Coalitional Solution in a Game Theoretic Model of Territorial Environmental Production

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Abstract A game-theoretic model of territorial environmental production under Cournot competition is studied. The process is modeled as cooperative differential game with coalitional structure. The Nash equilibrium in the game played by coalitions is computed and then the value of each coalition is allocated according to some given mechanism between its members. The numerical example is given.

Keywords: optimal control, nonlinear system, dynamic programming.

1. Introduction

A game-theoretic model of territorial environmental production is considered. The model is based on the research of Petrosyan and Zaccour (2003). In the paper of Petrosyan and Zaccour (2003) the international environmental agreement is modeled, which provides a time-consistent allocation of total costs for all players under which the pollution is reduced.

The model of territorial environmental production is an extension of above mentioned model (Petrosyan and Zaccour, 2003). The region market is considered, where all firms produce homogeneous product under Cournot competition. The production process damages to the environment. Emission of each player is proportional to its output. Any firm has three types of costs: production costs, abatement costs and damage costs.

We consider the voluntary approach to environmental regulation, which became popular in a series of countries. The cooperation of firms leads to increase their profits and decrease of pollution, but the price of product is increased.

The approach of this paper is different. The more general coalitional setting is considered, when not only the grand coalition, but also a coalitional partition of players can be formed. This kind of approach was considered before in . Coalitional values for static games have been studied in a series of papers (Bloch, 1966; Owen, 1997). In a recent contribution, Owen (1997) proposed a characterization of the Owen value for static games under transferable utility. Owen (1997) defined the coalitional value for static simultaneous games with transferable payoffs by generalizing the Shapley value to a coalitional framework. In particular, the coalitional value was defined by applying the Shapley value first to the coalition partition and then to cooperative games played inside the resulting coalitions. This approach assumed that coalitions in the first level can cooperate (as players) and form the grand coalition. The game played with coalition partitioning becomes cooperative one with specially defined characteristic function: The Shapley value computed for this characteristic function is then the Shapley-Owen value for the game.

The present paper emerges from idea that it is more natural not to assume that coalitions on the first level can form a grand coalition. At first step the Nash equilibrium in the game played by coalitions is computed. Secondly, the value of each coalition is allocated according to the Shapley value in the form of PMS-vector, that was derived in the paper of Petrosyan and Mamkina (2006) . The approach was considered earlier in (Kozlovskaya et al., 2010). The main result of this paper is the calculation of this solution (PMS-vector). The main result of the paper is construction the dynamic PMS-value in the model of territorial environmental production.

2. Problem Statement

Consider a region market with n firms which produce for simplicity the same product. Let I be the set of firms involved in the game: $I = \{1, 2, \dots, n\}$.

Denote by $q_i = q_i(t)$ the output of firm i at the instant of time t. The price of the product $p = p(t)$ is defined as follows

$$p(t) = a - bQ(t) \; , \tag{1}$$

where $a > 0,\ b > 0,\ Q(t) = \sum\limits_{i=1}^{n} q_i(t)$ – the total output. The price function $p(t)$ is inverse demand function:

$$Q = Q(t) = \frac{a - p(t)}{b} \; .$$

The production cost of any firm equals

$$C_i(q_i(t)) = cq_i(t), \quad c > 0,\ i \in I.$$

The game $\Gamma(s_0, t_0)$ starts at the instant of time t_0 from the initial state s_0, where $s_0 = s(t_0)$ is the stock of pollution at time t_0. Let us denote by $e_i(t)$ the emission of firm i at time t. The emission of firms are linear subject to output:

$$e_i(q_i(t)) = \alpha q_i(t), \quad \alpha > 0. \tag{2}$$

Denote by $\bar{e}_i$ maximum permissible emission for firm i:

$$0 \le e_i(q_i(t)) \le \bar{e}_i. \tag{3}$$

We get from (3) that maximal permissible output of firm i is equal to

$$q_i^{max} = \frac{\bar{e}_i}{\alpha},$$

then maximal permissible total output equals

$$Q^{max} = \frac{\bar{e}}{\alpha},$$

where $\bar{e} = \sum\limits_{i=1}^{n} \bar{e}_i$. Suppose the parameters of model are such that the following inequality is true

$$a - c - \frac{b}{\alpha}\bar{e} \ge 0,$$

which guarantees the nonnegativity of price (1).

Denote by $s = s(t)$ the total stock of accumulated pollution by time t. The dynamics of pollution accumulation is defined by the following differential equation:

$$\dot{s}(t) = \alpha \sum_{k=1}^{n} q_i(t) - \delta s(t),$$
$$s(t_0) = s_0, \tag{4}$$

where δ is the rate of pollution absorption, $\alpha > 0$ is a known parameter. Any firm has two types of costs, which are not directly connected with the production process: abatement costs and damage costs. The abatement costs at moment of time t equals

$$E_i(q_i(t)) = \frac{\gamma}{2} e_i(t)(2\bar{e}_i - e_i(t)) = \frac{\gamma}{2} \alpha q_i(2\bar{e}_i - \alpha q_i),$$
$$\gamma > 0, \qquad 0 \le e_i(t) \le \bar{e}_i.$$

The cost function $E_i(q_i)$ increases and reaches the maximum at $q_i = q_i^{max}$. The function $E_i(q_i)$ is concave. Damage costs depends on the stock of pollution:

$$D_i(s(t)) = \pi_i s(t), \qquad \pi_i > 0, \qquad i \in I.$$

The firm i tries to maximize the profit

$$\Pi_i(s_0, t_0; q) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\} dt, \tag{5}$$

where $q = q(t) = (q_1(t), q_2(t), \ldots, q_n(t)),\ t \ge t_0$ is trajectory of production output, $0 < \rho < 1$ is a discount rate, p is defined by (1).

