Analysing the Folk Theorem for Linked Repeated Games

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Abstract We deal with the linkage of infinitely repeated games. Results are obtained by analysing the relations between the feasible individually rational payoff regions of the isolated games and the linked game. In fact we have to handle geometric problems related to Minkowski sums, intersections and Pareto boundaries of convex sets.

Key words: asymmetries, convex set, feasible individually rational payoff region, Folk theorem, full cooperation, linking, Minkowski sum, Pareto boundary, tensor game.

1. Introduction

There has developed an interest in the theory and applications of linking, also called 'interconnection'. The basic idea is the following. Consider a group of decision makers who are simultaneously involved in several different real world problems (issues). The standard approach is to consider the decision making process for each problem in isolation. In practice, however, the decision making process with respect to one problem is usually influenced by the decision making processes with respect to the other problems (spill-over effects or links). Discarding the links among the issues and analyzing the decision process on each issue separately rather than in a multi-issue decision making context is likely to lead to biased outcomes. Particularly, a single issue approach ignores the possibility that if the issues have compensating asymmetries of similar magnitudes, an exchange of concessions may allow and enhance cooperation which extends beyond cooperation in the single issue context.

Some well-known real world examples of linking are the negotiations 'on land for peace' between Israel and Palestina and the deal on WTO membership and participation in the Kyoto agreement between the EU and Russia. In the economics literature the notion of linking has been applied in the context of multimarket behaviour in oligopolistic markets (see e.g. Bernheim and Whinston, 1990; Spagnolo, 1999) and of international environmental problems (see e.g. Folmer et al., 1993; Botteon and Carraro, 1998; Carraro and Siniscalco, 1999; Finus, 2001).

A game theoretical framework for the linking of repeated games was developed by Folmer et al. (1993) and by Folmer and von Mouche (1994). In Folmer and von Mouche (2000) the following themes for linking of discounted infinitely repeated games were suggested:

– linking may sustain more cooperation;[1]
– linking may eliminate social welfare losses;
– linking may bring Pareto improvements;
– linking may facilitate cooperation.

We observe that 'may' is used here to indicate that the characteristics of linking of repeated games mentioned do no hold unconditionally but depend on the particular nature of the problem at hand. However, to our best knowledge, the conditions under which these characteristics hold have not yet been thoroughly analysed which is a major omission in the light of the practical and theoretical relevance of linking. Admittedly, some results about the conditions under which the characteristics of more cooperation and Pareto improvements hold can be found in Ragland (1995) and Just and Netanyahu (2000). However, these results are limited in scope because the settings in these publications concern the special case of linking of two repeated 2×2-bimatrix games.

The main purpose of the article[2] is to identify classes of isolated stages games for which the themes 'linking may sustain more cooperation' and 'linking may bring Pareto improvements' materialize or not; special attention is paid to the role of asymmetries. As these themes refer to properties of subgame perfect Nash equilibria of the linked and isolated games, Folk theorems, and in particular feasible individually rational payoff regions, come into the picture. In fact we formalize the two themes in terms of these regions and analyse how these regions for the isolated games relate to that of the linked game. Our results apply to the linking of an arbitrary finite number of discounted infinitely repeated games with an arbitrary finite number of (the same) players.

From a mathematical point of view analysing the two themes concerns the handling of two geometric problems. As these problems are in their own interesting and make sense without their game theoretic motivation, we organize the article as follows. In Section 2 we introduce notations and present some useful general results about Minkowski sums, normal cones and Pareto boundaries with which we shall handle the two geometric problems. The material in this section may have some interest in its own, especially as we cannot give good references for it in the literature. In Section 3 we state and analyse the two geometric problems in their pure form. Next, in Section 4 we show how the results in Section 3 induce results for the two themes for linked repeated games.

2. Convexity and Geometry

For the whole article we fix positive numbers m, n and write

$$N := \{1, \dots, n\}, \ \ M := \{1, \dots, m\}.$$

For $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ we write $\mathbf{a} \geq \mathbf{b}$ if $a_i \geq b_i$ for all $i \in N$. We write $\mathbf{a} > \mathbf{b}$ if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. And we write $\mathbf{a} \gg \mathbf{b}$ if $a_i > b_i$ for all $i \in N$.

[1] This is the counterpart of the theme 'repetition enables cooperation' for repeated games. 'More' is relative to the single issue case.

[2] The article concerns an improved version of Folmer and von Mouche (2007) and deals with a research question proposed in Folmer and von Mouche(2000).

The relation $\geq$ on $\mathbb{R}^n$ is a partial order. For $A \subseteq \mathbb{R}^n$ its *(strong) Pareto boundary*

$$\mathrm{P}(A)$$

is defined as the set of maximal elements of A, i.e. as the set of elements $\mathbf{a}$ of A for which there does not exist $\mathbf{c} \in A$ with $\mathbf{c} > \mathbf{a}$. And for $A \subseteq \mathbb{R}^n$ its *weak Pareto boundary*

$$\mathrm{P}_w(A)$$

is defined as the set of elements $\mathbf{a}$ of A for which there does not exist $\mathbf{c} \in A$ with $\mathbf{c} \gg \mathbf{a}$. Of course, $\mathrm{P}(A) \subseteq \mathrm{P}_w(A)$.

Proposition 1. *Let A be a compact subset of $\mathbb{R}^n$. For every $\mathbf{a} \in A$ there exists $\mathbf{b} \in \mathrm{P}(A)$ with $\mathbf{b} \geq \mathbf{a}$.* $\diamond$

Proof. $Z := \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} \geq \mathbf{a}\}$ is closed. This implies that $Z \cap A$ is compact. As $\mathbf{a} \in Z \cap A$ w have $Z \cap A \neq \emptyset$ and therefore also $\mathrm{P}(Z \cap A) \neq \emptyset$. Take $\mathbf{b} \in \mathrm{P}(Z \cap A)$. Then $\mathbf{b} \in Z \cap A \subseteq Z$, so $\mathbf{b} \geq \mathbf{a}$. Now we prove by contradiction that $\mathbf{b} \in \mathrm{P}(A)$. So suppose there would exist $\mathbf{c} \in A$ with $\mathbf{c} > \mathbf{b}$. Then we had $\mathbf{c} > \mathbf{b} \geq \mathbf{a}$, so $\mathbf{c} \in Z \cap A$ and $\mathbf{c} > \mathbf{b}$, which is a contradiction with $\mathbf{b} \in \mathrm{P}(Z \cap A)$. Q.E.D.

Proposition 2. *Let $B, C \subseteq \mathbb{R}^n$. Suppose for no $\mathbf{c} \in C$ there exists $\mathbf{d} \in \mathbb{R}^n \setminus C$ with $\mathbf{d} > \mathbf{c}$. Then $\mathrm{P}(B \cap C) = \mathrm{P}(B) \cap C$.* $\diamond$

Proof. '$\subseteq$': by contradiction. So suppose $\mathbf{a} \in \mathrm{P}(B \cap C)$ and $\mathbf{a} \notin \mathrm{P}(B) \cap C$. As $\mathbf{a} \in B \cap C \subseteq C$, it follows that $\mathbf{a} \notin \mathrm{P}(B)$. As $\mathbf{a} \in B$, there is $\mathbf{b} \in B$ with $\mathbf{b} > \mathbf{a}$. As $\mathbf{a} \in \mathrm{P}(B \cap C)$, it follows that $\mathbf{b} \notin B \cap C$. Thus $\mathbf{b} \in \mathbb{R}^n \setminus C$, $\mathbf{a} \in C$ and $\mathbf{b} > \mathbf{a}$, which is a contradiction.

'$\supseteq$': suppose $\mathbf{d} \in \mathrm{P}(B) \cap C$. So $\mathbf{d} \in B \cap C$. If we would have $\mathbf{a} \in B \cap C$ such that $\mathbf{a} > \mathbf{d}$, then, noting that $\mathbf{a} \in B$ and $\mathbf{d} \in B$, we would have a contradiction. Q.E.D.

Denote the set of permutations of N by

$$S_n.$$

For $\pi \in S_n$, the mapping $T_\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_\pi(x_1, \ldots, x_n) \; := \; (x_{\pi(1)}, \ldots, x_{\pi(n)})$$

is a linear isomorphism. We have

$$T_{\pi_2} \circ T_{\pi_1} = T_{\pi_1 \circ \pi_2}, \;\; T_{\mathrm{id}} = \mathrm{id}, \;\; (T_\pi)^{-1} = T_{\pi^{-1}}.$$

We call $A \subseteq \mathbb{R}^n$ *permutation-symmetric* if $T_\pi(A) = A$ for all permutations $\pi \in S_n$. So each subset of $\mathbb{R}$ is permutation symmetric.

