A New Characterization of the Pre-Kernel for TU Games Through its Indirect Function and its Application to Determine the Nucleolus for Three Subclasses of TU Games*

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Abstract The main goal is twofold. Thanks to the so-called indirect function known as the dual representation of the characteristic function of a coalitional TU game, we derive a new characterization of the pre-kernel of the coalitional game using the evaluation of its indirect function on the tails of pairwise bargaining ranges arising from a given payoff vector. Secondly, we study three subclasses of coalitional games of which its indirect function has an explicit formula and show the applicability of the determination of the pre-kernel (nucleolus) for such types of games using the indirect function. Three such subclasses of games concern the 1−convex and 2-convex n person games and clan games. A clan game with the clan to be s singleton is known as a big boss game.

Keywords: dual representation, indirect function, pre-kernel, 1- and 2-convex n person games, clan games, big boss games.

1. Introduction and notions

As shown in (Driessen et al., 2010; Driessen et al., 2011), certain practical problems such as co-insurance situations and library situations can be modeled as a cooperative game in characteristic function form. Formally, a cooperative game on player set N is a characteristic function $v : \mathcal{P}(N) \to R$ defined on $\mathcal{P}(N)$ satisfying $v(\emptyset) = 0$. Here $\mathcal{P}(N)$ denotes the power set of the finite player set N, given by $\mathcal{P}(N) = \{S | S \subseteq N\}$, and shortly called a game v on N. In (Martinez-Legaz, 1996), the dual representation of cooperative games based on Fenchel Moreau Conjugation has been introduced, with every game v on N, there is associated the indirect function $\pi^v : R^N \to R$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) \qquad \text{for all } \vec{y} = (y_k)_{k \in N} \in R^N, \tag{1}$$

The excess $e^v(S, \vec{y})$ of a non-empty coalition S at the *salary vector* $\vec{y}$ in the game v represents the net profit the (unique) employer would receive from the selection of

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coalition S, assuming the members of S will produce, using the resources that are available to the employer, a total amount of output the monetary utility of which is measured by $v(S)$, and the (possibly negative) salary required by the player i amounts y_i, $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In the game theory setting, the efficient salary vectors of which all the excess are non-positive, compose the multi-valued solution concept called Core, that is

$$Core(v) = \{\vec{y} \in R^N | e^v(N, \vec{y}) = 0, e^v(S, \vec{y}) \leq 0 \quad \text{for all } S \subseteq N,\, S \neq \emptyset\}, \qquad (2)$$

According to (Martinez-Legaz, 1996), the indirect function $\pi^v : R^N \to R$ of a game v on N is a non-increasing convex function which attains its minimum at level zero, i.e., $\min_{\vec{y} \in R^N} \pi^v(\vec{y}) = 0$.

In this paper, we use indirect function to determine the nucleolus for three subclasses of games concerning $1-$convex and $2-$convex games (Driessen, 1988) (Driessen and Hou, 2010) and clan games. The theory on $1-$convex n person games has been well developed by Theo Driessen. The key feature of this kind of games is the geometrically regular structure of its core. For $2-$convex games, its core coincides with a so-called core catcher associated with appropriately chosen lower and upper Core bounds. For clan games, there is a nonempty coalition called clan, of which each member has veto power; i.e., no coalition can attain any positive reward unless it contains all clan members. With the clan to be a singleton, the clan game reduces to a big boss game.

2. The indirect function of 1-convex and 2-convex n person games and clan games

Given a game (N, v), its corresponding benefits vector $\vec{b}^v = (b_i^v)_{i \in N}$ is defined by $b_i^v = v(N) - v(N \backslash \{i\})$, $i \in N$. Note that the vector $\vec{b}^v$ is an upper bound for core allocations in that $y_i \leq b_i^v$ for all $i \in N$, all $\vec{y} \in Core(v)$. In terms of the characteristic function v, the 1-convexity property requires that, concerning the division problem, the worth $v(N)$ is sufficiently large to meet the coalitional demand amounting its worth $v(S)$, as well as the desirable marginal benefit by any individual not belonging to coalition S. For notation sake, write $\vec{z}(T)$ instead of $\sum_{k \in T} z_k$ for any coalition $T \subseteq N$ and any vector $\vec{z} = (z_k)_{k \in N} \in R^N$, where $\vec{z}(\emptyset) = 0$, and use $\vec{y} \leq \vec{b}^v$ instead of $y_i \leq b_i^v$ for all $i \in N$.