3. Coalitional Solution

Let $\Delta = (S_1, S_2, \ldots, S_m)$ be the partition of the set I, such that $S_i \cap S_j = \emptyset$, $\bigcup_{i=1}^{m} S_i = I$, $|S_i| = n_i$, $\sum_{i=1}^{m} n_i = n$.

Denote by M the set $M = \{1, 2, \ldots, , m\}$.

Suppose that each firm i from I is playing in interests of coalition S_k, to which it belongs, trying to maximize the sum of payoffs of its members, i.e.

$$\max_{q_j \in S_k} \sum_{j \in S_k} \Pi_j(s_0; q) =$$
$$= \max_{q_j \in S_k} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \sum_{j \in S_k} \{pq_j - C_j(q_j) - D_j(s) - E_j(q_j)\} dt, \tag{6}$$

where $q = q(t) = (q_1(t), q_2(t), \ldots, q_n(t)),\ t \ge t_0$ – trajectory of production output, $0 < \rho < 1$ – discount rate.

Without loss of generality it can be assumed that coalitions S_k are acting as players. Then at first stage the Nash equilibrium is computed. The total cost of coalition S_k is allocated among the players according to Shapley value of corresponding subgame $\Gamma(S_k)$. The game $\Gamma(S_k)$ is defined as follows: let S_k be the set of players involved in the game $\Gamma(S_k)$, $\Gamma(S_k)$ is a cooperative game.

Definition 1. The vector

$$PMS(x,t) = [PMS_1(x,t), PMS_2(x,t), \dots, PMS_n(x,t)],$$

is a PMS-vector, where $PMS_i(x,t) = Sh_i(S_k, x, t)$, if $i \in S_k$, where

$$Sh_i(S_k, x, t) = \sum_{M \supset i, M \subset S_k} \frac{(n_k - m)!(m-1)!}{n_k!}[V(M,x,t) - V(M \backslash \{i\}, x, t)]$$

and $(S_1, S_2, \dots, S_m)$ is the partition of the set I.

3.1. The Construction of Coalitional Solution

Step 1. Computation of the Nash equilibrium in the game of coalitions $S_k,\ k \in M$.
Each firm i from I is playing in interests of coalition S_k, to which it belongs, trying to maximize the sum of payoffs of its members (6).
The Nash equilibrium in the game of coalitions is computed by the solution of the following system:

$$\max_{q_j \in S_k} \sum_{j \in S_k} \mathit{\Pi}_j(s_0; q) = \max_{q_j \in S_k} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \sum_{j \in S_k} \{pq_j - \\ -C_j(q_j) - D_j(s) - E_j(q_j)\}dt \quad k \in M, \tag{7}$$

subject to equation dynamics (4).

Step 2. Computation of the characteristic function and the Shapley value in the game $\mathit{\Gamma}_V^{S_k}(s_0),\ k = 1, 2, \dots, m$. Computation of the characteristic function isn't standard (Petrosyan and Zaccour, 2003): when the characteristic function is calculated for K, the left-out players stick to their Nash strategies

Step 3. Construction of the PMS-vector.
Payoffs of all players $i \in I$ forms a PMS-vector (Petrosyan and Mamkina, 2006).
$PMS(s_0) = (PMS_1(s_0), PMS_2(s_0), \dots, PMS_n(s_0)),\ PMS_i(s_0) = Sh_i^{S_k}(s_0)$, where $Sh^{S_k}(s_0)$ is the Shapley value in the game $\mathit{\Gamma}_V^{S_k}(s_0)$

The Nash equilibrium is calculated with the help of Hamilton-Jacobi-Bellman equation (Dockner et al., 2000). The total cost of coalition S_k is allocated among the players according to Shapley value of corresponding subgame $\mathit{\Gamma}(S_k)$. The game $\mathit{\Gamma}(S_k)$ is defined as follows: let S_k be the set of players involved in the game $\mathit{\Gamma}(S_k)$, $\mathit{\Gamma}(S_k)$ is a cooperative game.

Computation of the characteristic function of this game isn't standard. When the characteristic function is computed for the coalition $K \in S_k$, the left-out players stick to their Nash strategies. Payoffs of all players $i \in I$ forms a PMS-vector (Petrosyan and Mamkina, 2006).

3.2. The Nash Equilibrium in the Game of Coalitions

The solution of the system (7) is equivalent to the solution if the system of Hamilton-Jacobi-Belman equations

$$\rho W_{S_k} = \max_{q_j, j \in S_k} \{ \sum_{j \in S_k} (q_j(a - bQ) - cq_j - \pi_j s + \frac{\gamma\alpha}{2} q_j(\alpha q_j - 2\bar{e}_j)) + \\ + \frac{\partial W_{S_k}}{\partial s}(\alpha Q - \delta s)\}, \quad k \in M, \tag{8}$$

where $Q = \sum\limits_{j\in I} q_j$, W_{s_k} is the Bellman function subject to equation od dynamics (4). By the first Step to find the Nash equilibrium, consider the system (8).