Define the function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{C}(\mathbf{a}) \; := \; \sum_{l \in N} a_l$$

and for a subset A of $\mathbb{R}^n$, denoting by $\mathcal{C} \restriction A$ the restriction of the function $\mathcal{C}$ to A,

$$S(A) \; := \; \mathrm{argmax}(C \restriction A), \;\; s(A) \; := \; \sup(\mathcal{C} \restriction A). \tag{1}$$

The following simple properties hold:

$$s(\mathrm{Conv}(A)) = s(A) \text{ and } S(\mathrm{Conv}(A)) = \mathrm{Conv}(S(A)); \tag{2}$$

$$\text{for all } \pi \in S_n: \quad s(T_\pi(A)) = s(A) \text{ and } S(T_\pi(A)) = T_\pi(S(A)). \tag{3}$$

Closedness (boundedness) of A implies closedness (boundedness) of $S(A)$. And, with Weierstrass' theorem,

$$A \text{ non-empty and compact } \Rightarrow S(A) \text{ non-empty and compact.} \tag{4}$$

The sets $S(A), \mathrm{P}(A), \mathrm{P}_w(A)$ are subsets of the topological boundary ∂A of A:

$$S(A) \subseteq \mathrm{P}(A) \subseteq \mathrm{P}_w(A) \subseteq \partial A.$$

So, by (4), $\mathrm{P}(A) \neq \emptyset$ if A is non-empty and compact.

Definition 1. Let A_k $(k \in M)$ be non-empty subsets of $\mathbb{R}^n$ and $\mathbf{a} \in A = \sum_{k \in M} A_k$.[3] We call $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in A_1 \times \cdots \times A_m$ a *decomposition* of $\mathbf{a}$ if $\mathbf{a} = \sum_{k \in M} \mathbf{a}^{(k)}$. ⋄

For this situation:

Proposition 3. *For every* $\mathbf{p} \in \mathbb{R}^n$ *and* $\mathbf{a} \in A$

$$\mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{a} \; (\mathbf{z} \in A) \;\Leftrightarrow\; \mathbf{p} \cdot \mathbf{z}^{(k)} \leq \mathbf{p} \cdot \mathbf{a}^{(k)} \; (k \in M, \; \mathbf{z}^{(k)} \in A_k). \; \diamond$$

Proof. '⇒': by contradiction, suppose there exists k and $\mathbf{z}^{(k)}$ such that $\mathbf{p} \cdot \mathbf{z}^{(k)} > \mathbf{p} \cdot \mathbf{a}^{(k)}$. Then $\mathbf{b} := \mathbf{z}^{(k)} + \sum_{l \in M \setminus \{k\}} \mathbf{a}^{(l)} \in A$ and $\mathbf{p} \cdot \mathbf{b} > \mathbf{p} \cdot \mathbf{a}$, which is a contradiction.

'⇐': suppose $\mathbf{z} \in A$. Let $(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)})$ be a decomposition of $\mathbf{z}$. By assumption $\mathbf{p} \cdot \mathbf{z}^{(k)} \leq \mathbf{p} \cdot \mathbf{a}^{(k)}$ $(k \in M)$. Summing over $k \in M$ gives $\mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{a}$. Q.E.D.

Proposition 4. *Let* A_k $(k \in M)$ *be subsets of* $\mathbb{R}^n$ *and* $A = \sum_{k \in M} A_k$.

1. *If* $A_k \neq \emptyset$ $(k \in M)$, *then* $s(A) = \sum_{k \in M} s(A_k)$.
2. $S(A) = \sum_{k \in M} S(A_k)$. ⋄

Proof. 1. As $A_k \neq \emptyset$ $(k \in M)$ we obtain

$$s(\sum_k A_k) = \sup(\mathcal{C}(\sum_k A_k)) = \sup(\sum_k \mathcal{C}(A_k)) = \sum_k \sup(\mathcal{C}_k(A_k)) = \sum_k s(A_k).$$

2. In case there is an k with $A_k = \emptyset$, the desired result holds. Now suppose $A_k \neq \emptyset$ $(k \in M)$. Taking $\mathbf{p} = (1, 1, \ldots, 1)$ in Proposition 3 gives for a decomposition $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ of $\mathbf{a} \in A$: $\mathbf{a} \in S(A) \;\Leftrightarrow\; \mathbf{a}^{(k)} \in S(A_k)$ $(k \in M)$, i.e. the desired result. Q.E.D.

Proposition 5. *Let* A_k $(k \in M)$ *be subsets of* $\mathbb{R}^n$ *and* $A = \sum_{k \in M} A_k$.

1. *Suppose every* A_k *is non-empty. For every decomposition* $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ *of* $\mathbf{a} \in A$ *it holds that* $\mathbf{a} \in \mathrm{P}(A) \;\Rightarrow\; \mathbf{a}^{(k)} \in \mathrm{P}(A_k)$ $(k \in M)$.
2. $\mathrm{P}(A) \subseteq \sum_{k \in M} \mathrm{P}(A_k)$. ⋄

[3] The sum here is a Minkowski sum.

Proof. 1. By contradiction, suppose $\mathbf{a} \in \mathrm{P}(A)$ and there exists l such that $\mathbf{b}^{(l)} \in A_l$ and $\mathbf{b}^{(l)} > \mathbf{a}^{(l)}$. With $y := \mathbf{b}^{(l)} + \sum_{k \in M \setminus \{l\}} \mathbf{a}^{(k)} \in A$ one has $\mathbf{y} = \mathbf{a} + (\mathbf{b}^{(l)} - \mathbf{a}^{(l)}) > \mathbf{a}$, a contradiction.

2. This follows from part 1. Q.E.D.

In general, the inclusion in Proposition 5(2) is not an equality. Here is a special case where equality holds:

Proposition 6. *If $m = 2$ and A_1 or A_2 has a maximiser, then* $\mathrm{P}(A_1 + A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2)$. $\diamond$

Proof. We may assume that A_2 has a maximiser, say $\mathbf{b}$. So we have

$$\mathbf{y} \leq \mathbf{b} \ (\mathbf{y} \in A_2) \tag{5}$$

This implies $P(A_2) = \{\mathbf{b}\}$. By Proposition 5(2) only '$\supseteq$' remains to be proved. This we do by contradiction. So suppose $\mathbf{c} \in \mathrm{P}(A_1) + \mathrm{P}(A_2)$, but $\mathbf{c} \notin \mathrm{P}(A_1 + A_2)$. Let $\mathbf{a} \in \mathrm{P}(A_1)$ such that $\mathbf{c} = \mathbf{a} + \mathbf{b}$. As $\mathbf{c} \in A_1 + A_2$ and $\mathbf{c} \notin \mathrm{P}(A_1 + A_2)$, there is $\mathbf{d} \in A_1 + A_2$ with $\mathbf{d} > \mathbf{c}$. Let $\mathbf{a}' \in A_1$ and $\mathbf{b}' \in A_2$ such that $\mathbf{d} = \mathbf{a}' + \mathbf{b}'$. Then, by (5), $\mathbf{a}' > \mathbf{a} + (\mathbf{b} - \mathbf{b}') \geq \mathbf{a}$, so $\mathbf{a}' > \mathbf{a}$. But $\mathbf{a} \in \mathrm{P}(A_1)$, a contradiction. Q.E.D.

Let A be a non-empty subset of $\mathbb{R}^n$ and $\mathbf{z} \in \overline{A}$, i.e. $\mathbf{z}$ is an element of the topological closure of A. Then

$$N_A(\mathbf{z}) := \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d} \cdot (\mathbf{a} - \mathbf{z}) \leq 0 \text{ for all } \mathbf{a} \in A\}.$$

$N_A(\mathbf{z})$ is a convex cone and is called the *normal cone* of A in $\mathbf{z}$. Moreover, we define for $\mathbf{z} \in \overline{A}$ the *positive normal cone* of A in $\mathbf{z}$ as

$$N_A^+(\mathbf{z}) := \{\mathbf{d} \in N_A(\mathbf{z}) \mid \mathbf{d} > \mathbf{0}\}.$$

Note that $\mathbf{0} \in N_A(\mathbf{z})$, but that $N_A^+(\mathbf{z})$ may be empty.

Proposition 7. *Let A_k $(k \in M)$ be non-empty subsets of $\mathbb{R}^n$, $A = \sum_{k \in M} A_k$ and $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ a decomposition of $\mathbf{a} \in A$. Then $N_A(\mathbf{a}) = \cap_{k \in M} N_{A_k}(\mathbf{a}^{(k)})$ and $N_A^+(\mathbf{a}) = \cap_{k \in M} N_{A_k}^+(\mathbf{a}^{(k)})$.* $\diamond$

Proof. We prove the first statement; then the second holds too.

$\subseteq$: suppose $\mathbf{d} \in N_A(\mathbf{a})$. So $\mathbf{d} \cdot \mathbf{z} \leq \mathbf{d} \cdot \mathbf{a}$ $(\mathbf{z} \in A)$. Proposition 3 implies $\mathbf{d} \cdot \mathbf{z}^{(k)} \leq \mathbf{d} \cdot \mathbf{a}^{(k)}$ $(k \in M,\ \mathbf{z}^{(k)} \in A_k)$. Thus $\mathbf{d} \in N_{A_k}(\mathbf{a}_k)$ $(k \in M)$.

$\supseteq$: suppose $\mathbf{d} \in \cap_{k \in M} N_{A_k}(\mathbf{a}^{(k)})$. So $\mathbf{d} \cdot \mathbf{z}^{(k)} \leq \mathbf{d} \cdot \mathbf{a}^{(k)}$ $(k \in M, \mathbf{z}^{(k)} \in A_k)$. Proposition 3 implies $\mathbf{d} \cdot \mathbf{z} \leq \mathbf{d} \cdot \mathbf{a}$ $(\mathbf{z} \in A)$. Thus $\mathbf{d} \in N_A(\mathbf{a})$. Q.E.D.