Definition 1. A game v on N is said to be 1-*convex* if it holds

$$\sum_{k \in N} b_k^v \geq v(N) \quad \text{and} \quad v(N) \geq v(S) + \sum_{k \in N \backslash S} b_k^v \qquad \text{for all } S \subseteq N,\, S \neq \emptyset. \qquad (3)$$

Example 1. Let the three-person game v on $N = \{1, 2, 3\}$ be given by $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 1$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 7$, $v(N) = 10$. It is left to the reader to check the 1-convexity of this game using the marginal benefit vector $b^v = (3, 4, 6)$. It turns out that core coincides with the triangle with the three vertices $(0, 4, 6)$, $(3, 1, 6)$, $(3, 4, 3)$. In fact, $(y_1, y_2, y_3) \in Core(v)$ is equivalent to $y_1 + y_2 + y_3 = 10$ and $y_1 \leq 3, y_2 \leq 4, y_3 \leq 6$. Under the latter upper core bound assumption $y \leq b^v$, the first part of the following theorem reports that the level equation $\pi^v(y) = c$ for its indirect function π^v is solved by the hyperplane equation

$y_1 + y_2 + \ldots + y_n = v(N) - c$m provided $c > 0$. Here the larger the strictly positive level c, the smaller $v(N) - c$. In case $c = 0$, then its level equation $\pi^v(y) = 0$ is solved by any hyperplane equation $y_1 + y_2 + \ldots + y_n = d$ where the real number d ranges from $b^v(N)$ to $v(N)$. The lowest hyperplane with $d = v(N)$ represents the core of the 1−convex game.

Theorem 1. *Let v be a 1-convex game on N and we study the indirect function of this game with respect to the following two types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b}^v$.
Type 2: There exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right] \quad \textit{for vectors of type 1.}$$

$$(ii)\pi^v(\vec{y}) = \max\left[0, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right]$$
$$= \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b^v_\ell\right] \quad \textit{for vectors of type 2.}$$

Proof. (i) Let $S \subseteq N$, $S \neq \emptyset$, and $\vec{y} \in R^N$ with $y_i \leq b^v_i$ for all $i \in N$. From (3), we derive

$$\begin{aligned} v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + \vec{y}(N\backslash S) \\ &\leq v(S) - \vec{y}(N) + \vec{b}^v(N\backslash S) \leq v(N) - \vec{y}(N) \end{aligned} \tag{4}$$

Thus, the restriction of the indirect function π^v to the comprehensive hull of the marginal benefit vector $\vec{b}^v$ attains its maximum either for $S = N$ or $S = \emptyset$.
(ii) For every $\vec{y} \in R^N$ such that there exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$, it holds that, on the one hand, $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ for all $S \subseteq N$ with $\ell \in S$ because the above chain (4) of inequalities still holds due to $\ell \notin N\backslash S$. For all $S \subseteq N$ with $\ell \notin S$, it holds

$$\begin{aligned} v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_\ell + \vec{y}(N\backslash(S \cup \{\ell\})) \\ &\leq v(S) - \vec{y}(N) + y_\ell + \vec{b}^v(N\backslash(S \cup \{\ell\})) \\ &= v(S) - \vec{y}(N) + y_\ell - b^v_\ell + \vec{b}^v(N\backslash S) \\ &\leq v(N) - \vec{y}(N) + y_\ell - b^v_\ell = v(N\backslash\{\ell\}) - \vec{y}(N\backslash\{\ell\}). \end{aligned} \tag{5}$$

In this setting, the indirect function π^v attains its maximum either for $S = N$, $S = N\backslash\{\ell\}$ or $S = \emptyset$, but $S = N$ cancels.

Corollary 1. *For every 1-convex game v on N and the payoff vector $\vec{y} = (y_k)_{k\in N} \in R^N$, it holds:*

$$\vec{y} \in Core(v) \Leftrightarrow \vec{y}(N) = v(N), \pi^v(\vec{y}) = 0 \Leftrightarrow \vec{y}(N) = v(N), \vec{y} \leq \vec{b}^v.$$

The former if and only if implication is trivial, while the latter if and only if implication is shown by the (partial) determination of the indirect function for 1-convex games according to Theorem 1.

