Consistency to the Values for Games in Generalized Characteristic Function Form

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Abstract In the classical game space, Evans (1996) introduced a procedure, such that the solution of a game determined endogenously as the expected outcome of a reduction of the game to a two-person bargaining problem, is just the Shapley value. This approach is not suitable for games in generalized characteristic function form, in which the order of players entering into the game affects the worth of coalitions. Based on Evans's approach, in this paper we propose a new procedure which induces the generalized Shapley value defined by Sanchez and Bergantinos (1997). Moreover, this generalized procedure can be adapted to characterize the class of values satisfying efficiency, linearity and symmetry, for games in generalized characteristic function form.

Keywords: cooperative game, orders, value, consistency, procedure

1. Introduction

In the classical case, a cooperative game with transferable utility, or shortly, a TU game, is an ordered pair $\langle N, v\rangle$, where N is a nonempty, finite set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. An element $i \in N$ is called a player, and a subset $S \subseteq N$ is called a coalition. The associated real number $v(S)$ is called the worth of coalition S. The size of coalition S is denoted by $|S|$, or shortly by s if no ambiguity arises. Particularly $|N|$, or equivalently n denotes the size of the grand coalition N. We denote by $\mathcal{G}_N$ the set of all cooperative TU games with player set N and by $\mathcal{G}$ the space of all cooperative TU games with arbitrary player set. A value $\phi = (\phi_i)_{i\in N}$ on $\mathcal{G}$ is a mapping which assigns to every TU game $\langle N, v\rangle$ exactly one element $\phi(N, v) \in \mathbb{R}^N$. One of the most well-known values is the Shapley value (Shapley, 1953a).

Evans (1996) introduced the following procedure: given an n-player cooperative game and a feasible "wage" n-vector, suppose that the players in a cooperative game are randomly split into two coalitions, each with a randomly chosen leader; the two leaders bargain bilaterally and each pays, out of his share, a wage to each member of his coalition as specified by the wage vector. More precisely for an arbitrary cooperative game $\langle N, v\rangle$ in $\mathcal{G}$, the following procedures are done sequentially:

(A) the players in the grand coalition N are randomly split into two coalitions, say S and $N \setminus S$ ($S \neq \emptyset, N$);
(B) each coalition generates randomly a leader, say leader i represents S and leader j represents $N \setminus S$, $i \in S$, $j \in N \setminus S$;
(C) player i and j play a two-person bargaining process based on coalition S and $N \setminus S$ separately. The rule is that each leader pays to each member of his coalition, an certain part of what he gets in the two-person bargaining process.

A value is said to be *consistent* with the above procedure if it is equal to the expected payoff. Under such consistency condition, Evans proved that the Shapley value is the unique solution, if all randomly chosen processes are with respect to the uniform distribution, and the two-person bargaining result is *standard* according to Hart and Mas-Colell (1989). Remind that a value ϕ on $\mathcal{G}$ is standard for two-person game, means for an arbitrary two-person game $\langle \{i,j\}, v\rangle$,

$$\phi_k(\{i,j\},v) = v(\{k\}) + \frac{1}{2}\left(v(\{i,j\}) - v(\{i\}) - v(\{j\})\right) \quad \text{for } k \in \{i,j\}. \tag{1.1}$$