Proposition 8. *Let A be a non-empty convex subset of $\mathbb{R}^n$. Then $\mathbf{z} \in \mathrm{P}_w(A) \Rightarrow N_A^+(\mathbf{z}) \neq \emptyset$.* $\diamond$

Proof. Define $B := \{\mathbf{b} \in \mathbb{R}^n \mid \mathbf{b} \geq \mathbf{z}\}$. For $\mathring{B}$, i.e. for the topological interior of B one has $\mathring{B} = \{\mathbf{b} \in \mathbb{R}^n \mid \mathbf{b} \gg \mathbf{z}\}$ and thus $\mathring{B} \cap A = \emptyset$. The sets $\mathring{B}$ and A are convex, non-empty and disjoint. Using a separation theorem, there exists an affine hyperplane that separates A and $\mathring{B}$. Therefore there exists $\mathbf{d} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{d} \cdot \mathbf{a} \leq \mathbf{d} \cdot \mathbf{b}$ $(\mathbf{a} \in A,\ \mathbf{b} \in \mathring{B})$. Even now

$$\mathbf{d} \cdot \mathbf{a} \leq \mathbf{d} \cdot \mathbf{b} \ (\mathbf{a} \in A, \mathbf{b} \in B). \tag{6}$$

With $\mathbf{b} = \mathbf{z}$ it follows that $\mathbf{d} \cdot \mathbf{a} \le \mathbf{d} \cdot \mathbf{z}$ ($\mathbf{a} \in A$). Now we prove by contradiction that $\mathbf{d} > 0$. So (remembering that $\mathbf{d} \neq \mathbf{0}$) suppose $d_i < 0$ for some i. For $\mathbf{b} \in B$ defined by $b_j := z_j$ ($j \neq i$) and $b_i := c$ where $c \ge a_i$, we have

$$\mathbf{d} \cdot \mathbf{b} = \sum_{j \in N \setminus \{i\}}^{n} d_j z_j + d_i c.$$

For c large enough $\mathbf{d} \cdot \mathbf{b} \le \mathbf{d} \cdot \mathbf{z}$, which is a contradiction with (6). Q.E.D.

Proposition 9. *For non-empty subsets A and B of $\mathbb{R}^n$ with $A \subseteq B$ one has: B compact and $\mathrm{P}(B) \subseteq A \;\Rightarrow\; N_B^+(\mathbf{z}) = N_A^+(\mathbf{z})$ ($\mathbf{z} \in \overline{A}$).* $\diamond$

Proof. Because $A \subseteq B$ one has $N_B^+(\mathbf{z}) \subseteq N_A^+(\mathbf{z})$. By contradiction we prove that $N_B^+(\mathbf{z}) \supseteq N_A^+(\mathbf{z})$. So suppose $\mathbf{d} \in N_A^+(\mathbf{z}) \setminus N_B^+(\mathbf{z})$. Now $(\mathbf{w} - \mathbf{z}) \cdot \mathbf{d} \le 0$ for all $\mathbf{w} \in A$, but not for all $\mathbf{z} \in B$. This implies that there is a $\mathbf{w} \in B \setminus A$ such that $\mathbf{d} \cdot (\mathbf{w} - \mathbf{z}) > 0$. As B is compact, there is, by Proposition 1, $\mathbf{b} \in \mathrm{P}(B)$ such that $\mathbf{b} \ge \mathbf{w}$. As $\mathbf{d} > \mathbf{0}$, also $\mathbf{d} \cdot (\mathbf{b} - \mathbf{z}) > 0$. So $\mathbf{b} \notin A$. But $\mathbf{b} \in \mathrm{P}(B) \subseteq A$, which is a contradiction. Q.E.D.

3. Two Geometric Problems

3.1. Stating the Problems

In this section fix non-empty subsets $U_1, \ldots, U_m$ of $\mathbb{R}^n$ and define, denoting by $\mathbb{R}^n_+$ the closed positive octant of $\mathbb{R}^n$,

$$F_k := \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+ \ (k \in M), \quad F := \sum_{k \in M} F_k,$$

$$U := \sum_{k \in M} U_k, \quad F_\star := \mathrm{Conv}(U) \cap \mathbb{R}^n_+.$$

Note that $\mathrm{Conv}(U) = \sum_{k \in M} \mathrm{Conv}(U_k)$. It is easy to see (also see Proposition 10(1)) that

$$F \subseteq F_\star. \tag{7}$$

Problem 1. Provide interesting conditions under which $F \subset F_\star$.

Our second problem deals with the following object

$$\mathrm{EXP} = \{\mathbf{v} \in \mathrm{P}(F) \mid \text{ there exists } \mathbf{w} \in F_\star \text{ with } \mathbf{w} \gg \mathbf{v}\}.$$

We refer to the elements of EXP as *expansion points* of PB(F). Note that So

$$F = F_\star \;\Rightarrow\; \mathrm{EXP} = \emptyset. \tag{8}$$

Of course, also $P(F) = \emptyset$ implies that $\mathrm{EXP} = \emptyset$. But $\mathrm{EXP} = \emptyset$ is also possible if $P(F) \neq \emptyset$ and $F \subset F_\star$ as Figure 4 below shows.

Problem 2. Provide interesting conditions under which $\mathrm{EXP} = \emptyset$, under which $\emptyset \subset \mathrm{EXP} \subset \mathrm{P}(F)$ and under which $\emptyset \subset \mathrm{EXP} = \mathrm{P}(F)$.

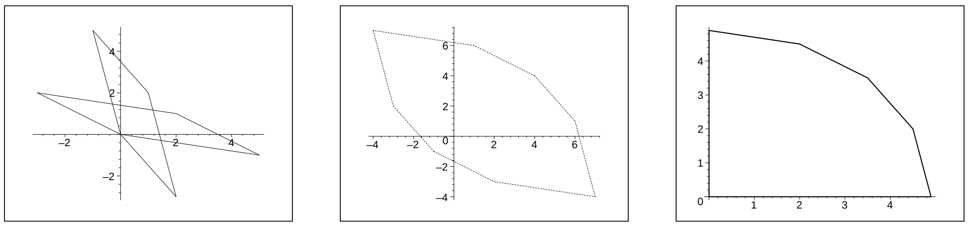
Now we illustrate these two problems (in case $m = n = 2$) with some figures.[4]

Remark. (1) In all these figures every $\mathrm{Conv}(U_k)$ are polygons. This makes that F_k ($k \in m$), F, U and $F_\star$ are polygons.

Figure 1 relates to

$$U_1 = \{(2,1), (-3,2), (5,-1), (0,0)\}, \;\; U_2 = \{(1,2), (-1,5), (2,-3), (0,0)\}.$$

Figure 1 (and also Figures 2 – 5) are to be interpreted as follows. Four polygons are drawn: the sets $\mathrm{Conv}(U_1)$ and $\mathrm{Conv}(U_2)$, the Minkowski sum of these two sets and the set $F = F_1 + F_2$ being the boldfaced polygon. These four polygons are respectively drawn in the following three figures:



We note that in the case of Figure 1

$$U_1 + U_2 = \{(3,3), (1,6), (-2,4), (-4,7), (4,-2), (2,1), (-1,-1), (-3,2),$$
$$(6,1), (4,4), (1,2), (-1,5), (7,-4), (5,-1), (2,-3), (0,0)\}.$$

Figure 2 relates to

$$U_1 = \{(0,2), (3,1), (-3,0), (0,0)\}, \;\; U_2 = \{(0,1), (1,1/2), (-2,0), (0,0)\}.$$

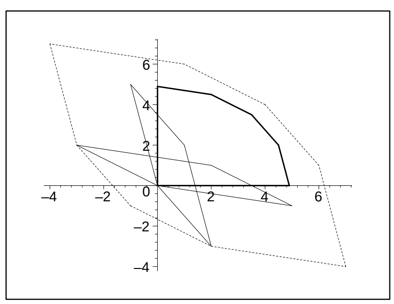


Fig. 1: $F \subset F_\star$ and $\emptyset \subset \mathrm{EXP} = P(F)$. (Neither U_1 nor U_2 is permutation-symmetric).

Figure 3 relates to

$$U_1 = \{(7,1), (-3,3), (10,-2), (0,0)\}, \;\; U_2 = \{(1,7), (-2,10), (3,-3), (0,0)\}.$$

Figure 4 relates to

$$U_1 = \{(2,2), (-2,4), (4,-2), (0,0)\}, \;\; U_2 = \{(2,2), (-1,1), (1,-1), (0,0)\}.$$

[4] Figures 1, 3, 4, 5 are taken from Folmer and von Mouche (2000).

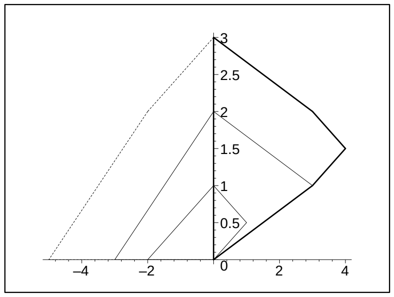


Fig. 2: $F = F_\star$ and $\text{EXP} = \emptyset$. (Neither U_1 nor U_2 is permutation-symmetric).