In the remainder of this section, we switch from 1-convex to 2-convex games. In this framework, it is useful to introduce the so-called *gap function* $g^v : \mathcal{P}(N) \to R$ of a game v on N, given by $g^v(S) = \vec{b}^v(S) - v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and $g^v(\emptyset) = 0$. In view of (3), a game v on N is 1-convex if and only if the nonnegative gap function attains its minimum at the grand coalition, i.e., $0 \leq g^v(N) \leq g^v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Definition 2. (Driessen, 1988) A game v on N is said to be 2-*convex* if the following two conditions hold:

$$g^v(\{i\}) + g^v(\{j\}) \geq g^v(N) \geq g^v(\{i\}) \quad \text{for any players } i, j \in N,\ i \neq j \tag{6}$$

$$v(N) \geq v(S) + \sum_{k \in N \setminus S} b_k^v \qquad \text{for all } S \subseteq N,\ |S| \geq 2 \tag{7}$$

For 2-convexity, the main condition (3) is kept except for singletons, of which the gap is leveled below the gap of the grand coalition, whereas the sum of two such gaps majorizes the gap of the grand coalition.

Theorem 2. *Let v be a 2-convex game on N and we study the indirect function of this game with respect to the following four types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b}^v$.
Type 2: There exists a unique $\ell \in N$ with $y_\ell > b_\ell^v \geq v(\{\ell\})$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$, $i \neq \ell$.
Type 3: There exists a unique $j \in N$ with $y_j < v(\{j\}) \leq b_j^v$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$, $i \neq j$.
Type 4: There exist unique $j, \ell \in N$ with $y_\ell > b_\ell^v \geq v(\{\ell\})$, $y_i \leq b_i^v$ for all $i \in N$, $i \neq \ell$, and $y_j < v(\{j\}) \leq b_j^v$, $y_i \geq v(\{i\})$ for all $i \in N$, $i \neq j$. Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, v(N) - \sum_{k \in N} y_k, \quad (v(\{i\}) - y_i)_{i \in N}\right] \text{ for vectors of type 1.}$$

$$(ii)\pi^v(\vec{y}) = \max\left[0, v(N \setminus \{\ell\}) - \sum_{k \in N \setminus \{\ell\}} y_k\right]$$

$$= \max\left[0, v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 2.}$$

$$(iii)\pi^v(\vec{y}) = \max\left[v(N) - \sum_{k \in N} y_k, \quad v(\{j\}) - y_j\right] \text{ for vectors of type 3.}$$

$$(iv)\pi^v(\vec{y}) = \max\left[v(N \setminus \{\ell\}) - \sum_{k \in N \setminus \{\ell\}} y_k, \quad v(\{j\}) - y_j\right]$$

$$= \max\left[v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v, \quad v(\{j\}) - y_j\right] \text{ for vectors of type 4.}$$

The proof is similar to the previous proof of Theorem(1) and is left to the reader.

Corollary 2. *Let v be a 2-convex game on N and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$.*

The former if and only if statement is general and the latter is shown by the structure of the indirect function.

Definition 3. (Potters et al., 1989; Muto et al., 1988; Branzei et al., 2008, page 59) A game v on N is said to be a *clan game* if $b_i^v \geq v(\{i\})$ for all $i \in N$ and there exists a coalition $T \subseteq N$, called the *clan*, such that $v(S) = 0$ whenever $T \not\subseteq S$ and

$$v(N) \geq v(S) + \sum_{k\in N\setminus S} b_k^v \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \text{ with } T \subseteq S \tag{8}$$

A clan game v with an empty clan reduces to an 1-convex game, provided $g^v(N) \geq 0$. A clan game with the clan to be a singleton is known as a big boss game. Although both subclasses are interrelated, the description of its indirect function requires to distinguish two cases (either a singleton or a multi-person clan).