The above results are derived in the classical game space, where the characteristic function assigns to each group of players a fixed single number, regardless of how players are ordered in the group. However, in modeling some economic situations or some special relationships among players, the earning of a group of players may depend not only on its members, but also on the sequential ordering of players joining the game. So a better approximation to some real life situation is to consider games where the so-called characteristic function is defined on all possible orders for coalitions of players. Such generalized model is introduced first by Nowak and Radzik (1994). They redefined the efficiency, null player property and strong monotonicity in this new game space, and axiomatized an adapted Shapley value by using two groups of redefined properties. The first group of properties contains efficiency, null player property and additivity, and the uniqueness proof follows the approach given by Shapley (1953a). The second group of properties are efficiency and strong monotonicity, and the proof proceeds according to the one given by Young (1985). The symmetry property compared to the classical case, was included in the definition of the null player in the null player property as well as the marginal contribution in the strong monotonicity. Sanchez and Bergantinos (1997) found the inaccuracy of the symmetry property, and gave more "suitable" definitions for the null player, symmetric player and marginal contribution, in the new game space. In this way, a new Shapley value was characterized, by using these new defined properties. Later such Shapley value was generalized to games with a priori unions in the same way that Owen (1977) did for the classical Shapley value, by Sanchez and Bergantinos (1999), who also characterized the weighted Shapley value in the new game space (Bergantinos and Sanchez, 2001), based on the results in the classical game space given by Shapley (1953b) as well as Kalai and Samet (1987).

In our point of view, the properties used in Sanchez and Bergantinos' papers are more fair and attractive, since it considered all possibilities (positions) how a single player joining into a coalition. We follow the notation given by Sanchez and Bergantinos (1997).

For any subset $S \subseteq N$, denote by $H(S)$ the set of all orders of players in S. The element $S' \in H(S)$ is called an *ordered coalition*. For notational convenience, we use S to represent a general coalition with size s regardless of order and $S' \in H(S)$ to represent an ordered coalition with the same player set. Note that $H(\emptyset) = \emptyset$ as well as $H(\{i\}) = \{\{i\}\}$ for all $i \in N$. Denote by Ω the set of all ordered coalitions, that is,

$$\Omega = \{S' | S' \in H(S), S \subseteq N, S \neq \emptyset\}.$$

Obviously, the total number of ordered coalitions in Ω equals

$$m := \sum_{s=1}^{n} s! C_n^s \quad \text{where} \quad C_n^s = \binom{n}{s} = \tfrac{s!(n-s)!}{n!} \quad \text{for all } 1 \le s \le n. \tag{1.2}$$

Definition 1.1. A game in generalized characteristic function form, or a generalized game is an ordered pair $\langle N, v\rangle$, where N is a non-empty, finite set of players and $v : \Omega \to \mathbb{R}$ is a generalized characteristic function that assigns to each $S' \in \Omega$, the real-valued worth $v(S')$ as the utility obtained by players in S according to the order S', such that $v(\emptyset) = 0$.

Denote by $\mathcal{G}'_N$ the set of all generalized cooperative games with player set N, and $\mathcal{G}'$ the set all generalized cooperative games with arbitrary player set. A value ϕ on $\mathcal{G}'_N$ is a mapping assigning exactly one element $(\phi_i(N, v))_{i\in N} \in \mathbb{R}^N$ to every $v \in \mathcal{G}'_N$. The following definition will play an important role in our solution theory for generalized TU games.

Definition 1.2. Let $S' \in H(S), S \subsetneqq N$ be given. A set T' is called an extension of S' of size $t, t > s$ if a set of $t - s$ players in $N \setminus S$ is inserted among the players of S' in such a way that the players in S appear in T' in the same order as in S'. We denote by $V(S')$ the set of all extensions of S'.

As a special case we define an extension $T' = (S', i^h)$ with $i \notin S$, $t = s + 1$ as follows. Given player $i \in N$, coalition $S \subseteq N \setminus \{i\}$ of size s, ordered coalition $S' \in H(S)$, and height $h \in \{1, 2, \ldots, s + 1\}$, then (S', i^h) denotes the $(s + 1)$-person ordered coalition with player i inserted in the h-th position, that is, if $S' = (i_1, \ldots, i_s)$, then $(S', i^1) = (i, i_1, \ldots, i_s)$; $(S', i^{s+1}) = (i_1, \ldots, i_s, i)$; and $(S', i^h) = (i_1, \ldots, i_{h-1}, i, i_h, \ldots, i_s)$ for all $2 \le h \le s$.