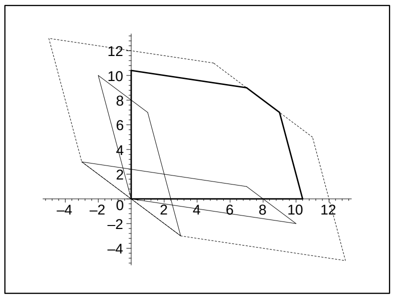


Fig. 3: $F \subset F_\star$ and $\emptyset \subset \text{EXP} \subset P(F)$ (U_1 and U_2 are not permutation-symmetric).

Figure 5 relates to

$$U_1 = \{(2,2),\ (-2,10),\ (10,-2),\ (0,0)\},\ \ U_2 = \{(3,3),\ (-3,4),\ (4,-3),\ (0,0)\}.$$

3.2. On Problem 1

Now let us return to Problem 1.

Proposition 10. *1. $F \subseteq F_\star$. And $F = F_\star$ if and only if $\sum_{k\in M}(\text{Conv}(U_k)\cap \mathbb{R}^n_+) \supseteq \mathbb{R}^n_+ \cap \sum_{k\in M} \text{Conv}(U_k)$.*
2. If $m = 1$, then $F = F_\star$. ⋄

Proof. 1. $F = \sum_{k\in M}(\text{Conv}(U_k) \cap \mathbb{R}^n_+) \subseteq \sum_{k\in M} \mathbb{R}^n_+ \cap \sum_{k\in M} \text{Conv}(U_k) = \mathbb{R}^n_+ \cap \sum_{k\in M} \text{Conv}(U_k) = F_\star$.
2. $F = F_1 = \text{Conv}(U_1) \cap \mathbb{R}^n_+ = \text{Conv}(U) \cap \mathbb{R}^n_+ = F_\star$. Q.E.D.

Of course, because of (7), if $F_\star = \emptyset$, then $F = F_\star$ holds. The next proposition identifies two little bit less trivial cases for this to hold:

Proposition 11. *Each of the following conditions is sufficient for $F = F_\star$ to hold.*

1. *There exist $r_k > 0$ ($k \in M$) and $c \in \mathbb{R}^n$ such that $U_k = r_k(U_1 + c)$ ($k \in M$).*
2. *$U_k \subseteq \mathbb{R}^n_+$ ($k \in M$).* ⋄

Proof. 1. We have $\text{Conv}(U_k) \cap \mathbb{R}^n_+ = \text{Conv}(r_k(U_1 + c)) \cap \mathbb{R}^n_+ = r_k\,\text{Conv}(U_1 + c) \cap r_k\mathbb{R}^n_+ = r_k(\text{Conv}(U_1 + c)\cap\mathbb{R}^n_+)$; here the last equality holds as $r_k \neq 0$. This implies, with $r := \sum_k r_k$ and with sums on $k \in M$

$$\sum(\text{Conv}(U_k) \cap \mathbb{R}^n_+) = \sum r_k(\text{Conv}(U_1 + c) \cap \mathbb{R}^n_+) = r(\text{Conv}(U_1 + c) \cap \mathbb{R}^n_+);$$

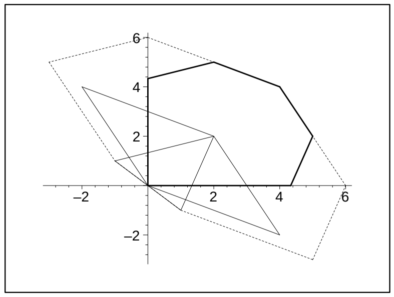


Fig. 4: $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$ (U_1 and U_2 are permutation-symmetric).

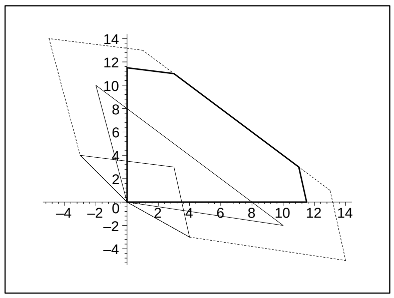


Fig. 5: $F \subset F_\star$ and $\emptyset \subset \mathrm{EXP} \subset P(F)$ (U_1 and U_2 are permutation-symmetric).

here the last equality holds as $\mathbb{R}^n_+ \cap \mathrm{Conv}(U_1 + c)$ is convex and the r_k are non-negative. Further

$$r(\mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+) = r\ \mathrm{Conv}(U_1 + c) \cap r\ \mathbb{R}^n_+ = r\ \mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+$$

$$= \mathbb{R}^n_+ \cap \sum (r_k\ \mathrm{Conv}(U_1 + c)) = \mathbb{R}^n_+ \cap \sum \mathrm{Conv}(U_k).$$

So the proof is complete by Proposition 10(1).

2. Using $U_k \subseteq \mathbb{R}^n_+$ and $\sum_k \mathrm{Conv}(U_k) \subseteq \mathbb{R}^n_+$ we obtain $\sum_k(\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+) = \sum_k \mathrm{Conv}(U_k) = \mathrm{Conv}(\sum_k U_k) = \mathrm{Conv}(\sum_k U_k) \cap \mathbb{R}^n_+ = \mathbb{R}^n_+ \cap \sum_k \mathrm{Conv}(U_k)$. Q.E.D.

Figure 2 shows that there are situations with $F = F_\star$ that are not covered by Proposition 11. In all other figures $F \subset F_\star$ holds. Theorem 1 below gives our main result for $F \subset F_\star$ to hold. This theorem is based on the following principle:

Proposition 12. *Suppose there exists $l \in M$ such that $\mathrm{Conv}(S(U_l)) \cap \mathbb{R}^n_+ = \emptyset$ and $S(U) \cap \mathbb{R}^n_+ \neq \emptyset$, then $F \subset F_\star$.* $\diamond$

Proof. We shall prove that $S(U) \cap \mathbb{R}^n_+ \subseteq F_\star \setminus F$ (and then the desired result follows). So fix $\mathbf{b} \in S(u) \cap \mathbb{R}^n_+$. Of course, $\mathbf{b} \in F_\star$. Now we shall prove by contradiction that $\mathbf{b} \notin F$. So suppose $\mathbf{b} \in F = \sum_k F_k$. Take $\mathbf{h}^k \in \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+$ such that $\mathbf{b} = \sum_k \mathbf{h}^k$. Using (2), we have for every $k \in M$

$$\sum_j h^k_j \leq s(\mathrm{Conv} U_k) = s(U_k). \tag{9}$$

Because $\mathbf{h}^l \in \mathbb{R}^n_+$ it follows that $\mathbf{h}^l \notin \mathrm{Conv}(S(U_l))$ and so $\mathbf{h}^l \in \mathrm{Conv}(U_l) \setminus \mathrm{Conv}(S(U_l))$. By virtue of (2) we have $\mathrm{Conv}(S(U_l)) = S(\mathrm{Conv}(U_l))$ and so $\mathbf{h}^l \in$

$\mathrm{Conv}(U_l) \setminus S(\mathrm{Conv}(U_l))$. Therefore, in (9) we have a strict inequality for $k = l$. Because $\mathbf{b} \in S(U)$, one has $\sum_j b_j = s(U)$. With Proposition 4 it follows that $s(U) = \sum_k s(U_k) > \sum_k \sum_j F_j^k = \sum_j \sum_k F_j^k = \sum_j b_j = s(U)$, which is a contradiction. Q.E.D.

Now we shall identify a more concrete situation (i.e. in terms of the U_k) that satisfies this principle. In order to do so we introduce some notions in the following two definitions.

Definition 2. Let A_k $(k \in M)$ be subsets of $\mathbb{R}^n$. The sets A_k $(k \in M)$ have *compensating asymmetries of exactly the same magnitude* if $m = n$ and there are $\pi_k \in S_n$ $(k \in M)$ with $\pi_1 = \mathrm{Id}$ such that

$$\{\pi_1(j), \ldots, \pi_n(j)\} = N \ (j \in N) \text{ and } A_k = T_{\pi_k}(A_1) \ (k \in M). \ \diamond$$

<u>Remarks</u>. (2) If $m = n = 1$, then A_1 has compensating asymmetries of exactly the same magnitude.

(3) If at least one A_k is permutation-symmetric, then A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude if and only if all A_k are identical.

(4) If A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude, their Minkowski sum A is not necessarily permutation-symmetric as the following example shows;[5] but it is if $m = 2$ as Proposition 13(4) shows.

