Pricing and Transportation Costs in Queueing System

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Abstract A non-cooperative four-person game which is related to the queueing system $M/M/2$ is considered. There are two competing stores and two competing transport companies which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The stream forms the Poisson process with intensity λ. The problem of pricing and determining the optimal intensity for each player in the competition is solved.

Keywords: Duopoly, equilibrium prices, queueing system.

1. Introduction

A non-cooperative four-person game which is related to the queueing system $M/M/2$ is considered. There are two competing stores P_1 and P_2 and two competing transport companies C_1 and C_2 which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The stream forms the Poisson process with intensity λ.. Suppose that $\lambda < \mu_1 + \mu_2$. Let shops declare the price for the produced product. After that transport companies declare the price of the service and carry passengers to the store, and the company C_1 carries passengers to P_1, when the company C_2 carries passengers to P_2. Customers choose the service with minimal costs. This approach was used in the Hotelling's duopoly (Hotelling, 1929; D'Aspremont et al., 1979; Mazalova, 2012) to determine the equilibrium price in the market. But the costs of each customer are calculated as the price of the product and transport charges. In this model, costs are calculated as the sum of prices for services and product plus losses of staying in the queue. Thus, the incoming stream is divided into two Poisson flows with intensities λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \lambda$. So the problem is following, what price for the service, the price for the product and the intensity of services is better to announce for the companies and shops. Such articles as (Altman and Shimkin, 1998; Levhari and Luski, 1978; Hassin and Haviv, 2003; Mazalova, 2013; Koryagin, 2008; Luski, 1976) are devoted to the similar game-theoretic problems of queuing processes.

Game-theoretic model of pricing. Consider the following game. Players P_1 and P_2 declare the price for the produced product p_1 and p_2 respectively. The customers have to use a transport to get to the shop. There are two competing transport companies C_1 and C_2 which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The transport companies declare the price of the service c_1 and c_2 respectively and carry passengers to the store, and the company C_1 carries passengers to P_1, when the company C_2 carries passengers to P_2. So the customers choose the service with minimal costs, and the incoming stream is divided into two Poisson flows with intensities λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \lambda$. In this case the costs of each customer will be

$$c_i + p_i + \frac{1}{\mu_i - \lambda_i}, \quad i = 1, 2,$$

where $1/(\mu_i - \lambda_i)$ is the expected time of staying in a queueing system (Saati, 1961). Then the intensities of the flows λ_1 and $\lambda_2 = \lambda - \lambda_1$ for the corresponding services can be found from

$$c_1 + p_1 + \frac{1}{\mu_1 - \lambda_1} = c_2 + p_2 + \frac{1}{\mu_2 - \lambda_2}. \tag{1}$$

So, the payoff functions for each player are

$$H_1(c_1, c_2, p_1, p_2) = \lambda_1 c_1, \quad H_2(c_1, c_2, p_1, p_2) = \lambda_2 c_2,$$

$$K_1(c_1, c_2, p_1, p_2) = \lambda_1 p_1, \quad K_2(c_1, c_2, p_1, p_2) = \lambda_2 p_2.$$

We are interested in the equilibrium in this game.

Symmetric model. Let start from the symmetric case, when the services are the same, i. e. $\mu_1 = \mu_2 = \mu$. Assuming that the stores fixed their prices p_1 and p_2, let us find the the equilibrium behavior for the transport companies. The equation (1) for the intensity λ_1 is

$$c_1 + p_1 + \frac{1}{\mu - \lambda_1} = c_2 + p_2 + \frac{1}{\mu - \lambda + \lambda_1}. \tag{2}$$

Differentiating (2) by c_1 we can find

$$1 + \frac{1}{(\mu - \lambda_1)^2} \frac{d\lambda_1}{dc_1} = -\frac{1}{(\mu - \lambda + \lambda_1)^2} \frac{d\lambda_1}{dc_1},$$

from which

$$\frac{d\lambda_1}{dc_1} = -\left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right)^{-1}. \tag{3}$$

Now we can find Nash equilibrium strategies c_1^* and c_2^* for fixed p_1, p_2 and c_2, i. e. we can find the maximum of $H_1(c_1, c_2, p_1, p_2)$ by c_1. The first order condition for the maximum of payoff function is

$$\frac{dH_1(c_1, c_2, p_1, p_2)}{dc_1} = \lambda_1 + c_1 \frac{d\lambda_1}{dc_1} = 0,$$

wherefrom

$$c_1^* = \frac{\lambda_1}{\frac{d\lambda_1}{dc_1}}. \tag{4}$$

substituting (3) to (4), we will get

$$c_1^* = \lambda_1 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right). \tag{5}$$