Differentiating with respect to $q_i,\ i \in S_k$ the right hand side of the equation (8) leads to

$$a - bQ - b\sum_{j\in S_k} q_j - c + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_{S_k}}{\partial s} = 0, \quad i \in I,\ k \in M. \tag{9}$$

Let us denote $Q_{S_k} = \sum\limits_{j\in S_k} q_j$.Then $Q = \sum\limits_{j=1}^{m} Q_{S_j}$, the system (9) is obtained in the following form

$$a - b\sum_{j=1}^{m} Q_{S_j} - bQ_{S_k} - c + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_{S_k}}{\partial s} = 0, \quad i \in I,\ k \in M. \tag{10}$$

Summing equations (10) with respect to S_k gives

$$n_k(a - c - b\sum_{j=1}^{m} Q_{S_j}) - n_k bQ_{S_k} + \gamma\alpha^2 Q_{S_k} - \gamma\alpha\bar{e}^{S_k} + \alpha n_k\frac{\partial W_{S_k}}{\partial s} = 0, \quad k \in M, \tag{11}$$

where $\bar{e}^{S_k} = \sum\limits_{j\in S_k} \bar{e}_j$. Solving (11) subject to Q_{S_k}, find

$$Q_{S_k} = \frac{n_k(a - c - bQ) - \gamma\alpha\bar{e}^{S_k} + \alpha n_k\frac{\partial W_{S_k}}{\partial s}}{bn_k - \alpha^2\gamma}, \quad k \in M. \tag{12}$$

Summing (12) with respect to the set M leads to

$$Q = \sum_{j=1}^{m} \frac{n_j(a - c - bQ) - \gamma\alpha\bar{e}^{S_j} + \alpha n_j\frac{\partial W_{S_j}}{\partial s}}{bn_j - \alpha^2\gamma} =$$
$$= \sum_{j=1}^{m} \frac{n_j(a - c + \alpha\frac{\partial W_{S_j}}{\partial s}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma} - Q\sum_{j=1}^{m}\frac{bn_j}{bn_j - \alpha^2\gamma},$$

then one can find:

$$Q = \frac{\sum\limits_{j=1}^{m} \dfrac{n_j(a - c + \alpha\frac{\partial W_{S_j}}{\partial s}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma}}{1 + \sum\limits_{j=1}^{m}\dfrac{bn_j}{bn_j - \alpha^2\gamma}}. \tag{13}$$

Substituting (13) in (12) gives the formula for Q_{S_k}. Then solving (9) leads to:

$$q_i = \frac{\bar{e}_i}{\alpha} - \frac{1}{\alpha^2\gamma}(a - c - bQ - bQ_{S_k} + \alpha\frac{\partial W_{S_k}}{\partial s}), \quad i \in S_k \tag{14}$$

It can be shown by the usual way that the Bellman function

$$W_{S_k} = A_{S_k}x + B_{S_k}, \quad k = 1, 2, \dots, m, \tag{15}$$

satisfies the Hamilton-Jacobi-Bellman equation (8) [13]. One can notice that

$$\frac{\partial W_{S_k}}{\partial x} = A_{S_k}. \tag{16}$$

Substituting (15) and (16) in formula (14) gives:

$$\hat{q}_i = \frac{\bar{e}_i}{\alpha} - \frac{1}{\alpha^2\gamma}(a - c - b\hat{Q} - b\hat{Q}_{S_k} + \alpha A_{S_k}), \quad i \in S_k,$$

where

$$\hat{Q}_{S_k} = \frac{n_k(a - c - bQ) - \gamma\alpha\bar{e}^{S_k} + \alpha n_k A_{S_k}}{bn_k - \alpha^2\gamma}, \quad k \in M,$$

and

$$\hat{Q} = \frac{\sum\limits_{j=1}^{m} \dfrac{n_j(a - c + \alpha A_{S_j}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma}}{1 + \sum\limits_{j=1}^{m} \dfrac{bn_j}{bn_j - \alpha^2\gamma}},$$

it means that

$$q_i^n = \begin{cases} \hat{q}_i, & \hat{q}_i \in [0, \frac{\bar{e}_i}{\alpha}] \\ \frac{\bar{e}_i}{\alpha}, & \hat{q}_i > \frac{\bar{e}_i}{\alpha} \\ 0, & \hat{q}_i < 0 \end{cases} \qquad i \in I. \tag{17}$$

Substituting (15) and (16) into the formula (8) leads to:

$$\begin{gathered} \rho A_{S_k}s + \rho B_{S_k} = Q_{S_k}^n(a - c - Q^n) - \sum_{j \in S_k} \pi_j s + \\ + \frac{\gamma\alpha}{2} \sum_{j \in S_k} q_j^n(\alpha q_j^n - 2\bar{e}_j) + A_{S_k}(\alpha Q^n - \delta s), \quad k \in M. \end{gathered} \tag{18}$$