Example 1. Let $m = n = 3$, $A_1 = \{(3, 0, 1), (0, 2, 4)\}$ and (using cycle notations) $\pi_1 = \mathrm{id}$, $\pi_2 = (132)$, $\pi_3 = (123)$. So $A_2 = \pi_2(A_1) = \{(1, 3, 0), (4, 0, 2)\}$, $A_3 = \pi_3(A_1) = \{(0, 1, 3), (2, 4, 0)\}$. The sets A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude and

$$A = \{(4, 4, 4), \ (6, 7, 1), \ (7, 1, 6), \ (9, 4, 3), \ (1, 6, 7), (3, 9, 4), (4, 3, 9), (6, 6, 6)\}. \ \diamond$$

Proposition 13. *Suppose A_k $(k \in M)$ are subsets of $\mathbb{R}^n$ that have compensating asymmetries of exactly the same magnitude. Let $A = \sum_{k \in M} A_k$ and $l \in M$.*

1. *$S(A_l) \neq \emptyset \Leftrightarrow S(A) \neq \emptyset$. And $\#S(A_l) = 1 \Leftrightarrow \#S(A) = 1$.*
2. *If $S(A_l) \neq \emptyset$, then $(s(A_l), \ldots, s(A_l)) \in S(A)$.*
3. *$s(A) = ns(A_l)$.*
4. *If $m = 2$, then A is permutation-symmetric.* $\diamond$

Proof. It is easy to see that we may suppose $l = 1$.

Let π_k $(k \in M)$ be as in Definition 2. By Proposition 4(2) and (3)

$$S(A) = \sum_{k \in M} T_{\pi_k}(S(A_1)). \tag{10}$$

1. By (10).

2. Let $\mathbf{a} \in S(A_1)$. By (10), $\mathbf{b} := \sum_k T_{\pi_k}(\mathbf{a}) \in S(A)$. For $i \in N$ we have $b_i = \sum_k a_{\pi_k(i)} = \sum_k a_k$. Thus $b_1 = \cdots = b_n$. As $nb_1 = s(A) = ns(A_1)$ it follows that $\mathbf{b} = (s(A_1), \ldots, s(A_1)) \in S(A)$.

[5] Indeed, there $T_{(12)}A \neq A$.

3. This holds if $A_1 = \emptyset$. Now suppose $A_1 \neq \emptyset$. By Proposition 4, $s(A) = s(\sum_k T_{\pi_k}(A_1)) = \sum_k s(T_{\pi_k}(A_1)) = \sum_k s(A_1) = ns(A_1)$.

4. Let $\pi \in S_n$. We shall prove that $T_\pi(A) = A$. Well, $T_\pi(A) = T_\pi(A_1 + T_{\pi_2}(A_1)) = T_\pi(A_1) + T_\pi(T_{\pi_2}(A_1)) = T_\pi(A_1) + (T_{\pi_2 \circ \pi})(A_1)$. As $S_2 = \{\pi_1, \pi_2\}$, we obtain $T_\pi(A) = A_1 + T_{\pi_2}(A_1) = A$. Q.E.D.

The notion in the following definition is taken from Folmer and von Mouche (2000).

Definition 3. Let X be a subset of $\mathbb{R}^n$. For $j \in N$, X has a *j-defect* if $y_j < 0$ for all $\mathbf{y} \in S(X)$. And X has a *defect* if it has a j-defect for some j. $\diamond$

Proposition 14. *Let X be a subset of $\mathbb{R}^n$ with a defect.*

1. *If X has a j-defect and $\pi \in S_n$, then $T_\pi(X)$ has a $\pi^{-1}(j)$-defect.*
2. *If $S(X) \neq \emptyset$ and $X \cap \mathbb{R}^n_+ \neq \emptyset$, then X is not permutation-symmetric.*
3. $\mathrm{Conv}(S(X)) \cap \mathbb{R}^n_+ = \emptyset$. $\diamond$

Proof. We suppose that X has a j-defect.

1. Suppose $\mathbf{b} \in S(T_\pi(X))$. By (3), $\mathbf{b} \in T_\pi(S(X))$. Take $\mathbf{a} \in S(X)$ such that $\mathbf{b} = T_\pi(\mathbf{a})$. So $b_{\pi^{-1}(j)} = a_j$. Using that X has a j-defect, we see that $b_{\pi^{-1}(j)} < 0$.

2. By contradiction, suppose X is permutation-symmetric. By part 1, for each $\pi \in S_n$ the set $T_\pi(X)$ has a $\pi^{-1}(j)$-defect. As $T_\pi(X) = X$, the set X has an i-defect for every $i \in N$. Take $\mathbf{y} \in S(X)$. Now $y_i < 0$ ($i \in N$). Let $\mathbf{w} \in X \cap \mathbb{R}^n_+$. Then one has $\sum_{j=1}^n w_j \geq 0 > \sum_{i=1}^n y_i$, a contradiction with $\mathbf{y} \in S(X)$.

3. With $I_j := \{\mathbf{a} \in \mathbb{R}^n \mid a_j < 0\}$, X having a j-defect is equivalent with $S(X) \subseteq I_j$. As I_j is convex, this in turn is equivalent with $\mathrm{Conv}(S(X)) \subseteq I_j$. As $I_j \cap \mathbb{R}^n_+ = \emptyset$, it follows that $\mathrm{Conv}(S(X)) \cap \mathbb{R}^n_+ = \emptyset$. Q.E.D.

Proposition 15. *Suppose U_k ($k \in M$) have compensating asymmetries of exactly the same magnitude. Let $l \in M$.*

1. $[S(U_l) \neq \emptyset$ *and* $s(U_l) \geq 0] \Leftrightarrow S(U) \cap \mathbb{R}^n_+ \neq \emptyset$.
2. $[S(U_l) \neq \emptyset$ *and* $s(U_l) > 0] \Leftrightarrow S(U) \cap \mathbb{R}^n_{++} \neq \emptyset$. $\diamond$

Proof. 1. '$\Leftarrow$': so $S(U) \neq \emptyset$. By Proposition 4, $S(U_l) \neq \emptyset$. Take $\mathbf{u} \in S(U) \cap \mathbb{R}^n_+$. Then $s(U) = \mathcal{C}(\mathbf{u}) \geq 0$. Proposition 13(3) implies $s(U_l) \geq 0$.

'$\Rightarrow$': with Proposition 13(2), $(s(U_l), \ldots, s(U_l)) \in S(U) \cap \mathbb{R}^n_+$.

2. Analogous to part 1. Q.E.D.

Theorem 1. *Suppose U_k ($k \in M$) have compensating asymmetries of exactly the same magnitude, $S(U_1) \neq \emptyset$ and $s(U_1) \geq 0$.*[6]

1. (a) *$S(U) \cap \mathbb{R}^n_+ \neq \emptyset$, so U does not have a defect.*
   (b) *if U_1 has a defect, then $F \subset F_\star$.*
2. *Suppose $U_1 \cap \mathbb{R}^n_+ \neq \emptyset$. Fix $\mathbf{n} \in U_1 \cap \mathbb{R}^n_+$ and $\mathbf{y} \in S(U_1)$.*
   (a) *No U_k is permutation-symmetric.*
   (b) *With*[7] $\mathbf{a} := \sum_k T_{\pi_k}(\mathbf{n})$ *and* $\mathbf{b} := \sum_k T_{\pi_k}(\mathbf{y})$ *we have* $\mathbf{a} \in U \cap \mathbb{R}^n_+$, $\mathbf{b} \in S(U)$ *and* $\mathbf{b} \gg \mathbf{a}$.

[6] This implies that $n \neq 1$ and therefore also that $m \neq 1$.
[7] Here π_k ($k \in M$) are as in Definition 2.

(c) $S(U) \cap \mathbb{R}^n_{++} \neq \emptyset$. $\diamond$

Proof. 1a. By Proposition 15(1).

1b. By the principle (i.e. Proposition 12). It applies by virtue of Proposition 14(3) and part 1a.

2. a. By Proposition 14(2), U_1 is not permutation symmetric. Now further apply also the first part of this proposition.

b. Note that $T_{\pi_k}(\mathbf{n}), T_{\pi_k}(\mathbf{y}) \in U_k$ $(k \in M)$. Also $T_{\pi_k}(\mathbf{n}) \in \mathbb{R}^n_+$ $(k \in M)$. So

$$\mathbf{a} = \sum_k T_{\pi_k}(\mathbf{n}) \in U \cap \mathbb{R}^n_+, \quad \mathbf{b} = \sum_k T_{\pi_k}(\mathbf{y}) \in U.$$

By (3), $T_{\pi_k}(\mathbf{y}) \in S(T_{\pi_k}(U_1)) = S(U_k)$ $(k \in M)$. By Proposition 4, $\mathbf{b} \in S(U)$. For $i \in N$ we have $a_i = \sum_k n_{\pi_k(i)} = \sum_k n_k$ and $b_i = \sum_k y_{\pi_k(i)} = \sum_k y_k$. So $a_1 = a_2 = \cdots = a_n =: a$ and $b_1 = b_2 = \cdots = b_n =: b$ follows. As U_1 has a defect and $\mathbf{n} \in \mathbb{R}^n_+$, $\mathbf{n} \notin S(U_1)$ holds. Proposition 3 now implies that $\mathbf{a} \notin S(U)$. It follows that $na < nb$. Therefore $a < b$ which implies that $\mathbf{b} \gg \mathbf{a}$.

c. By part 2b. Q.E.D.

Theorem 1(1b) explains $F \subset F_\star$ in Figure 1. In this figure also the assumptions of Theorem 1(2) and therefore also its conclusions hold.

Although in Theorem 1(2) no U_k is permutation symmetric, we observe from Figures 4 and 5 that $F \subset F_\star$ is compatible with every U_k permutation-symmetric.