Definition 1.3. (Sanchez and Bergantinos, 1997) For any generalized TU game $\langle N, v\rangle$, the generalized Shapley value $Sh'(N, v) = (Sh'_i(N, v))_{i\in N}$ is given by

$$Sh'_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} \frac{p^n_s}{(s + 1)!} \sum_{S'\in H(S)} \sum_{h=1}^{s+1} \left[v(S', i^h) - v(S')\right] \quad \text{for all } i \in N.$$

We can rewrite the equation above in view of the extension (see Definition 1.2) in the following way:

$$Sh'_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} p^n_s \sum_{S'\in H(S)} (s!)^{-1} \sum_{\substack{T'\in V(S'), T'\ni i,\\ t=s+1}} \frac{v(T') - v(S')}{s + 1}. \tag{1.3}$$

Shapley (1953a) introduced p^n_s as the classical probability measure over the collection of (unordered) coalitions not containing any fixed player. The difference compared with the classical case is that in this new setting, any player $i \in N$ has $(s + 1)$ ways to join any ordered coalition S' of size s, $S' \in H(S)$, $S \subseteq N \setminus \{i\}$, yielding various marginal contributions $v(T') - v(S')$ for all $T' \in V(S')$ containing player i, of size $t = s + 1$. The expected payoff to any player i with respect to the underlying classical probability measure is obtained through averaging over all the player's marginal contributions as well as over all $s!$ possible ordered coalitions with player set S.

Definition 1.4. (Nowak and Radzik, 1994; Sanchez and Bergantinos, 1997) A value ϕ on $\mathcal{G}'$ satisfies the

(i) efficiency, if

$$\sum_{i\in N}\phi_i(N,v)=\frac{1}{n!}\sum_{N'\in H(N)}v(N') \quad \text{for all generalized game } \langle N,v\rangle; \qquad (1.4)$$

(ii) symmetry, if $\phi_i(N,v)=\phi_j(N,v)$ for all symmetric players i and j. Two players $i,j\in N$ are symmetric in $\langle N,v\rangle$ if for every ordered coalition S' such that $S'\not\ni i,j$, we have that $v(S',i^h)=v(S',j^h)$ for all $h\in\{1,2,\ldots,s+1\}$;

(iii) null player property, if $\phi_i(N,v)=0$ for every generalized game $\langle N,v\rangle$, and every null player $i\in N$. Player i is called a null player in $\langle N,v\rangle$ if for every ordered coalition S' not containing i, we have that $v(S',i^h)=v(S')$ for every $h\in\{1,2,\ldots,s+1\}$.

Denote by $\bar{v}(N)$ the average sum of the worths for all permutations $N'\in H(N)$, i.e.,

$$\bar{v}(N)=\frac{1}{n!}\sum_{N'\in H(N)}v(N'),$$

then the efficiency condition (1.4) is equivalent to $\sum_{i\in N}\phi_i(N,v)=\bar{v}(N)$. Sanchez and Bergantinos (1997) proved that the Shapley value is the unique value on $\mathcal{G}'$ satisfying efficiency, additivity, symmetry and null player property.

Definition 1.5. Let $S'\in H(S), S\subseteq N$ be given. A set T' is called an restriction[1][2] of S' if $T'\in H(T)$, $T\subseteq S$, and the order of players in T' is in accordance with that in S'. We denote by $R(S')$ the set of all restrictions of S'.

Although Evans's procedure proceeds well on the classical game space $\mathcal{G}$, it is not suitable to characterize a solution on the generalized game space $\mathcal{G}'$. The problem is that, when players are randomly split into two coalitions, there is no order information about the two subcoalitions, so the leader does not know what he actually owns to bargain with his opponent. In Section 2 we will give a generalized procedure, based on Evans's approach, to characterize the generalized Shapley value of form (1.3). In Section 3, the procedure we derived in section 2 is modified to characterize a class of values satisfying efficiency, linearity and symmetry on the generalized game space $\mathcal{G}'$.