From (18), we get the coefficients A_{S_k} and B_{S_k}:

$$\begin{gathered} A_{S_k} = -\frac{\sum\limits_{j \in S_k} \pi_j}{\rho + \delta} \\ B_{S_k} = \frac{1}{\rho}\Big(Q_{S_k}^n(a - c - bQ^n) + A_{S_k}Q^n + \frac{\gamma\alpha}{2} \sum_{j \in S_k} q_j^n(\alpha q_j^n - 2\bar{e}_j)\Big), \end{gathered} \tag{19}$$

where q_i^n is defined by the formula (17), and

$$Q_{S_k}^n = \sum_{j \in S_k} q_j^n, \quad k \in M, \tag{20}$$

$$Q^n = \sum_{j \in M} Q_{S_j}^n, \tag{21}$$

3.3. Computation of the Characteristic function

Computation of the characteristic function of this game is not standard. When the characteristic function is computed for the coalition $K \subset I$, we suppose that the left-out players have used their Nash equilibrium strategies. The advantage of this approach is the following: such characteristic function is easier to compute. This approach requires to solve only one equilibrium problem, all others being standard dynamic optimization problems, while standard approach requires to solve $2^n - 2$ equilibrium problems, which are harder then a dynamic optimization one. But this approach has a limitation, because in general the characteristic function is not superadditive. The superadditivity of the characteristic function was considered in (Kozlovskaya et al., 2010; Zenkevich and Kozlovskaya, 2010).

Suppose that for parameters of the model the following conditions hold:

$$\begin{aligned} \frac{1}{b(n+1)-\alpha^2\gamma}\Big(a-c-\frac{b\alpha(A-\gamma\bar{e})}{b-\alpha^2\gamma}\Big) \le \frac{1}{b-\alpha^2\gamma}\Big(\frac{b}{\alpha}\bar{e}_i-\alpha A_i\Big), \\ \frac{\bar{e}_i}{\alpha}+\frac{1}{2bn-\alpha^2\gamma}(a-c+\alpha A-\frac{2b}{\alpha}\bar{e}) \ge 0, \quad i \in I, \end{aligned} \tag{22}$$

where

$$\bar{e} = \sum_{j\in I} \bar{e}_j,$$
$$A = -\frac{\sum\limits_{j\in I}\pi_j}{\rho+\delta}.$$

Conditions (22) are the sufficient conditions of superadditivity of the characteristic function.

Computation of th Nash equilibrium in the game $\boldsymbol{\Gamma_V^{S_k}(s_0)}$ To find the Nash equilibrium the system of Hamilton-Jacobi-Bellman equations must be solved:

$$\max_{e_i} \Pi_i(s;q) = \max_{e_i} \int_t^{\infty} e^{-\rho(\tau-t)}\{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\}d\tau, \quad i \in S_k. \tag{23}$$

The solution of the system (23) is equivalent to the solution of the system of Hamilton-Jacobi-Bellman equations.

$$\begin{aligned} \rho W_i = \max_{q_i}\{q_i(a-bQ)-cq_i-\pi_i s+\frac{\gamma\alpha^2}{2}q_i^2-\gamma\alpha\bar{e}_i q_i+ \\ +\frac{\partial W_i}{\partial s}(\alpha Q-\delta s)\}, \quad i\in S_k. \end{aligned} \tag{24}$$

Differentiaiting the right hand side (24) with respect to q_i and equating to 0 leads to

$$a-bQ-bq_i-c+\gamma\alpha^2 q_i-\gamma\alpha\bar{e}_i+\alpha\frac{\partial W_i}{\partial s}=0, \quad i\in S_k$$

Recall that players from $I \backslash S_k$ stick to the strategies (17), where $Q^n_{S_j}$ is defined by the formula (20)

$$a-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ_{s_k}-bq_i-c+\gamma\alpha^2q_i-\gamma\alpha\bar{e}_i+\alpha\frac{\partial W_i}{\partial s}=0,\quad i\in S_k \tag{25}$$

Summing (25) by S_k gets

$$n_k(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j})-n_kbQ_{s_k}-bQ_{S_k}+\gamma\alpha^2Q_{S_k}-\gamma\alpha\bar{e}^{S_k}+\alpha\sum_{j\in S_k}\frac{\partial W_j}{\partial s}=0,$$

We obtain

$$Q^N_{S_k}=\frac{n_k(a-c-b\sum\limits_{j\in M\setminus\{k\}}Q^n_{S_j})-\gamma\alpha\bar{e}^{S_k}+\alpha\sum\limits_{j\in S_k}\frac{\partial W_j}{\partial s}}{b(n_k+1)-\alpha^2\gamma}. \tag{26}$$

One can find from (25), that

$$q^N_i=\frac{\bar{e}_i}{\alpha}+\frac{1}{b-\alpha^2\gamma}(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ_{S_k}+\alpha\frac{\partial W_i}{\partial s}-\frac{b}{\alpha}\bar{e}_i). \tag{27}$$