For another transport company it is

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right). \tag{6}$$

Now we can find the Nash equilibrium for players P_1 and P_2. Let us find the maximum of $K_1(c_1, c_2, p_1, p_2)$ by p_1 when p_2 is fixed, assuming that transport companies use the equilibrium strategies. The first order condition for the maximum of payoff function is

$$\frac{dK_1(c_1, c_2, p_1, p_2)}{dp_1} = \lambda_1 + p_1 \frac{d\lambda_1}{dp_1} = 0,$$

from where

$$p_1^* = \frac{\lambda_1}{\frac{d\lambda_1}{dp_1}}.$$

substituting the equilibrium prices of the transport companies (5)-(6) to (2) and differentiating it by p_1, we will get

$$\frac{d\lambda_1}{dp_1} = -\left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda + \lambda_1)^2} + (2\lambda_1 - \lambda)\left(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda + \lambda_1)^3}\right) \right)^{-1}. \tag{7}$$

So,

$$p_1^* = \lambda_1 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_1 - \lambda)\left(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda_2)^3}\right) \right).$$

For another store it is

$$p_2^* = \lambda_2 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_2 - \lambda)\left(\frac{2}{(\mu - \lambda_2)^3} - \frac{2}{(\mu - \lambda_1)^3}\right) \right).$$

Thus we get the system of equations that defines the equilibrium prices as transport companies and stores.

$$\begin{aligned}
c_1 + p_1 + \frac{1}{\mu - \lambda_1} &= c_2 + p_2 + \frac{1}{\mu - \lambda_2} \\
c_1^* &= \lambda_1 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda_2)^2} \right) \\
c_2^* &= \lambda_2 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda_2)^2} \right) \\
p_1^* &= \lambda_1 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_1 - \lambda)\left(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda_2)^3}\right) \right) \\
p_2^* = \lambda_2 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_2 - \lambda)\left(\frac{2}{(\mu - \lambda_2)^3} - \frac{2}{(\mu - \lambda_1)^3}\right) \right) \lambda_1 + \lambda_2 &= \lambda.
\end{aligned}$$

Using the symmetry of the problem, the solution of this system is

$$\begin{aligned}\lambda_1 = \lambda_2 &= \frac{\lambda}{2}\\ c_1^* = c_2^* &= \frac{\lambda}{(\mu - \frac{\lambda}{2})^2}\\ p_1^* = p_2^* &= \frac{3\lambda}{(\mu - \frac{\lambda}{2})^2}\end{aligned} \tag{8}$$

It is easy to check, that the second order condition for the maximum of payoff function is also satisfied.

$$\frac{d^2H_1}{dc_1^2} = 2\frac{d\lambda_1}{dc_1} + c_1\frac{d^2\lambda_1}{dc_1^2}.$$

$$\frac{d^2K_1}{dp_1^2} = 2\frac{d\lambda_1}{dp_1} + p_1\frac{d^2\lambda_1}{dp_1^2}.$$

Differentiating (3) by c_1 and (7) by p_1 we find

$$\frac{d^2\lambda_1}{dc_1^2} = \left(\frac{d\lambda_1}{dc_1}\right)\left[\frac{2}{(\mu-\lambda_1)^3} - \frac{2}{(\mu-\lambda+\lambda_1)^3}\right].$$

$$\frac{d^2\lambda_1}{dp_1^2} = \left(\frac{d\lambda_1}{dp_1}\right)\left[\frac{10}{(\mu-\lambda_1)^3} - \frac{10}{(\mu-\lambda+\lambda_1)^3} + (2\lambda_1-\lambda)\Big(\frac{6}{(\mu-\lambda_1)^4} + \frac{6}{(\mu-\lambda+\lambda_1)^4}\right].$$

In the equilibrium $\lambda_1 = \lambda/2$, from which $\frac{d^2\lambda_1}{dc_1^2} = 0$ è $\frac{d^2\lambda_1}{dp_1^2} = 0$. So,

$$\frac{d^2H_1(c_1^*, c_2^*, p_1^*, p_2^*)}{dc_1^2} = 2\frac{d\lambda_1}{dc_1} = -\left(\mu - \frac{\lambda}{2}\right)^2 < 0.$$

$$\frac{d^2K_1(c_1^*, c_2^*, p_1^*, p_2^*)}{dp_1^2} = 2\frac{d\lambda_1}{dp_1} = -\frac{\left(\mu - \frac{\lambda}{2}\right)^2}{3} < 0.$$

So, if one of the players uses the strategy (8), the maximum of payoff of another player is reached at the same strategy. That means that this set of strategies is equilibrium.