Theorem 3. *Let v be a big boss game on N, say player 1 is the big boss and we study the indirect function of this game with respect to the following four types of vector, given $\vec{y} \in R^n$.*
Type 1: $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1\}$.
Type 2: There exists a unique $\ell \in N\setminus\{1\}$ with $y_\ell > b_\ell^v \geq 0$ and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,\ell\}$.
Type 3: There exists a unique $\ell \in N\setminus\{1\}$ with $y_\ell < 0 \leq b_\ell^v$ and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,\ell\}$.
Type 4: There exist unique $j,\ell \in N\setminus\{1\}$ with $y_\ell > b_\ell^v \geq 0$, $y_j < 0 \leq b_j^v$, and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,j,\ell\}$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right] \text{ for vectors of type 1.}$$

$$(ii)\pi^v(\vec{y}) = \max\left[0, \quad v(N\setminus\{\ell\}) - \sum_{k\in N\setminus\{\ell\}} y_k\right]$$

$$= \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 2.}$$

$$(iii)\pi^v(\vec{y}) = \max\left[-y_\ell, \quad v(N) - \sum_{k\in N} y_k\right] \text{ for vectors of type 3.}$$

$$(iv)\pi^v(\vec{y}) = \max\left[-y_j, \quad v(N\setminus\{\ell\}) - \sum_{k\in N\setminus\{\ell\}} y_k\right]$$

$$= \max\left[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 4.}$$

Proof. Let $\vec{y} = (y_k)_{k \in N} \in R^N$.
(i) Suppose that $0 \leq y_i \leq b_i^v$ for all $i \in N \backslash \{1\}$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $1 \in S$, then $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \leq b_i^v$ for all $i \in N \backslash \{1\}$. This proves part (i).
(ii) Suppose that there exists a unique $\ell \in N \backslash \{1\}$ with $y_\ell > b_\ell^v \geq 0$ and $0 \leq y_i \leq b_i^v$ for all $i \in N \backslash \{1, \ell\}$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $\{1, \ell\} \subseteq S$, then $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \leq b_i^v$ for all $i \in N \backslash \{1, \ell\}$. In case $1 \in S$, $\ell \notin S$, then (5) applies once again. This proves part (ii).
(iii) Suppose that there exists a unique $\ell \in N \backslash \{1\}$with $y_\ell < 0 \leq b_\ell^v$ and $0 \leq y_i \leq b_i^v$ for all $i \in N \backslash \{1, \ell\}$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq -y_\ell$. In case $1 \in S$, then $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \leq b_i^v$ for all $i \in N \backslash \{1\}$. This proves part (iii).
(iv) Suppose that there exist unique $j, \ell \in N \backslash \{1\}$ with $y_\ell > b_\ell^v \geq 0$, $y_j < 0 \leq b_j^v$, and $0 \leq y_i \leq b_i^v$ for all $i \in N \backslash \{1, j, \ell\}$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq -y_j$. In case $1 \in S$, the proof proceeds similar to the proof of part (ii).

Corollary 3. *Let v be a big boss game on N and let $\vec{y} = (y_k)_{k \in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $0 \leq y_i \leq b_i^v$ for all $i \in N \backslash \{1\}$.*

Theorem 4. *Let v be a clan game on N, say coalition $T \subseteq N$ with at least two players is the clan. and we study the indirect function of this game with respect to the following four types of vector, given $\vec{y} \in R^n$.*
Type 1: $y_i \geq 0$ for all $i \in N$ and $y_i \leq b_i^v$ for all $i \in N \backslash T$.
Type 2: There exists a unique $\ell \in N \backslash T$ with $y_\ell > b_\ell^v \geq 0$, $y_i \leq b_i^v$ for all $i \in N \backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$.
Type 3: There exists a unique $\ell \in N$ with $y_\ell < 0$, $y_i \geq 0$ for all $i \in N \backslash \{\ell\}$, and $y_i \leq b_i^v$ for all $i \in N \backslash T$.
Type 4: There exist unique $j \in N$, $\ell \in N \backslash T$ with $y_j < 0$, $y_i \geq 0$ for all $i \in N \backslash \{j\}$, and $y_\ell > b_\ell^v \geq 0$, $y_i \leq b_i^v$ for all $i \in N \backslash T$, $i \neq \ell$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k \in N} y_k\right] \textit{ for vectors of type 1.}$$

$$(ii)\pi^v(\vec{y}) = \max\left[0, \quad v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k\right]$$

$$= \max\left[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v\right] \textit{ for vectors of type 2.}$$

$$(iii)\pi^v(\vec{y}) = \max\Big[-y_\ell, \quad v(N) - \sum_{k\in N} y_k\Big] \text{ for vectors of type 3.}$$

$$(iv)\pi^v(\vec{y}) = \max\Big[-y_j, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\Big]$$
$$= \max\Big[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b^v_\ell\Big] \text{ for vectors of type 4.}$$

The proof of Theorem 4 is similar as the proof of Theorem 3 and is left to the reader.