2. Generalization of Evans's procedure

Following Evans's procedure, we assume that for a set of fixed players, each player has the same probability to be chosen as a leader in all possible permutations of the

[1] Sanchez and Bergantinos (1997) use the notation $T'=S'/T'$ to express the restriction T' of S'. We change the notation to avoid the possible confusion with the set-minus sign "\".

[2] In order to explain such 'restriction set', we introduce the notion of predecessors and successors. Consider an arbitrary ordered coalition $S'\in\Omega$, $S'=\{i_1,\ldots,i_{k-1},i_k,i_{k+1},\ldots,i_s\}$. For any $k\in\{2,\ldots,s\}$, denote the predecessors of i_k according to S' by $pre(i_k,S')$. For any $k\in\{1,\ldots,s-1\}$, denote the successors of i_k according to S' by $suc(i_k,S')$. Then $pre(i_k,S')=\{i_1,\ldots i_{k-1}\}$ as well as $suc(i_k,S')=\{i_{k+1},\ldots i_s\}$ hold. For any two player $i,j\in T'$ where $T'\in H(T)$, $T\subseteq N$, the restriction set $T'\in R(S')$ means $T\subseteq S$, and if $i\in pre(j,S')$ then $i\in pre(j,T')$, or if $i\in suc(j,S')$ then $i\in suc(j,T')$.

set of players, i.e. for any $S \subseteq N$, $S', S'' \in H(S)$, the probability of i to be chosen as a leader in S' is the same as that in S'' for all $i \in S$. Remind the problem lying in Evans's procedure for the generalized case is the lack of order information for the two partitioned coalitions in (A). In order to fix the orders of the two coalitions in the two-person bargaining process, we first fix one permutation $N' \in H(N)$ with some probability, then a partition $\{S', N' \setminus S'\}$ can be chosen based on N' where $S', N' \setminus S' \in R(N')$, $S' \in H(S)$, $S \subsetneq N$ and $S \neq \emptyset$.

Let $\theta : \mathcal{G}' \to \mathbb{R}^2$ be the payoff of the two-person bargaining process between S' and $N' \setminus S'$, say S' gets $\theta_{S'}^{N'}(v)$ and $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$. According to Evans's procedure, the leader of each ordered coalition is then obliged to pay to each member of his coalition a prespecified feasible allocation $x = (x_i)_{i \in N} \in \mathbb{R}^N$. If i is chosen as the leader of S', then what he gets is

$$\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k.$$

Similarly if j is the leader of $N' \setminus S'$ then he gets

$$\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k.$$

Denote by f the probability distribution that determines the choice of the permutation N', the partition $\{S', N' \setminus S'\}$, and which two players are the leader of S' and $N' \setminus S'$ respectively. Given the triple (f, θ, x), denote by $E_f\,(\mathit{\Pi}_i | \theta, x)$ the expected payoff to player i. We generalize in the following the consistency concept defined by Evans:

Definition 2.1. Given a pair (f, θ), a feasible payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ satisfies Evans's consistency with respect to (f, θ) if $x_i = E_f\,(\mathit{\Pi}_i | \theta, x)$ for $i \in N$.

We assume that the distribution f is uniform. Then the whole procedure under the uniform distribution can be described as follows:

(i) Choose a permutation N' from the set $H(N)$ with probability $1/n!$;
(ii) Choose the size of the first coalition S' with each possible size $\{1, 2, \ldots, n-1\}$ being equally likely, hence with probability $1/(n-1)$. Suppose s is the chosen size;
(iii) Choose an ordered coalition S' of size s in $R(N')$. Since the positions of players in N' are all fixed, we only need to fix s players with probability $1/C_n^s$. Once S' is fixed, its complement $N' \setminus S'$ according to N' is also fixed;
(iv) Choose a leader i from S' (already fixed in (iii)) with probability $1/s$, and a leader j from its complement $N' \setminus S'$ (already fixed in (iii)) with probability $1/(n-s)$;
(v) Leader i and j play a two-person bargaining game based on coalition S' and $N' \setminus S'$ respectively. Coalition S' gets $\theta_{S'}^{N'}(v)$ while $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$;
(vi) Leader i gets $\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k$ after assigning each of his member x_k for all $k \in S' \setminus \{i\}$, meanwhile leader j gets $\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k$ after assigning each of his member x_k for all $k \in N \setminus (S \cup \{j\})$.