The next result generalises Theorem 1(1): indeed, there in case $s(U_1) \geq 0$ it is possible to take $W_k = U_k$ $(k \in M)$ and $v^{(k)} = T_{\pi_k}(\mathbf{y})$ $(k \in M)$.

Theorem 2. *Suppose U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude and $S(U_1) \neq \emptyset$. Fix $\mathbf{y} \in S(U_1)$. Suppose $W_1, \ldots, W_n$ are subsets of $\mathbb{R}^n$ such that for every $k \in M$ there exists $\mathbf{v}^{(k)} \in S(W_k)$ such that*

$$v_i^{(k)} \geq y_{\pi_k(i)} - \frac{s(U_1)}{n} \; (i \in N). \tag{11}$$

Let $W := \sum_k W_k$,

1. *(a) $S(W) \cap \mathbb{R}^n_+ \neq \emptyset$, so W does not have a defect;*
   (b) if some W_k has a defect, then $F \subset F_\star$.
2. *if the inequalities in (11) are strict, then $S(W) \cap \mathbb{R}^n_{++} \neq \emptyset$.* $\diamond$

Proof. 1a. Let $\mathbf{v} := \sum_k \mathbf{v}^{(k)}$. By Proposition 4(2), $\mathbf{v} \in S(W)$. For $i \in N$ we have

$$v_i = \sum_k v_i^{(k)} \geq \sum_k (y_{\pi_k(i)} - \frac{s(U_1)}{n}) = \sum_k y_k - s(U_1) = s(U_1) - s(U_1) = 0.$$

Thus also $\mathbf{v} \in \mathbb{R}^n_+$.

1b. By the principle. It applies by virtue of Proposition 14(3) and part 1a.

2. Analogous to part 1a. Q.E.D.

Figure 3 shows that there are situations where the U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude where $F \subset F_\star$ holds that are not covered by Theorem 1.

3.3. On Problem 2

Now let us return to problem 2.

Proposition 16. *1.* $\mathrm{P}(F_\star) = \mathrm{P}(\mathrm{Conv}(U)) \cap \mathbb{R}^n_+$ *and* $\mathrm{P}(F_k) = \mathrm{P}(\mathrm{Conv}(U_k)) \cap \mathbb{R}^n_+$ *(*$k \in M$*).*
2. $\mathrm{EXP} = \mathrm{P}(F) \setminus \mathrm{P}_w(F_\star)$. $\diamond$

Proof. 1. By Proposition 2.

2. '$\subseteq$': suppose $\mathbf{u} \in \mathrm{EXP}$. Then $\mathbf{u} \in \mathrm{P}(F)$ and there exists $\mathbf{w} \in F_\star$ such that $\mathbf{w} \gg \mathbf{u}$. By (7), $\mathbf{u} \in F_\star$. Therefore $\mathbf{u} \notin \mathrm{P}_w(F_\star)$.

'$\supseteq$': suppose $\mathbf{u} \in \mathrm{P}(F) \setminus \mathrm{P}_w(F_\star)$. By (7), $\mathbf{u} \in F_\star$. As $\mathbf{u} \notin \mathrm{P}_w(F_\star)$, there is an $\mathbf{w} \in F_\star$ with $\mathbf{w} \gg \mathbf{u}$. Thus $\mathbf{u} \in \mathrm{EXP}$. Q.E.D.

Theorem 1(1b) also explains $F \subset F_\star$ in the following example and shows that $\mathrm{EXP} = \emptyset$ can hold under the general assumptions of Theorem 1.

Example 2. $m = n = 2$, $U_1 = \{(-1,1),(-1,-2)\}$, $U_2 = \{(1,-1),(-2,-1)\}$. Now $U = \{(0,0),(-3,0),(0,-3),(-3,-3)\}$, $F_1 = \emptyset$, $F_2 = \{-1\} \times [0,1], F = \emptyset$, $F_\star = \{(0,0)\}$, $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$. $\diamond$

Proposition 17. *If* $\mathbf{a} \in \mathrm{P}(F)$*, then* $\mathbf{a} \in \mathrm{EXP} \Leftrightarrow N^+_{\mathrm{Conv}(U)}(\mathbf{a}) = \emptyset$. $\diamond$

Proof. '$\Rightarrow$': let $\mathbf{c} \in F_\star$ such that $\mathbf{c} \gg \mathbf{a}$. For all $\mathbf{d} > \mathbf{0}$ one has $\mathbf{d} \cdot (\mathbf{c} - \mathbf{a}) > 0$. Because $\mathbf{c} \in \mathrm{Conv}(U)$, it follows that $\mathbf{d} \notin N^+_{\mathrm{Conv}(U)}(\mathbf{a})$.

'$\Leftarrow$': by Proposition 8, $\mathbf{a} \notin \mathrm{P}_w(\mathrm{Conv}(U))$. So there exists $\mathbf{c} \in \mathrm{Conv}(U)$ with $\mathbf{c} \gg \mathbf{a}$. Since $\mathbf{a} \in \mathbb{R}^n_+$, also $\mathbf{c} \in \mathbb{R}^n_+$. This implies $\mathbf{c} \in F_\star$. Thus $\mathbf{a} \in \mathrm{EXP}$. Q.E.D.

We have already seen that if $F = F_\star$ holds, then $\mathrm{EXP} = \emptyset$. A natural question now is whether $F \subset F_\star$ implies that $\mathrm{EXP} = \emptyset$. The answer is 'no' as Figure 3 shows. Proposition 11(2) implies that the condition $U_k \subseteq \mathbb{R}^n_+$ ($k \in M$) is sufficient for $\mathrm{EXP} = \emptyset$ to hold. This condition is quite strong. In the next proposition, which also explains $\mathrm{EXP} = \emptyset$ in Figure 4, there are more interesting conditions.

Proposition 18. *If, in case* $m = 2$, $\mathrm{P}(\mathrm{Conv}(U)) \subseteq \mathbb{R}^n_+$ *and* $\mathrm{Conv}(U_1)$ *or* $\mathrm{Conv}(U_2)$ *has a maximiser which belongs to* $\mathbb{R}^n_+$*, then*

1. $\mathrm{P}(F_\star) \supseteq \mathrm{P}(F)$*;*
2. $\mathrm{EXP} = \emptyset$. $\diamond$

Proof. 1. We may assume that $\mathrm{Conv}(U_2)$ has a maximiser, say $\mathbf{b}$. This implies $P(\mathrm{Conv}(U_2)) = \{\mathbf{b}\}$. As $\mathbf{b} \in \mathbb{R}^n_+$, we have $\mathbf{b} \in F_2$. This implies that $\mathbf{b}$ also is a maximiser of F_2 and therefore $P(F_2) = \{\mathbf{b}\}$. Now with Proposition 16(1) and Proposition 6

$$\mathrm{P}(F_\star) = \mathrm{P}(\mathrm{Conv}(U)) \cap \mathbb{R}^n_+ = \mathrm{P}(\mathrm{Conv}(U)) = \mathrm{P}(\mathrm{Conv}(U_1) + \mathrm{Conv}(U_2))$$

$$= \mathrm{P}(\mathrm{Conv}(U_1)) + \mathrm{P}(\mathrm{Conv}(U_2)) \supseteq \mathrm{P}(\mathrm{Conv}(U_1)) \cap \mathbb{R}^n_+ + \mathrm{P}(\mathrm{Conv}(U_2))$$

$$= \mathrm{P}(F_1) + \mathrm{P}(F_2) = \mathrm{P}(F_1 + F_2) = \mathrm{P}(F).$$

2. By part 1 and Proposition 16(2). Q.E.D.

Remark. (5) Figure 4 shows that the general conditions of Proposition 18 are compatible with $F \subset F_\star, \mathrm{EXP} = \emptyset$ and $\mathrm{P}(F_\star) \subset \mathrm{P}(F)$.

The next theorem explains $\mathrm{EXP} = \emptyset$ in Figure 2.

Theorem 3. *Suppose* $\mathrm{Conv}(U_k)$ $(k \in M)$ *are compact. If* $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathbb{R}^n_+$ $(k \in M)$*, then* $\mathrm{EXP} = \emptyset$. $\diamond$

Proof. According to Proposition 17 the proof is complete if we can prove that $N^+_{\mathrm{Conv}(U)}(\mathbf{z}) \neq \emptyset$ for all $\mathbf{z} \in \mathrm{P}(F)$. So suppose $\mathbf{z} \in \mathrm{P}(F) = \mathrm{P}(\sum_k F_k)$. By Proposition 8 one has $N^+_{\sum_k F_k}(\mathbf{z}) \neq \emptyset$. As $\mathbf{z} \in \sum_k F_k$, there exists $\mathbf{z}^{(k)} \in F_k$ $(k \in M)$ such that $\mathbf{z} = \sum_k \mathbf{z}^{(k)}$. With Proposition 7 one obtains

$$\emptyset \neq N^+_{\sum_k F_k}(\mathbf{z}) = \cap_k N^+_{F_k}(\mathbf{z}).$$

By assumption $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathbb{R}^n_+$ for all k. Therefore $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+ = F_k$. So we can apply Proposition 9 with $A = F_k$, $B = \mathrm{Conv}(U_k)$ and $\mathbf{z} = \mathbf{z}^{(k)}$ and get

$$N^+_{\mathrm{Conv}(U_k)}(\mathbf{z}^{(k)}) = N^+_{F_k}(\mathbf{z}^{(k)}) \ (k \in M)$$

and therefore $\cap_k N^+_{\mathrm{Conv}(U_k)}(\mathbf{z}) \neq \emptyset$. Applying Proposition 7, $N^+_{\mathrm{Conv}(U)}(\mathbf{z}) \neq \emptyset$ follows. Q.E.D.