On account of (22), $0\le q^N_i\le\frac{\bar{e}_i}{\alpha}$. The Bellman functions have the linear form:

$$W_i=A_is+B_i,\quad i\in S_k. \tag{28}$$

Substituting (28) into (24), we obtain

$$\begin{aligned}\rho A_is+\rho B_i&=q^N_i(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k})-\pi_is+\\&+\frac{\gamma\alpha}{2}q^N_i(\alpha^N_i-2\bar{e}_i)+A_i(\alpha(\sum_{j\in M\setminus\{k\}}Q^n_{S_j}+Q^N_{S_k})-\delta s)\end{aligned} \tag{29}$$

from (29) one can find

$$\begin{aligned}A_i&=-\frac{\pi_i}{\rho+\delta}\\B_i&=\frac{1}{\rho}(q^N_i(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k})+\frac{\gamma\alpha}{2}q^N_i(\alpha^N_i-2\bar{e}_i)+\\&\quad+\alpha A_i(\sum_{j\in M\setminus\{k\}}Q^n_{S_j}+Q^N_{S_k})),\end{aligned} \tag{30}$$

where $Q^n_{S_j}$ is defined by (3.2.), q^N_i is defined by (3.3.) and

$$Q^N_{S_k}=\frac{n_k(a-c-b\sum\limits_{j\in M\setminus\{k\}}Q^n_{S_j})-\gamma\alpha\bar{e}^{S_k}+\alpha A_{S_k}}{b(n_k+1)-\alpha^2\gamma},$$

$$q^N_i=\frac{\bar{e}_i}{\alpha}+\frac{1}{b-\alpha^2\gamma}(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k}+\alpha A_i-\frac{b}{\alpha}\bar{e}_i).$$

Computation of the characteristic function for the intermidiate coalition L in the game $\Gamma_V^{S_k}(s_0)$ Let $L \in S_k,\ |L| = l,\ |S_k| = n_k$. Players from L maximize

$$\max_{q_i, q_i \in L} \Pi_i(s;q) = \max_{q_i, q_i \in L} \int_t^{\infty} e^{-\rho(t-\tau)}\{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\}d\tau, \tag{31}$$

on the assumption of the left-out players stick to their Nash equilibrium strategies q_i^N. The solution of (31) is equivalent to the solution of the following Hamilton-Jacobi-Bellman equation.

$$\begin{aligned}\rho W_L = \max_{q_j \in L}\{\sum_{j\in L} q_j(a - bQ) - c\sum_{j\in L} q_j - \sum_{j\in L}\pi_j s+ \\ +\frac{\gamma\alpha^2}{2}\sum_{j\in L} q_j^2 - \gamma\alpha\sum_{j\in L}\bar{e}_j q_j + \frac{\partial W_L}{\partial s}(\alpha Q - \delta s)\}.\end{aligned} \tag{32}$$

Differentitating the right hand side of (32) with respect to q_i and equating to 0 gives:

$$a - c - bQ - b\sum_{j\in L} q_j + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_L}{\partial s} = 0. \tag{33}$$

Suppose the players from $I\backslash S_k$ stick to q_i^n (17) and t he players from $S_k\backslash L$ stick to q_i^N (3.3.), so from (33) it can be obtained

$$a - c - b\sum_{j\in M\backslash\{k\}} Q_{S_j}^n - b\sum_{j\in S_k\backslash\{L\}} q_j^N - 2b\sum_{j\in L} q_j + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_L}{\partial s} = 0. \tag{34}$$

By the same way it can be found:

$$q^L = \sum_{j\in L} q_j^L = \frac{l(a - c - b(\sum\limits_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum\limits_{j\in S_k\backslash\{L\}} q_j^N)) - \gamma\alpha\bar{e}^L + \alpha A_L}{2bl - \alpha^2\gamma}, \tag{35}$$

and then

$$q_i^L = \frac{\bar{e}_i}{\alpha} + \frac{1}{b - \alpha^2\gamma}(a - c - b(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L) + \alpha A_L - \frac{b}{\alpha}\bar{e}^L). \tag{36}$$

Because of the condition (22), $0 \le q_i^L \le \frac{\bar{e}_i}{\alpha}$. The characteristic function is defined by the following formula:

$$W_L = A_L s + B_L, \tag{37}$$

where

$$A_L = -\frac{\sum\limits_{j\in L}\pi_j}{\rho + \delta}$$

$$\begin{aligned}B_L = \frac{1}{\rho}(q^L(a - c - b(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L)))+ \\ +\frac{\gamma\alpha}{2}\sum_{j\in L} q_j^L(\alpha q_j^L - 2\bar{e}_j) + \alpha A_L(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L).\end{aligned}$$

3.4. Characteristic function

We have proved that characteristic function of the game $\Gamma_V^{S_k}(s_0)$ is given by the following formula:

$$V(K,s) = \begin{cases} 0, & K = \emptyset, \\ W_i(s), & K = \{i\}, \\ W_{S_k}(s), & K = S_k, \\ W_L(s), & K = L, \end{cases}$$

where $W_i(s)$, $W_L(s)$, $W_{S_k}(s)$ is defined by (15), (37), (28).

3.5. The PMS-vector in the game $\Gamma_V^{S_k}(s_0)$

Let $s^n(t),\ t \geq t_0$ be the coaltiotnal trajectory, and players from coalition S_k players are agreed to divide the total payoff $V(S_k, s_0)$ according to Shapley value:

$$Sh(s) = (Sh_1(s), Sh_2(s), \ldots, Sh_n(s)),$$

where $SH_i(s)$ is defined by (38). The structure of the Shapley value is the following

$$Sh_i(s^n(t)) = A_i s^n(t) + Bsh_i, \tag{38}$$

4. The Numerical Example of the Coalitional Solution

All computations were executed in MAPLE 10.