According to the above procedure the probability that player i will find himself leader of coalition S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot 2,$$

and the probability of being a follower in S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{s-1}{s} \cdot 2.$$

Player i could be either in the first coalition or in the second one, hence we add the factor 2 in the probability. Now everything is well-defined except for θ in the generalized case.

In contrast to the standard two-person bargaining solution (1.1), we give the following definition:

Definition 2.2. For any two-person generalized game $\langle \{i,j\}, v\rangle$, the generalized standard bargaining solution $\phi : \mathcal{G}' \to \mathbb{R}^2$ is defined by

$$\phi_k(\{i,j\}, v) = v(\{k\}) + \frac{1}{2}\left(\bar{v}(\{i,j\}) - v(\{i\}) - v(\{j\})\right) \quad \text{for } k \in \{i,j\}.$$

Clearly ϕ satisfies the efficiency condition (1.4). Hence the solution θ of the two-person bargaining process between S' and $N' \setminus S'$ in game $\langle N, v\rangle$ is

$$\begin{aligned} \theta_{S'}^{N'}(v) &= v(S') + \frac{1}{2}\left(\bar{v}(N) - v(S') - v(N' \setminus S')\right); \\ \theta_{N'\setminus S'}^{N'}(v) &= v(N' \setminus S') + \frac{1}{2}\left(\bar{v}(N) - v(N' \setminus S') - v(S')\right). \end{aligned} \tag{2.5}$$

Lemma 2.3. *The generalized Shapley value of form (1.3) for any generalized game $\langle N, v\rangle$ is equivalent to:*

$$Sh'_i(N,v) = \sum_{\substack{S' \in \Omega, \\ S' \ni i}} \frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!}\right) \quad \textit{for all } i \in N. \tag{2.6}$$

The proof of this lemma can be found in the appendix.

Theorem 2.4. *A feasible payoff vector $x \in \mathbb{R}^N$ is consistent with (f, θ) for the generalized game $\langle N, v\rangle$ if and only if x is the generalized Shapley value of form (1.3).*

Proof. According to the procedure, player i's expected payoff x_i is

$$x_i = \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'), \\ S' \ni i, |S'| \neq n}} 2 \cdot \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \left[\frac{1}{s}\left(\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k\right) + \frac{s-1}{s} x_i\right]. \tag{2.7}$$

We first show that x satisfies the efficiency condition (1.4):

$$\begin{aligned}\sum_{i\in N} x_i &= \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ s\neq n,0}} \sum_{i\in S} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\left[\frac{1}{s}\left(\theta_{S'}^{N'}(v)-x(S)\right)+x_i\right]\\ &= \sum_{N'\in H(N)} \frac{1}{n!} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ s\neq n,0}} \frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\theta_{S'}^{N'}(v)\\ &= \sum_{N'\in H(N)} \frac{1}{n!}\sum_{s=1}^{n-1} C_n^s\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{2}\cdot\bar{v}(N) = \bar{v}(N).\end{aligned}$$

Note that (2.7) is equivalent to

$$0 = \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\left(\theta_{S'}^{N'}(v)-x(S)\right), \qquad (2.8)$$

since

$$\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s} = \sum_{N'\in H(N)} \sum_{s=1}^{n-1} C_{n-1}^{s-1}\cdot 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s} = 1.$$