Note that in Figure 2 even $F = F_\star$ holds. However, under the conditions of Theorem 3, $F \subset F_\star$ may hold as the following example shows.

Example 3. In case $m = 3, n = 1$, $U_1 = \{-1, 1\}$, $U_2 = \{2\}$, $U_3 = \{3\}$ one has $F = [5,6], F_\star = [4,6]$. Thus $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$. $\diamond$

The above results partially solve Problem 2.

4. Application to Linked Repeated Games

4.1. Games in strategic form

Consider a game in strategic form Γ among n players. That is, for each *player* $i \in N = \{1, \ldots, n\}$ we have a non-empty (*action*) set X^i and a real-valued (*payoff*) function f^i on the set of *action profiles* $\mathbf{X} := X^1 \times \cdots \times X^n$. For $\mathbf{x} \in \mathbf{X}$, $\mathbf{f}(\mathbf{x}) := (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x}))$ is called the *payoff vector* at $\mathbf{x}$ and $f^i(\mathbf{x})$ is called the *payoff* of player i at $\mathbf{x}$. We call

$$B := \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}\}$$

the *set of basic payoff vectors*. Its convex hull $\mathrm{Conv}(B)$ is called the *feasible set*. The minimax payoff of player i is defined by

$$\overline{v}^i := \inf_{\mathbf{z} \in X^1 \times \cdots \times X^{i-1} \times X^{i+1} \times \cdots \times X^n} \ \sup_{x^i \in X^i} f^i(z^1, \ldots, z^{i-1}, x^i, z^{i+1}, \ldots, z^n).$$

An element $\mathbf{w}$ of $\mathbb{R}^n$ is called *individually rational* if $w^i \geq v^i$ $(i \in N)$ and *strictly individually rational* if $w^i > v^i$ $(i \in N)$

We call the game *regular* if each payoff function is bounded and each player has minimax payoff 0.[8]

A *Nash equilibrium* $\mathbf{e}$ is an action profile with the property that for every $i \in N$ the function $f^i(e^1, \dots, e^{i-1}, \cdot, e^{i+1}, \dots, e^n)$ has e_i as maximiser. Payoff vectors at Nash equilibria are individually rational. An action profile that maximises the total payoff function $\sum_{i \in N} f^i$ is called *fully cooperative*. Denoting the set of fully cooperative strategy profiles by Y we have (in terms of (1)

$$S(B) = \mathbf{f}(Y), \tag{12}$$

So sufficient for Y to be non-empty is that B is compact.

For $\pi \in S_n$, i.e. a permutation of N, the game in strategic form $\pi(\Gamma)$ (called a *permuted game* of Γ) is defined as the game in strategic form where the action set Z^i of player i is $X^{\pi(i)}$ and his payoff function h^i is given by $h^i(z^1, \dots, z^n) = f^{\pi(i)}(z^{\pi^{-1}(1)}, \dots, z^{\pi^{-1}(n)})$. Note that

$$\text{the set of basic payoff vectors of } \pi(\Gamma) \text{ equals } T_\pi(B). \tag{13}$$

The game Γ is called *symmetric* if each player has the same action set and if for every $\pi \in S_n$ one has $\Gamma = \pi(\Gamma)$. If Γ is symmetric, then $T_\pi(B) = B$ for all $\pi \in S_n$, i.e. (see section 2) B is permutation-symmetric.

4.2. Repeated games

A repeated game is specified by a game in strategic form Γ, called the *stage game*, a number T (positive or $+\infty$) and a number $\delta \in [0, 1]$. Such a game simply will be denoted by

$$< \Gamma > .$$

T is called the *number of repetitions* and δ is called a *discount factor*.[9] When $T = \infty$, we always suppose to avoid convergence problems that $\delta < 1$ and that payoff functions are bounded. Itself $< \Gamma >$ is a game in strategic form with player set N where the action set of player i now is called his *strategy set*, denoted by $[X^i]$, and defined as the collection of sequences of mappings $\sigma^i = (\sigma^i_t)_{0 \le t < T}$ with $\sigma^i_t : \prod_{\tau=0}^{t-1} \mathbf{X} \to X^i$. And the payoff function of player i in $< \Gamma >$ is the function $[f^i] : [X^1] \times \cdots \times [X^n] \to \mathbb{R}$ defined by

$$[f^i](\boldsymbol{\sigma}) := \sum_{t=0}^{T-1} \delta^t f^i(\mathbf{a}_t(\boldsymbol{\sigma})),$$

where $a^j_t(\boldsymbol{\sigma}) \in X^j$ $(0 \le t < T)$, called *outcome path* for player j, inductively is defined by $a^j_0(\boldsymbol{\sigma}) := \sigma^j_0$ and $a^j_t(\boldsymbol{\sigma}) := \sigma^j_t(\mathbf{a}_0(\boldsymbol{\sigma}), \mathbf{a}_1(\boldsymbol{\sigma}), \dots, \mathbf{a}_{t-1}(\boldsymbol{\sigma}))$ $(1 \le t < T)$.

For a regular game in strategic form Γ the intersection of $\mathbb{R}^n_+$ and its feasible set is an important object. One calls it the *feasible individually rational payoff region* of the game. The feasible individually rational payoff region plays an important role in Folk theorems which relate to the geometric structure of the set of (average)

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[8] Note that for a regular game in strategic form it is possible that its feasible set does not contain $\mathbf{0}$. Indeed, this for example holds for the regular bimatrix game $\begin{pmatrix} -2;\ 2 & 0;\ -4 \\ 1;\ -3 & -2;\ 0 \end{pmatrix}$.

[9] Notice that in our setting a discount factor is player independent.

subgame perfect Nash equilibrium payoff vectors for infinitely repeated games $< \Gamma >$. For the purpose of this paper it is not necessary to go into the details of the Folk theorems.[10] For this, we refer to, for example, Benoît and Krishna (1996).

4.3. Direct sum games

Consider games in strategic form ${}_1\Gamma, \ldots, {}_m\Gamma$ with (the same) n players. We refer to them as *isolated stage games*. $M = \{1, \ldots, m\}$ is the set of *issues*. Denote, for $k \in M$, by

$$U_k$$

the set of basic payoff vectors of ${}_k\Gamma$. So $U_k \subseteq \mathbb{R}^n$. Let, for $k \in M$ and $j \in N$, ${}_kX^j$ be the action set and ${}_kf^j$ the payoff function of player j in ${}_k\Gamma$. Define for each $k \in M$

$${}_k\mathbf{X} \ := \ {}_kX^1 \times \cdots \times {}_kX^n$$

and for each player $j \in N$

$${}_*X^j \ := \ {}_1X^j \times \cdots \times {}_mX^j.$$

Moreover, define the mapping $\Psi : {}_1\mathbf{X} \times \cdots \times {}_m\mathbf{X} \to {}_*X^1 \times \cdots \times {}_*X^n$ by

$$\Psi({}_1\mathbf{x}, \cdots, {}_m\mathbf{x}) \ := \ ({}_*x^1, \ldots, {}_*x^n).$$

Ψ is called the *canonical mapping*. Note that the canonical mapping is a bijection. The *trade-off direct sum game* $(\oplus\Gamma)_\alpha$ is defined as the game in strategic form where player j has action set ${}_*X^j$ and his payoff function is given by[11]

$$f^j_\alpha({}_*x^1, \ldots, {}_*x^n) \ := \ \sum_{k \in M} {}_kf^j({}_kx^1, \ldots, {}_kx^n).$$

(In the case of two bimatrix games $(\oplus\Gamma)_\alpha$ is the tensor sum of the individual bimatrix games.) The set of basic payoffs vectors U of $(\oplus\Gamma)_\alpha$ equals the Minkowski sum of the U_k:

$$U = \sum_{k \in M} U_k.$$

Let, for $k \in M$, ${}_kE$ be the set of Nash equilibria of ${}_k\Gamma$, ${}_kY$ the set of fully cooperative action profiles of ${}_k\Gamma$. And let E_α be the set of Nash equilibria of $(\oplus\Gamma)_\alpha$ and Y_α the set of fully cooperative action profiles of $(\oplus\Gamma)_\alpha$. It can be shown that (see Folmer and von Mouche (1994))

$$\Psi({}_1E \times \cdots \times {}_mE) = E_\alpha, \tag{14}$$

$$\Psi({}_1Y \times \cdots \times {}_mY) = Y_\alpha \tag{15}$$

and also that regularity of each ${}_k\Gamma$ implies regularity of $(\oplus\Gamma)_\alpha$. In this case the feasible individually rational payoff region of ${}_k\Gamma$ is

$$F_k \ := \ \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+$$

[10] Especially one has to specify the types of strategies.