4.1. Parameters of the Model

Consider the game of territorial environmental production of 7 players: $I = \{1, 2, 3, 4, 5, 6, 7\}$. Let the parameters of the model be the following:
$t_0 = 0$ – the initial instant of time ,
$s_0 = 0$ – the initial stock of pollution,
$p(t) = 8000 - 10 \sum\limits_{i=1}^{7} q_i(t)$ - the price function,
$c = 3$ – specific production costs,
$\rho = 0.07$ – discount rate,
$\alpha = 4$ – coefficient that characterizes the specific emission volume,
$\delta = 0.2$ – natural rate of pollution absorption,
$\gamma = 0.055$ – abatement costs coefficient
$\bar{e} = (600, 450, 510, 480, 550, 410, 430)$ - maximum permissible emissions,
$\pi = (4.7, 5.3, 5, 5.1, 4.8, 5.2, 5.05)$ - damage costs coefficients.
It follows from (2) and (3) that maximum permissible outputs of players are equal to

$$q^{max} = (150, 112.5, 127.5, 120, 137.5, 102.5, 107.5)$$

4.2. Results

Consider the following cases:

1. the Nash equilibrium
2. full cooperation
3. coalitional partition $\Delta_1 = (\{1, 2, 3\}, \{4, 5\}, \{6, 7\})$

4. coalitional partition $\Delta_2 = (\{1,2\},\{3,4\},\{5,6,7\})$
5. coalitional partition $\Delta_3 = (\{1,2,3,4\},\{5,6,7\})$
6. coalitional partition $\Delta_4 = (\{1,2\},\{3,4\},\{5\},\{6\},\{7\})$
7. coalitional partition $\Delta_5 = (\{1,2,3,4\},\{5\},\{6\},\{7\})$

Table 1: Results

	max	NE	COO	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
p	-575	1085.99	4292.47	2155.02	2156.65	2869.6	2098.54	3173.57
q_1	150	96.64	80.46	82.34	117.6	85.3	79.7	56.29
q_2	112.5	99.28	42.96	44.84	80.1	47.8	42.2	18.79
q_3	127.5	98.32	57.96	59.84	102.9	62.8	65	33.79
q_4	120	98.89	50.46	90.21	95.4	55.3	57.5	26.29
q_5	137.5	97.68	67.96	107.7	84.5	109	135.7	137.5*
q_6	102.5	100.41	32.96	97.28	49.5	74	102.5*	102.5*
q_7	107.5	100.17	37.96	102.3	54.5	79	107.5*	107.5*

The first string of the table contains the prices of product in all 7 cases. The price of product is the highest in the case off full cooperation, the price is the lowest, when the players compete. The dynamics of pollution in any of 7 cases are the following:

$$\begin{aligned}
s^N(t) &= 13828.02 - 13828.02e^{-02t}, \\
s^I(t) &= 7415.07 - 7415.07e^{-02t}, \\
s^{\Delta_1}(t) &= 11689.95 - 11689.95e^{-02t}, \\
s^{\Delta_2}(t) &= 11686.7 - 11686.7e^{-02t}, \\
s^{\Delta_3}(t) &= 10260.8 - 10260.8e^{-02t}, \\
s^{\Delta_4}(t) &= 11802.9 - 11802.9e^{-02t}, \\
s^{\Delta_5}(t) &= 9652.9 - 9652.9e^{-02t}.
\end{aligned}$$

Functions $s^{\Delta_1}(t)$, $s^{\Delta_2}(t)$ $s^{\Delta_4}(t)$ are almost coincides, so let us denote it by $s^{\Delta_1}(t)$ (Pic. 1). The emissions are maximin in the case of competition at any t and minimum in the case of cooperation. On Fig. 2-8 profits of any player are represented. The profit is lowest in the Nash equilibrium (competitive case) for any player. On Fig. 9 and 10 the profit functions of players in the case of cooperation and competition are represented. On Fig. 11-15 the the profit functions of players in the case of coalitional partitions are represented.