We now simplify the formula for x_i given by (2.8). Note that $x(S) = x_i + x(S\setminus\{i\})$. Then the coefficient of x_i on the right hand side of (2.8) is

$$-\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s} = -\sum_{N'\in H(N)} \sum_{s=1}^{n-1} C_{n-1}^{s-1}\cdot 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s} = -\frac{2}{n},$$

while the item concerning $x(S\setminus\{i\})$ on the right hand side of (2.8) is

$$\begin{aligned}&-\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\cdot x(S'\setminus\{i\})\\ &= -\sum_{N'\in H(N)} \sum_{j\in N\setminus\{i\}} x_j \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i,j, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\\ &= -\sum_{N'\in H(N)} \sum_{j\in N\setminus\{i\}} x_j \sum_{s=2}^{n-1} C_{n-2}^{s-2}\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\\ &= -\frac{n-2}{n(n-1)} \sum_{j\in N\setminus\{i\}} x_j = -\frac{n-2}{n(n-1)}\left(\bar{v}(N)-x_i\right).\end{aligned}$$

The latter equation is because of the efficiency of x. The only thing that is not treated yet on the right hand side of (2.8) is

$$\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\cdot\theta_{S'}^{N'}$$

$$= \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S') \tag{2.9}$$

$$- \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(N'\setminus S') \tag{2.10}$$

$$+ \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot \bar{v}(N). \tag{2.11}$$

In is easy to derive that the result of (2.11) is $\bar{v}(N)/n$. By changing the order of summations, (2.9) is equivalent to

$$\sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \sum_{\substack{N'\in H(N),\\ N'\in V(S')}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S') = \sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \frac{n!}{s!}\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S')$$

$$= \frac{1}{n-1} \sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \frac{(s-1)!(n-s)!}{n!}\cdot\frac{1}{s!}\cdot v(S').$$

Let $T' = N'\setminus S'$, then (2.10) is equivalent to

$$- \sum_{N'\in H(N)} \sum_{\substack{T'\in\Omega, T'\in R(N'),\\ T'\not\ni i, |T'|\neq 0}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^{n-t}}\cdot\frac{1}{n-t}\cdot v(T')$$

$$= - \sum_{\substack{T'\in\Omega,\\ T'\not\ni i}} \sum_{\substack{N'\in H(N),\\ N'\in V(T')}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^{n-t}}\cdot\frac{1}{n-t}\cdot v(T')$$

$$= - \frac{1}{n-1} \sum_{\substack{T'\in\Omega,\\ T'\ni i}} \frac{(t-1)!(n-t)!}{n!}\cdot\frac{1}{(t-1)!}\cdot v(T'\setminus\{i\}).$$

In summary we derive that

$$x_i = \sum_{\substack{S'\in\Omega,\\ S'\ni i}} \frac{(s-1)!(n-s)!}{n!}\cdot\left(\frac{v(S')}{s!} - \frac{v(S'\setminus\{i\})}{(s-1)!}\right). \tag{2.12}$$

Then by Lemma 2.3, we have $x = Sh'(N,v)$ for all generalized game $\langle N, v\rangle$.

In fact Theorem 2.4 can be restated as follows, where κ (which was defined above as the "uniform" distribution over two-configuration for the given game $\langle N, v\rangle$) is to be understood now as a function from games to such probability distributions.

Corollary 2.5. *The generalized Shapley value is the unique value on $\mathcal{G}'$ that is both consistent with κ and standard on two-person games.*

3. Consistency to the class of generalized values

Remind in the classical game space, a value ϕ on $\mathcal{G}$ is said to satisfy

(i) efficiency, if $\sum_{i\in N} \phi_i(N,v) = v(N)$ for all $v \in \mathcal{G}_N$;
(ii) linearity, if $\phi(N, a\cdot v + b\cdot w) = a\cdot\phi(N,v) + b\cdot\phi(N,w)$ for all $a,b\in\mathbb{R}$, all $v,w\in\mathcal{G}_N$;
(iii) symmetry, if $\phi_i(N,v) = \phi_j(N,v)$ for all symmetric players i and j in game $v\in\mathcal{G}_N$. Players i and j are called symmetric players in $\langle N,v\rangle$ if $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S\subseteq N\setminus\{i,j\}$.