Asymmetric model. Let us assume now, that transport services are not equal, i. e. $\mu_1 \neq \mu_2$, suppose that $\mu_1 > \mu_2$. Let us find the equilibrium in the pricing problem in this case. Let us fix p_1, p_2 and c_2 and find the best reply of the player C_1. As well as in the symmetric case we get

$$\frac{dH_1(c_1, c_2, p_1, p_2)}{dc_1} = \lambda_1 + c_1\frac{d\lambda_1}{dc_1} = 0,$$

wherefrom

$$c_1^* = \frac{\lambda_1}{d\lambda_1/dc_1}.$$

Differentiating (1),we find

$$c_1^* = \lambda_1\left(\frac{1}{(\mu_1-\lambda_1)^2} + \frac{1}{(\mu_2-\lambda_2)^2}\right).$$

For another transport company it is

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right).$$

Table 1: The value of (c_1^*, c_2^*), (p_1^*, p_2^*) and (λ_1, λ_2) at $\lambda = 10$

		μ_2				
μ_1		6	7	8	9	10
6	$(c_1^*;c_2^*)$	(10;10)				
	$(p_1^*;p_2^*)$	(30;30)				
	$(\lambda_1;\lambda_2)$	(5;5)				
7	$(c_1^*;c_2^*)$	(5,918;5,804)	(2,5;2,5)			
	$(p_1^*;p_2^*)$	(17,035;16,707)	(7,5;7,5)			
	$(\lambda_1;\lambda_2)$	(5,049;4,951)	(5;5)			
8	$(c_1^*;c_2^*)$	(4,953;4,797)	(1,781;1,743)	(1,11;1,11)		
	$(p_1^*;p_2^*)$	(13,636;13,208)	(5,26;5,15)	(3,33;3,33)		
	$(\lambda_1;\lambda_2)$	(5,08;4,92)	(5,053;4,947)	(5;5)		
9	$(c_1^*;c_2^*)$	(4,553;4,375)	(1,494;1,437)	(0,866;0,848)	(0,625;0,625)	
	$(p_1^*;p_2^*)$	(12,165;11,689)	(4,3;4,136)	(2,597;2,533)	(1,875;1,875)	
	$(\lambda_1;\lambda_2)$	(5,1;4,9)	(5,097;4,903)	(5,054;4,946)	(5,5)	
10	$(c_1^*;c_2^*)$	(4,342;4,15)	(1,346;1,276)	(0,743;0,713)	(0,514;0,503)	(0,4;0,4)
	$(p_1^*;p_2^*)$	(11,371;10,869)	(3,781;3,586)	(2,176;2,088)	(1,535;1,502)	(1,2;1,2)
	$(\lambda_1;\lambda_2)$	(5,113;4,887)	(5,132;4,868)	(5,103;4,897)	(5,055;4,945)	(5;5)

Now we can find the best replies for the P_1 and P_2.

$$\frac{dK_i(c_1, c_2, p_1, p_2)}{dp_i} = \lambda_i + p_i \frac{d\lambda_i}{dp_i} = 0, \quad i = 1, 2,$$

from which

$$p_i^* = \frac{\lambda_i}{d\lambda_i / dp_i} i = 1, 2.$$

Using the same arguments as in the symmetric model, we obtain the system of equations that determine the equilibrium prices as transport companies and stores.

$$c_1 + p_1 + \frac{1}{\mu_1 - \lambda_1} = c_2 + p_2 + \frac{1}{\mu_2 - \lambda_2}$$

$$c_1^* = \lambda_1 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right)$$

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right)$$

$$p_1^* = \lambda_1 \left(\frac{3}{(\mu_1 - \lambda_1)^2} + \frac{3}{(\mu_2 - \lambda_2)^2} + (2\lambda_1 - \lambda)\left(\frac{2}{(\mu_1 - \lambda_1)^3} - \frac{2}{(\mu_2 - \lambda_2)^3}\right) \right)$$

$$p_2^* = \lambda_2 \left(\frac{3}{(\mu_1 - \lambda_1)^2} + \frac{3}{(\mu_2 - \lambda_2)^2} + (2\lambda_2 - \lambda)\left(\frac{2}{(\mu_2 - \lambda_2)^3} - \frac{2}{(\mu_1 - \lambda_1)^3}\right) \right)$$

$$\lambda_1 + \lambda_2 = \lambda.$$

In Table 1 the values of the equilibrium prices with different μ_1, μ_2 at $\lambda = 10$ and are given.

2. Conclusion

It is seen from the table, that the higher the intensity of service of one transport company is, the higher payoff this transport company and the store, which is connected to this company, get. So, they can increase the price of the service and the price for the product.

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