Appendix

$$V(\{1\}, s^N(t)) = 443161.9 + 240710e^{-0.2t}$$
$$V(\{2\}, s^N(t)) = 410614.1 + 271439e^{-0.2t}$$
$$V(\{3\}, s^N(t)) = 436648.3 + 256074.5e^{-0.2t}$$
$$V(\{4\}, s^N(t)) = 434696.7 + 261196e^{-0.2t}$$
$$V(\{5\}, s^N(t)) = 454212 + 245831.5e^{-0.2t}$$
$$V(\{6\}, s^N(t)) = 460253.9 + 266317.5e^{-0.2t}$$
$$V(\{7\}, s^N(t)) = 479901 + 258635.2e^{-0.2t}$$
$$Sh_1(s^I(t)) = 2596830.2 + 129041.3e^{-0.2t}$$
$$Sh_2(s^I(t)) = 2534222.5 + 145514.7e^{-0.2t}$$
$$Sh_3(s^I(t)) = 2633017.4 + 137278e^{-0.2t}$$
$$Sh_4(s^I(t)) = 2643934.6 + 140023.6e^{-0.2t}$$
$$Sh_5(s^I(t)) = 2630935.5 + 131786.9e^{-0.2t}$$
$$Sh_6(s^I(t)) = 2693927.9 + 142769.1e^{-0.2t}$$
$$Sh_7(s^I(t)) = 2704317.1 + 138650.8e^{-0.2t}$$
$$PMS_1^1(s^1(t)) = 1004458.7 + 203491.8e^{-0.2t}$$
$$PMS_2^1(s^1(t)) = 9836653.3 + 229469.5e^{-0.2t}$$
$$PMS_3^1(s^1(t)) = 1019834.5 + 216480.6e^{-0.2t}$$
$$PMS_4^1(s^1(t)) = 2116292.1 + 220810.2e^{-0.2t}$$
$$PMS_5^1(s^1(t)) = 2116722 + 207821.4e^{-0.2t}$$
$$PMS_6^1(s^1(t)) = 2135539.1 + 225139.9e^{-0.2t}$$
$$PMS_7^1(s^1(t)) = 2149420.1 + 218645.4e^{-0.2t}$$
$$PMS_1^2(s^2(t)) = 2098449.6 + 203435.1e^{-0.2t}$$
$$PMS_2^2(s^2(t)) = 2106725.7 + 229405.6e^{-0.2t}$$
$$PMS_3^2(s^2(t)) = 2110817.6 + 216420.4e^{-0.2t}$$
$$PMS_4^2(s^2(t)) = 2116851.4 + 220748.8e^{-0.2t}$$
$$PMS_5^2(s^2(t)) = 1004927.8 + 207763.6e^{-0.2t}$$
$$PMS_6^2(s^2(t)) = 1021950.9 + 225077.2e^{-0.2t}$$
$$PMS_7^2(s^2(t)) = 1052447.7 + 218584.6e^{-0.2t}$$
$$PMS_1^3(s^3(t)) = 1618274.1 + 1786139e^{-0.2t}$$
$$PMS_2^3(s^3(t)) = 1620465.4 + 2014156.7e^{-0.2t}$$
$$PMS_3^3(s^3(t)) = 1634972.8 + 1900147.8e^{-0.2t}$$
$$PMS_4^3(s^3(t)) = 1638652.2 + 1938150.8e^{-0.2t}$$
$$PMS_5^2(s^3(t)) = 2698271.6 + 1824141.9e^{-0.2t}$$
$$PMS_6^3(s^3(t)) = 2740960.6 + 1976153.7e^{-0.2t}$$
$$PMS_7^3(s^3(t)) = 2289477 + 1919149.3e^{-0.2t}$$

$$PMS_1^4(s^4(t)) = 912508.3 + 205458.4e^{-0.2t}$$
$$PMS_2^4(s^4(t)) = 893504 + 231687e^{-0.2t}$$
$$PMS_3^4(s^4(t)) = 941322.5 + 218572.7e^{-0.2t}$$
$$PMS_4^4(s^4(t)) = 879177.1 + 222944.2e^{-0.2t}$$
$$PMS_5^4(s^4(t)) = 3133855.8 + 209829.8e^{-0.2t}$$
$$PMS_6^4(s^4(t)) = 2125635.4 + 2273156.4e^{-0.2t}$$
$$PMS_7^4(s^4(t)) = 2294008.6 + 2207584.7e^{-0.2t}$$
$$PMS_1^5(s^5(t)) = 718845.5 + 168031.3e^{-0.2t}$$
$$PMS_2^5(s^5(t)) = 702952 + 189482.1e^{-0.2t}$$
$$PMS_3^5(s^5(t)) = 724518 + 178756.7e^{-0.2t}$$
$$PMS_4^5(s^5(t)) = 724522.1 + 182331.8e^{-0.2t}$$
$$PMS_5^5(s^5(t)) = 5447153.4 + 171606.4e^{-0.2t}$$
$$PMS_6^5(s^5(t)) = 3859509.6 + 185907e^{-0.2t}$$
$$PMS_7^5(s^5(t)) = 4100063.7 + 180544.2e^{-0.2t}$$