The ELS value is denoted to the class of values on $\mathcal{G}_N$ satisfying efficiency, linearity and symmetry. Since additivity can be deduced from linearity, and additivity is equivalent to linearity for continuous values, the Shapley value clearly belongs to this class of values. The ELS value was firstly characterized by Ruiz, Valenciano and Zarzuelo (1998). Later Driessen (2002; 2010) gave the following characterization for the ELS value:

Theorem 3.1. *(Driessen and Radzik, 2002; Driessen, 2010) A value Φ on $\mathcal{G}_N$ satisfies the efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants $\mathcal{B} = \{b_s^n \mid n\in\mathbb{N}\setminus\{0,1\}, s=1,2,\ldots,n\}$ with $b_n^n = 1$ such that, for every n-person game $\langle N,v\rangle$ with at least two players,*

$$\Phi_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} p_s^n\cdot\left(b_{s+1}^n\cdot v(S\cup\{i\}) - b_s^n\cdot v(S)\right) \quad \textit{for all } i\in N. \tag{3.13}$$

Whenever $b_s^n = 1$ for all $s\in\{1,2,\ldots,n\}$, the expression on the right hand of (3.13) reduces to the Shapley value payoff of player i in the n-person game $\langle N,v\rangle$ itself. We now generalize the ELS value on the classical game space $\mathcal{G}$ to the generalized game space $\mathcal{G}'$ by considering all possible orders of coalitions.

Theorem 3.2. *There is a unique value $\Phi' : \mathcal{G}'_N \to \mathbb{R}^N$ satisfying the generalized efficiency, linearity and the generalized symmetry, such that for all $v\in\mathcal{G}'_N$ and all $i\in N$,*

$$\Phi'_i(N,v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \left(p_{s-1}^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)} v(S') - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} \left(p_s^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)} v(S). \tag{3.14}$$

The proof of this theorem is postponed to the appendix. Remind the procedure we used in the latter section to characterize the generalized Shapley value on $\mathcal{G}'$. If we change the standard two-person bargaining solution (2.5) by

$$\eta_{S'}^{N'}(v) = b_s^n\cdot v(S') + \frac{1}{2}\left(b_n^n\cdot\bar{v}(N) - b_s^n\cdot v(S') - b_{n-s}^n\cdot v(N'\setminus S')\right);$$
$$\eta_{N'\setminus S'}^{N'}(v) = b_{n-s}^n\cdot v(N'\setminus S') + \frac{1}{2}\left(b_n^n\cdot\bar{v}(N) - b_{n-s}^n\cdot v(N'\setminus S') - b_s^n\cdot v(S')\right),$$

then by a similar arguments as in the proof of Theorem 2.4, we can derive the following result:

Theorem 3.3. *A feasible payoff vector $x\in\mathbb{R}^N$ is consistent with (f,η) for the generalized game $\langle N,v\rangle$ if and only if x is the generalized ELS value of form (3.14).*

4. Conclusions

In this paper all characterizations are done in the generalized game space. The difference compared with the classical game space is that, the order of players entering into the game influences the worth of coalitions. So for a fixed set of players, different permutations of this set may take different worths, which makes the characterization more complicated.

In the classical game space, Evans (1996) introduced an approach, such that the solution of the game determined endogenously as the expected outcome of a reduction of the game to a two-person bargaining problem, is just the Shapley value. However this approach is not suitable for the generalized games. So we modify Evans's approach in the following way: for any generalized game $\langle N, v\rangle$, firstly choose one permutation $N' \in H(N)$, secondly choose two subcoalitions S' and $N' \setminus S'$ according to N', and then choose two leaders from these two subcoalitions separately. The two leaders play a two-person bargaining game and promise to give the left players some part of his earning. We prove if all the choosing processes are under uniform distribution, and the standard solution on two-person games is used, then the expectation under the procedure is the generalized Shapley value. This also means, the generalized Shapley value can be axiomatized by Evans's consistency and the standardness on two-person games.