Corollary 4. *Let v be a clan game with coalition $T \subseteq N$ as the clan and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $y_i \geq 0$ for all $i \in N$ and $y_i \leq b^v_i$ for all $i \in N\backslash T$.*

Finally, we remark that a geometrical characterization of a clan game, say with coalition $T \subseteq N$ as the clan, is shown in (Branzei et al., 2008, page 60) requiring that $v(N)\cdot\vec{e}_j \in Core(v)$ for all $j \in T$ and there exists $\vec{x} \in Core(v)$ such that $x_i = b^v_i$ for all $i \in N\backslash T$.

3. Solving the pre-kernel by means of the indirect function

In this section, we characterize the pre-kernel of a game on N by the evaluation of the indirect function of the game at pairwise bargaining ranges arising from the payoff vector involved. Formally, for every pair of players $i, j \in N, i \neq j$, the surplus $s^v_{ij}(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v on N is given by the maximal excess among coalitions containing player i, but not containing player j. That is,

Definition 4. Let v be a game on N and $\vec{y} = (y_k)_{k\in N} \in R^N$.

(i) For every pair of players $i, j \in N$, $i \neq j$, the *surplus* $s^v_{ij}(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v is given by

$$s^v_{ij}(\vec{y}) = \max\Big[e^v(S,\vec{y})| \quad S \subseteq N, \quad i \in S, \quad j \notin S\Big] \tag{9}$$

(ii) The *pre-kernel* $\mathcal{K}^*(v)$ of the game v consist of *efficient* salary vectors of which all the pairwise surpluses are in equilibrium, that is (Maschler et al., 1979)

$$\mathcal{K}^*(v) = \{\vec{y} \in R^N | e^v(N,\vec{y}) = 0, s^v_{ij}(\vec{y}) = s^v_{ji}(\vec{y}) \quad \text{for all } i, j \in N, \, i \neq j.\} \tag{10}$$

For the alternative description of the pre-kernel, with every payoff vector $\vec{x} = (x_k)_{k\in N} \in R^N$, every pair of players $i, j \in N$, $i \neq j$, and every transfer amount $\delta \geq 0$ from player i to player j, there is associated the modified payoff vector $\vec{x}^{ij\delta} = (\vec{x}^{ij\delta}_k)_{k\in N} \in R^N$ defined by $x^{ij\delta}_i = x_i - \delta$, $x^{ij\delta}_j = x_j + \delta$, and $x^{ij\delta}_k = x_k$ for all $k \in N\backslash\{i, j\}$.

Theorem 5. *Let v be a game on N and $\vec{x} = (x_k)_{k\in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$.*

(i) For every pair of players $i, j \in N$, $i \neq j$, the indirect function $\pi^v : R^N \to R$ satisfies $\pi^v(\vec{x}^{ij\delta}) = s^v_{ij}(\vec{x}) + \delta$, provided $\delta \geq 0$ is sufficiently large.

(ii) $\vec{x} \in \mathcal{K}^*(v)$ *if and only if the evaluation of the pariwise bargaining ranges arising from* $\vec{x}$ *through the indirect function are in equilibrium, that is, for every pair of players* $i, j \in N$, $i \neq j$, *the indirect function satisfies* $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ *for* δ *sufficiently large.*

