > p$, then the remaining elements of $\cup_{j=p+1}^{k} \mathcal{B}^j$ cover $(N \setminus N^0)$.

Let $j_0 \in N \setminus N^0$, $\mathcal{A}_{j_0} = \{Q_i\}_{i \in M}$, then $Q_i \in \mathcal{B}^i$, $i \in \{p + 1, \ldots, k\}$. Since $\mathcal{A}$ is mixed by Theorem 7, there exists $j_1 \in N$ such that $\mathcal{A}_{j_1} = \{S_i\}_{i \in M}$, then $j_1 \in N \setminus N^0$, hence Case 1 is impossible.

Case 2. If $S_j \in \mathcal{B}^j$ is taken out from $\mathcal{B}^j$, $j = p + 1, \ldots, k$, then the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ do not cover $N \setminus N^0$.

For each $j = p + 1, \ldots, k$, $S_j \in \mathcal{B}^j$, we have $S_j \cap (N \setminus N^0) \neq \emptyset$. Indeed, suppose that $S_{j_0} \subset N^0$ for some $j_0 > p$. Then if we take S_{j_0} and arbitrary $S_j \in \mathcal{B}^j$ for $j > p$,

$j \neq j_0$ out from $\cup_{j=p+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=p+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ as if $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C} = \{(N \setminus N^0) \cap S : \ S \in \mathcal{B}^j, \, |\mathcal{B}^j| > 1, \, j > p\}.$$

Note that $P, S \in \cup_{j=p+1}^k \mathcal{B}^j$, $P \neq S$, $P \cap (N \setminus N^0) \in \mathcal{C}$ imply $P \cap (N \setminus N^0) \neq S \cap (N \setminus N^0)$.

Indeed, suppose that $P \cap (N \setminus N^0) = S \cap (N \setminus N^0)$. There exists $P^1 \in \mathcal{A}$ such that $P \cap P^1 = \emptyset$. If we take S, P^1 and arbitrary $S_j \in \mathcal{B}^j$ for $j > p$ with $P \notin \mathcal{B}^j$ out from $\cup_{j=p+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=p+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ because $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0} \ni S$, but this is impossible in the considered case.

For arbitrary problem $(N, \mathcal{A}, c, v)$, where $\mathcal{A}$ is under the Case 2, consider the problem $(N \setminus N^0, \mathcal{C}, c, w)$, where $w(T) = v(S)$ for $T = S \cap (N \setminus N^0) \in \mathcal{C}$. As was proved above, w is well defined. Under the Case 2, due to Theorem 3, there exists y in the Weakly U–equal sacrifice solution of $(N \setminus N^0, \mathcal{C}, c, w)$. Let $x \in R^{|N|}$, $x_i = 0$ for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, $x(N^0) = 0$.

Let $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$, then $\tilde{x}$ belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, 1, \tilde{v})$ as if $U(\tilde{x}(S)) - U(\tilde{v}(S)) = 0$ for all $S \in \mathcal{A}$ and $\tilde{x}(N^0) > 0$. By Property 2 of qU–solutions, for z in the qU–solution, $z(S)$ are uniquely defined at each $S \in \mathcal{A}$, but $0 = x(T) \neq \tilde{x}(T) > 0$ for $T \in \mathcal{B}^p$. Thus, in Case 2, qU–solution does not coincide with the Weakly U–equal sacrifice solution for some problem. □

References

Bregman, L. M. and Naumova, N. I. (1984). *Arbitration solutions with ideal point generated by systems of coalitions.* Dokl. Akad. Nauk SSSR, **279**, 16–20. English translation in Soviet Math. Dokl. **30**, 583–587.

Bregman, L. M. and Naumova, N. I. (2002). *Goal programming solutions generated by utility functions.* Lecture Notes in Economic and Math. Systems, **510**, 495–514.

Bregman, L. M. and Romanovskij, J. V. (1975). *Allotment and optimization in allocation problems.* Operations Research and Statistical Modeling (editor J.V.Romanovskij), Vyp.3, Izdat. Leningrad. Univ.: Leningrad, 137–162. (Russian)

Katsev, I. and Yanovskaya, E. (2011). *The prenucleolus for games with restricted cooperation.* Mathematical Game Theory and Applications, **3**, N4. (Russian)

Katsev, I. and Yanovskaya, E. (2013). *The prenucleolus for games with restricted cooperation.* Mathematical Social Sciences. Forthcoming.

Moulin, H., (2002). *Axiomatic cost and surplus sharing.* Handbook of Social Choice and Welfare, Vol. 1, Chapter 6. Arrow K.J., Sen A.K. and Suzumura K. eds., 289–357.

Naumova, N. I. (1978). *M-systems of relations and their application in cooperative games.* Vestnik Leningrad. Univ. N 1 (Ser. Math. Mech. Astr.), 60–66; English translation. in Vestnik Leningrad. Univ. Math. 11, 1983, 67–73.

Naumova, N. I. (2008). *Associated consistency based on utility functions of coalitions.* Game Theory and Applications, **13**, Nova Publishers: New York, 115–125.

Naumova, N. (2011). *Claim problems with coalition demands.* In: Contributions to Game Theory and Management, **4**. The Fourth International Conference Game Theory and Management June 28-30 2010, St. Petersburg, Russia. Collected Papers. Graduate School of Management St.Petersburg University: St. Petersburg, 311–326.

Naumova, N. (2012). *Generalized proportional solutions to games with restricted cooperation.* In: Contributions to Game Theory and Management, **4**. The Fifth International Conference Game Theory and Management June 27-29 2011, St. Petersburg, Russia. Collected Papers. Graduate School of Management St.Petersburg University, St. Petersburg, 230–242.

Thomson, W. (2003). *Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey.* Mathematical Social Sciences, **45**, 249–297.

Yanovskaya, E. (2002). *Consistency for proportional solutions.* International Game Theory Review, **4**, 343–356.