[11] The α refers to the fact that in this formula the payoffs of the isolated games are added (with weights 1).

and the feasible individually rational payoff region of $(\oplus\Gamma)_\alpha$ is

$$F_\star = \mathrm{Conv}(U) \cap \mathbb{R}^n_+.$$

Finally, define the *aggregated feasible individually rational payoff region* as

$$F := \sum_{k\in M} F_k.$$

4.4. Tensor games

Let ${}_1\Gamma, \ldots, {}_m\Gamma$ be regular isolated stage games with (the same) n players and consider the infinitely repeated games $< {}_k\Gamma >$ $(k \in M)$.[12] Linking of the (isolated) repeated games $< {}_k\Gamma >$ $(k \in M)$ is done by combining them into a repeated game $(\otimes\Gamma)_\alpha$, a so-called *trade-off tensor game*. Formally $(\otimes\Gamma)_\alpha$ just is the infinitely repeated game with $(\oplus\Gamma)_\alpha$ as stage game. In Folmer et al. (1993) it is shown that Nash equilibria for each repeated game $< {}_k\Gamma >$ lead in a canonical way to a Nash equilibrium for the trade-off tensor game $(\otimes\Gamma)_\alpha$.[13] In general, the trade-off tensor game also has other (subgame perfect) Nash equilibria. Folk theorems are useful for investigating these equilibria. In fact, the effects of linking can be studied by comparing the sets F and $F_\star$. This has been done in Section 3. All the results there, in particular $F \subseteq F_\star$, apply. The five figures in Section 3 are compatible with the following regular games. Below we shall discuss game theoretic pendants of the results in Section 3.

Figure 1: ${}_1\Gamma = \begin{pmatrix} 2;\ 1 & -3;\ 2 \\ 5;\ -1 & 0;\ 0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 1;\ 2 & -1;\ 5 \\ 2;\ -3 & 0;\ 0 \end{pmatrix}$.

Figure 2: ${}_1\Gamma = \begin{pmatrix} 0;\ 2 & 3;\ 1 \\ -3;\ 0 & 0;0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 0;\ 1 & 1;\ 0.5 \\ -2;\ 0 & 0;\ 0 \end{pmatrix}$.

Figure 3: ${}_1\Gamma = \begin{pmatrix} 7;\ 1 & -3;\ 3 \\ 10;\ -2 & 0;0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 1;\ 7 & -2;\ 10 \\ 3;\ -3 & 0;\ 0 \end{pmatrix}$.

Figure 4: ${}_1\Gamma = \begin{pmatrix} 2;\ 2 & -2;\ 4 \\ 4;\ -2 & 0;0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 2;\ 2 & -1;\ 1 \\ 1;\ -1 & 0;\ 0 \end{pmatrix}$.

Figure 5: ${}_1\Gamma = \begin{pmatrix} 2;\ 2 & -2;\ 10 \\ 10;\ -2 & 0;0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 3;\ 3 & -3;\ 4 \\ 4;\ -3 & 0;\ 0 \end{pmatrix}$.

A strict inclusion $F \subset F_\star$ (see Problem 1 in Section 3) can be interpreted as 'linking sustains more cooperation'. And $\mathrm{EXP} \neq \emptyset$, i.e. the existence of an expansion point of the Pareto boundary $\mathrm{PB}(F)$ (see Problem 2 in Section 3), can be interpreted as 'Linking brings Pareto improvements'. So in this way we now have formalized for tensor games the themes 'linking may sustain more cooperation' and 'linking may bring Pareto improvements' from the introduction.

The results in Section 3 now can be formulate in terms of the above game theoretic situation. (8) implies that in the case linking brings Pareto improvements, it also sustains more cooperation. The reverse does not hold in general. Proposition 11 leads in an obvious way to two classes of isolated stage games for which linking does

[12] It is is implicitly assumed that in each of them the periods are the same and the discount factors are the same.

[13] It is straightforward to show that this statement remains valid if one replaces 'Nash equilibrium' by 'subgame perfect Nash equilibrium'.

not sustain more cooperation. The next theorem is the game theoretic pendant of Theorem 1 and is a formalisation of the basic idea that an exchange of concessions may enhance cooperation if the issues have compensating asymmetries of similar magnitude.

Theorem 4. *Consider isolated regular stage games* ${}_1\Gamma, \ldots, {}_m\Gamma$ *with* $m = n$ *players for which there are* $\pi_k \in S_n$ $(k \in M)$ *with* $\pi_1 = \mathrm{Id}$ *such that* $\{\pi_1(j), \ldots, \pi_n(j)\} = N$ $(j \in N)$ *and* ${}_k\Gamma = T_{\pi_k}({}_1\Gamma)$ $(k \in M)$. *Also suppose the basic payoff set* U_1 *is compact. Suppose* ${}_1\Gamma$ *has a Nash equilibrium and* U_1 *has a defect.*

1. *Then linking sustains more cooperation.*
2. *The game* $(\oplus\Gamma)_\alpha$ *has a Nash equilibrium* $\mathbf{e}$ *and a fully cooperative action profile* $\mathbf{y}$, *with strictly individually payoff vector, which is an unanimous Pareto improvement of* $\mathbf{e}$.[14] $\diamond$

Proof. Let $\mathbf{n}$ be a Nash equilibrium of ${}_1\Gamma$. As ${}_1\mathbf{f}(\mathbf{n})$ is individually rational, we have ${}_1\mathbf{f}(\mathbf{n}) \in U_1 \cap \mathbb{R}^n_+$. So $s(U_1) \geq 0$. As U_1 is compact, $S(U_1) \neq \emptyset$. By (12), $S(U_1) = {}_1\mathbf{f}({}_1Y)$. Fix $\mathbf{r} \in {}_1Y$. So ${}_1\mathbf{f}(\mathbf{r}) \in S(U_1)$.

1. (13) implies that U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude. Now apply Theorem 1(1b).

2. Now ${}_k\mathbf{x} := (n^{\pi_k(1)}, \ldots, n^{\pi_k(n)}) \in {}_kE$ $(k \in M)$. By (14), $\mathbf{e} := \Psi({}_1\mathbf{x}, \ldots, {}_m\mathbf{x}) \in E_\alpha$, i.e. $\mathbf{e}$ is a nash equilibrium of $(\oplus\Gamma)_\alpha$. Also ${}_k\mathbf{z} := (r^{\pi_k(1)}, \ldots, r^{\pi_k(n)}) \in {}_kY$ $(k \in M)$. By (15), $\mathbf{y} := \Psi({}_1\mathbf{z}, \ldots, {}_m\mathbf{z}) \in Y_\alpha$, i.e. $\mathbf{y}$ is a fully cooperative action profile of $(\oplus\Gamma)_\alpha$.

The payoff vector at $\mathbf{e}$ equals $\mathbf{a} := \sum_{k \in M} T_{\pi_k}({}_1\mathbf{f}(\mathbf{n}))$.[15] And that at $\mathbf{y}$ equals $\mathbf{b} := \sum_{k \in M} T_{\pi_k}({}_1\mathbf{f}(\mathbf{y}))$. Now apply Theorem 1(2b,2c). Q.E.D.

With Theorem 2 we have studied how far can one deviate in Theorem 1 from the situation of (exact) permuted games. In doing so, we have made more precise the above 'similar magnitude'. Concerning Pareto improvements, we identified in Proposition 18 and Theorem 3 classes where linking does not bring Pareto improvements. We also showed with Figure 5 that in the case all isolated stage game are symmetric (but not identical), more cooperation and even Pareto improvements are possible.

We note that the above isolated stage games related to Figures 1, 3 and 5 are prisoners' dilemma games.[16] Concerning this we mention that sufficient for the condition 'Suppose ${}_1\Gamma$ has a Nash equilibrium and U_1 has a defect' in Theorem 4 to hold is that ${}_1\Gamma$ is a 2×2-bimatrix prisoners' dilemma game with a *unique* fully cooperative action profile.[17]

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[14] I.e. $f^j_\alpha(\mathbf{y}) > f^j_\alpha(\mathbf{e})$ $(j \in N)$.

[15] Indeed: $\mathbf{a} = (\sum_k {}_kf^1({}_k\mathbf{x}), \ldots, \sum_k {}_kf^n({}_k\mathbf{x})) = \sum_k ({}_kf^1({}_k\mathbf{x}), \ldots, {}_kf^n({}_k\mathbf{x})) = \sum_k ({}_1f^{\pi_k(1)}(\mathbf{n}), \ldots, {}_1f^{\pi_k(n)}(\mathbf{n})) = \sum_k T_{\pi_k}({}_1f^1(\mathbf{n}), \ldots, {}_1f^n(\mathbf{n})) = \sum_k T_{\pi_k}({}_1\mathbf{f}(\mathbf{n}))$.

[16] We call a game in strategic form a *prisoners' dilemma game* if every player $i \in N$ has a strictly dominant action (i.e. a unique action that gives player i for every choice of actions of the other players a maximal payoff) and the unique Nash equilibrium is in the weak sense Pareto-inefficient (i.e. there exists an action profile in which every payoff is higher than in the equilibrium).

[17] Indeed, for this situation ${}_1\Gamma$ has a Nash equilibrium and a defect. The existence of a defect follows from the fact that for every 2×2-bimatrix prisoners' dilemma game for each player his payoff at the unique Nash equilibrium equals his minimax payoff 0.

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