Proof. Fix the pair of players $i, j \in N$, $i \neq j$. Firstly, we claim that coalitions not containing player i or containing player j are redundant for maximizing the excesses at the modified payoff vector $\vec{x}^{ij\delta}$, provided the transfer amount $\delta \geq 0$ is sufficiently large. For that purpose, for all coalitions $S \subseteq N\backslash\{i\}$, $T \subseteq N\backslash\{j\}$, note the following two equivalences:

$$v(S \cup \{i\}) - \sum_{k \in S \cup \{i\}} x_k^{ij\delta} \geq v(S) - \sum_{k \in S} x_k^{ij\delta} \quad \text{iff} \quad \delta \geq v(S) - v(S \cup \{i\}) + x_i \quad (11)$$

$$v(T \cup \{j\}) - \sum_{k \in T \cup \{j\}} x_k^{ij\delta} \leq v(T) - \sum_{k \in T} x_k^{ij\delta} \quad \text{iff} \quad \delta \geq v(T \cup \{j\}) - v(T) - x_j \quad (12)$$

From (1) and (11)–(12) respectively, we derive that

$$\pi^v(\vec{x}^{ij\delta}) = \max_{S \subseteq N} \Big[v(S) - \sum_{k \in S} x_k^{ij\delta} \Big] = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \Big[v(S) - \sum_{k \in S} x_k^{ij\delta} \Big] \qquad (13)$$

where the choice of δ can be improved by

$$\delta \geq \max \Big[\max_{S \subseteq N\backslash\{i\}} |v(S \cup \{i\}) - v(S) - x_i|, \quad \max_{T \subseteq N\backslash\{j\}} |v(T \cup \{j\}) - v(T) - x_j| \Big]$$

because of $|\alpha| \geq \alpha$ as well as $|\alpha| \geq -\alpha$ for all $\alpha \in R$. Finally, from (13), $x_i^{ij\delta} = x_i - \delta$, and (9) respectively, we conclude that, for $\delta \geq 0$ sufficiently large, the following chain of equalities holds:

$$\pi^v(\vec{x}^{ij\delta}) = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \Big[v(S) - \sum_{k \in S} x_k^{ij\delta} \Big] = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \Big[v(S) - \sum_{k \in S} x_k \Big] + \delta = s_{ij}^v(\vec{x}) + \delta$$

This proves part (i). Together with (10), part (ii) follows immediately.

4. Remarks about determination of the nucleolus

The aim of this section is to illustrate the significant role of the indirect function for three classes of games (1-convex, 2-convex and clan games) to determine its nucleolus through a uniform approach replacing its original computation approach. Under these circumstances, the nucleolus belongs always to the pre-kernel, and so it is sufficient to solve the system for its unique solution. Thus we avoid the formal definition of the nucleolus.

Remark 1. Suppose the game v on N is 1-convex. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $\vec{x} \leq \vec{b}^v$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : R^N \to R$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{ij\delta}$ is in accordance with Theorem 1(i)–(ii) dependent on the size of its j-th component $\vec{x}_j^{ij\delta} = x_j + \delta$ in comparison to player j-th marginal benefit b_j^v. From the explicit formula for the indirect function of 1-convex games, we conclude the following:

$$\pi^v(\vec{x}^{ij\delta}) = 0 \qquad \text{if} \quad x_j^{ij\delta} \leq b_j^v, \text{ that is } \delta \leq b_j^v - x_j$$

$$\pi^v(\vec{x}^{ij\delta}) = \max\Big[0, \quad x_j^{ij\delta} - b_j^v\Big] = x_j + \delta - b_j^v > 0 \quad \text{otherwise}$$

For sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ is met if and only if $x_j + \delta - b_j^v = x_i + \delta - b_i^v$, that is $x_j - b_j^v = x_i - b_i^v$ for all $i \neq j$. Together with the efficiency principle $\vec{x}(N) = v(N)$, the unique solution of this system of linear equations is given by

$$x_i = b_i^v - \frac{\alpha}{n} \qquad \text{for all } i \in N, \text{ where} \qquad \alpha = \vec{b}^v(N) - v(N) \geq 0$$

The latter solution is known as the nucleolus and turns out to coincide with the gravity of the core being the convex hull of n extreme points of the form $\vec{b}^v - \alpha \cdot \vec{e}_i$, $i \in N$. Here $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denotes the standard basis of R^n.