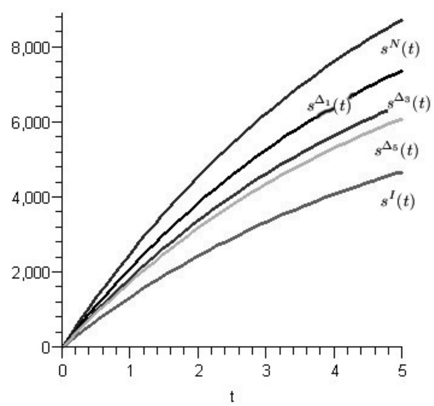

Fig. 1: Dynamics of pollution

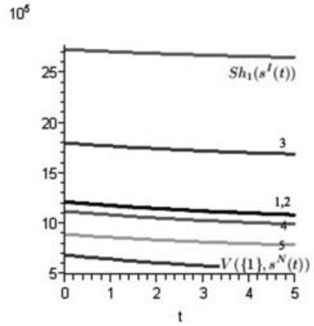

Fig. 2: Profit functions of 1st player

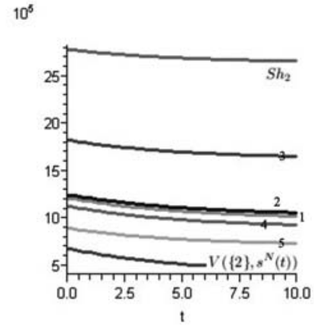

Fig. 3: Profit functions of 2nd player

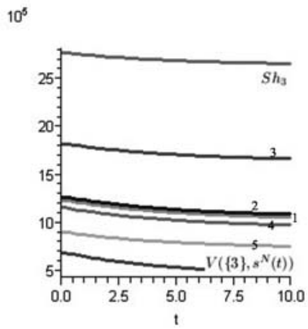

Fig. 4: Profit functions of 3st player

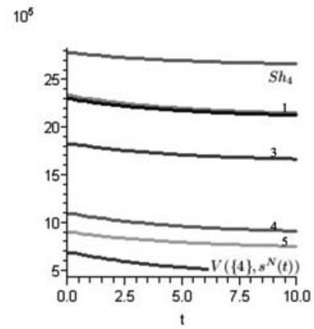

Fig. 5: Profit functions of 4nd player

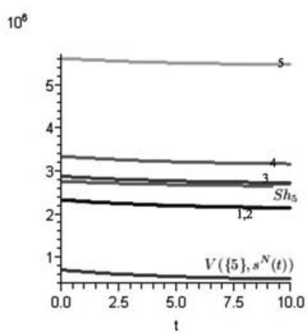

Fig. 6: Profit functions of 5st player

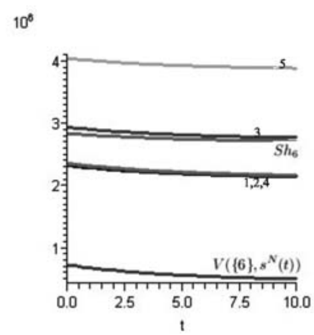

Fig. 7: Profit functions of 6nd player

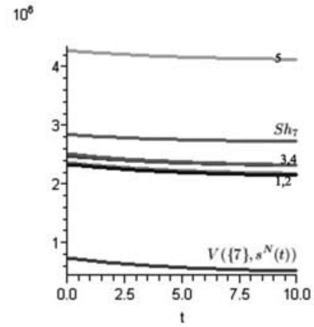

Fig. 8: Profit functions of 7st player

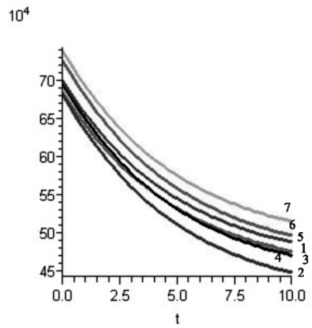

Fig. 9: Profit functions of player in the Nash equilibrium

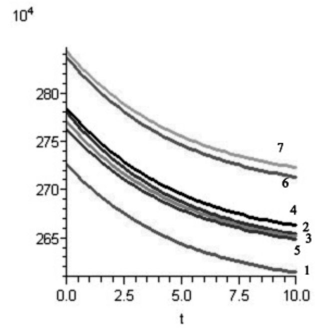

Fig. 10: Profit functions of player in the cooperation

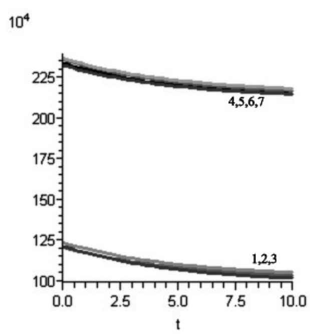

Fig. 11: Profit functions of player in the case Δ_1

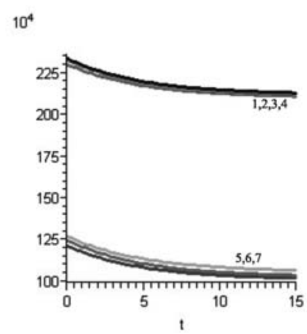

Fig. 12: Profit functions of player in the case Δ_2

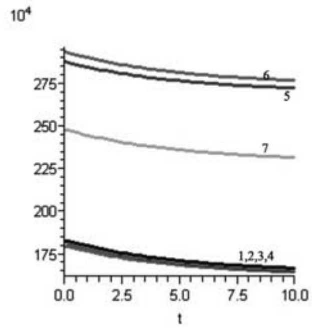

Fig. 13: Profit functions of player in the case Δ_3

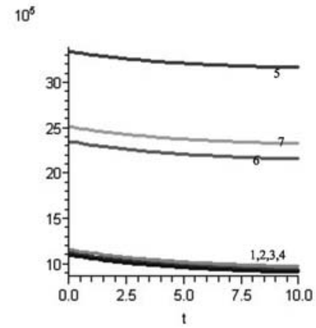

Fig. 14: Profit functions of player in the case Δ_4

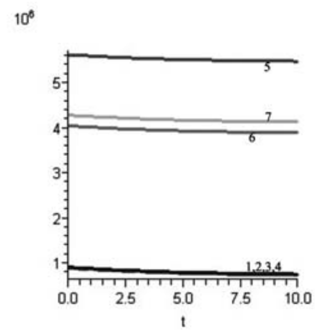

Fig. 15: Profit functions of player in the case Δ_5

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