We consider once again the 3-person game of the Example 1 in order to illustrate Remark 1 and Theorem 5. Let payoff vector $\vec{x}$ satisfy $\vec{x}(N) = v(N) = 10$ as well as $\vec{x} \leq \vec{b}^v = (3, 4, 6)$. From Remark 1, we obtain that $\pi^v(\vec{x}^{ij\delta}) = x_j + \delta - b_j^v$, $\pi^v(\vec{x}^{ji\delta}) = x_i + \delta - b_i^v$ for sufficiently large δ. By Theorem 5(ii), it holds that $\vec{x} \in \mathcal{K}^*(v)$ iff $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for δ sufficiently large. Thus, $\vec{x} \in \mathcal{K}^*(v)$ iff $x_j + \delta - b_j^v = x_i + \delta - b_i^v$ and due to efficiency, the nucleolus is given by $\vec{x} = (2, 3, 5)$.

Remark 2. Suppose the game v on N is a big boss game, with player 1 as the big boss. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $0 \leq x_k \leq b_k^v$ for all $k \in N \backslash \{1\}$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : R^N \to R$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{j\ell\delta}$ is in accordance with Theorem 3(i)–(iv) dependent on the size of its j-th component $\vec{x}_j^{j\ell\delta} = x_j - \delta$ in comparison to the zero level as well as its ℓ-th component $\vec{x}_\ell^{j\ell\delta} = x_\ell + \delta$ in comparison to player ℓ-th marginal benefit b_ℓ^v. From the explicit formula for the indirect function of big boss games, we conclude the following: for $\{j, \ell\} \subseteq N \backslash \{1\}$, and for $\delta \geq 0$ sufficiently large

$$\pi^v(\vec{x}^{j\ell\delta}) = \max\Big[-(x_j - \delta), \quad (x_\ell + \delta) - b_\ell^v\Big] = \delta - \min\Big[x_j, \quad b_\ell^v - x_\ell\Big]$$

$$\pi^v(\vec{x}^{1\ell\delta}) = \max\Big[0, \quad (x_\ell + \delta) - b_\ell^v\Big] = \delta + x_\ell - b_\ell^v$$

$$\pi^v(\vec{x}^{\ell 1\delta}) = \max\Big[0, \quad -(x_\ell - \delta)\Big] = \delta - x_\ell$$

For all $\ell \in N \backslash \{1\}$ and sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{1\ell\delta}) = \pi^v(\vec{x}^{\ell 1\delta})$ is met if and only if $x_\ell - b_\ell^v = -x_\ell$, that is $x_\ell = \frac{b_\ell^v}{2}$ for all $\ell \neq 1$.

Further, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ for any pair $\{j, \ell\} \subseteq N\backslash\{1\}$ is given by

$$\min\Big[x_j, \quad b_\ell^v - x_\ell\Big] = \min\Big[x_\ell, \quad b_j^v - x_j\Big] \quad \text{equalities which are satisfied trivially.}$$

Remark 3. Suppose the game v on N is a clan game, say coalition $T \subseteq N$ with at least two players is the clan. From the explicit formula for the indirect function of clan games, as presented in Theorem 4 (ii)–(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ reduces to the following system of equations: $x_i = x_j$ for all $i, j \in T$, and

$$x_i = \min\Big[b_i^v - x_i, \quad x_j\Big] \quad \text{whenever } i \notin T,\ j \in T$$

$$\min\Big[b_j^v - x_j, \quad x_i\Big] = \min\Big[b_i^v - x_i, \quad x_j\Big] \quad \text{whenever } i, j \notin T$$

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ for all $i \in T$ and $x_i = \min\Big[\lambda, \quad \frac{b_i^v}{2}\Big]$ for all $i \in N\backslash T$, where the parameter $\lambda \in R$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

Remark 4. Suppose the game v on N is 2-convex. From the explicit formula for the indirect function of 2-convex n-person games, as presented in Theorem 2(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ reduces to the following system of equations: for every pair of players $j, \ell \in N$, $j \neq \ell$,

$$\min\Big[b_\ell^v - x_\ell, \quad x_j - v(\{j\})\Big] = \min\Big[b_j^v - x_j, \quad x_\ell - v(\{\ell\})\Big]$$

As shown in (Driessen and Hou, 2010), the unique solution is of the parametric form $x_i = v(\{i\}) + \min\Big[\mu, \quad \frac{b_i^v - v(\{i\})}{2}\Big]$ for all $i \in N$, where the parameter $\mu \in R$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

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