The class of values satisfying efficiency, linearity and symmetry on the generalized game space is well-defined. By a simple change to the standard two-person bargaining solution, the procedure we used to characterize the generalized Shapley value can be further used to characterize the class of values.

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Appendix A: Proof of Lemma 2.3

Proof. We will show the value defined by (2.6) satisfies additivity, together with efficiency, symmetry, null player property (see Definition 1.4), since Sanchez and Bergantinos (1997) proved the Shapley value in Definition 1.3 is the unique value on $\mathcal{G}'$ satisfying these four properties. Additivity is clear. Denote by ϕ the value defined by (2.6), then

$$\begin{aligned}
\sum_{i\in N}\phi_i(N,v) &= \sum_{i\in N}\sum_{\substack{S'\in\Omega,\\ S'\ni i}}\frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!}-\frac{v(S'\setminus\{i\})}{(s-1)!}\right)\\
&= \sum_{S'\in\Omega}\sum_{i\in S}\frac{(s-1)!(n-s)!}{n!}\frac{v(S')}{s!}-\sum_{\substack{S'\in\Omega,\\ s\neq n}}\sum_{i\notin S}\frac{s!(n-s-1)!}{n!}\frac{v(S')}{s!}\\
&= \sum_{S'\in\Omega}\frac{s!(n-s)!}{n!}\frac{v(S')}{s!}-\sum_{\substack{S'\in\Omega,\\ s\neq n}}\frac{s!(n-s)!}{n!}\frac{v(S')}{s!}=\bar{v}(N).
\end{aligned}$$

This proves the efficiency. Now suppose player i is a null player in $\langle N, v\rangle$, that is, $v(S', i^h) = v(S')$ for all $S' \in \Omega, S' \not\ni i$, $h \in \{1, 2, \ldots, s+1\}$, then we have

$\phi_i(N,v) = 0$ since

$$\sum_{\substack{S' \in \Omega, \\ S' \ni i}} v(S') = \sum_{\substack{S' \in \Omega, \\ S' \ni i}} s \cdot v(S' \setminus \{i\}).$$

In order to explain the above equality, we consider a coalition $S \subseteq N$, $S \ni i$. Fix $S' \backslash \{i\} \in H(S \backslash \{i\})$, then $(S' \backslash \{i\}, i^h)$, $h \in \{1, 2, \ldots, s\}$ results s different $S' \in H(S)$. This proves the null player property. To prove the symmetry, consider two symmetric players $i, j \in N$, $i \neq j$, that is, $v(S', i^h) = v(S', j^h)$ for all $S' \in \Omega, S' \not\ni i, j$, $h \in \{1, 2, \ldots, s+1\}$. We can rewrite (2.6) in the following way:

$$\begin{aligned}
\phi_i(N,v) &= \left(\sum_{\substack{S' \in \Omega, \\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega, \\ S' \ni i, S' \not\ni j}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!} \right) \\
&= \left(\sum_{\substack{S' \in \Omega, \\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega, \\ S' \ni j, S' \not\ni i}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{j\})}{(s-1)!} \right) = \phi_j(N,v).
\end{aligned}$$

This proves the symmetry.

Appendix B: Proof of Theorem 3.2

Proof. Linearity is clear. Suppose the pair $i, j \in N$ are symmetric players. Then $v(S', i^h) = v(S', j^h)$ for all $S' \in \Omega$, $S' \not\ni i, j$, $h \in \{1, 2, \ldots, s+1\}$ gives

$$\sum_{\substack{S \subseteq N, \\ S \ni i, S \not\ni j}} \sum_{S' \in H(S)} v(S') = \sum_{\substack{S \subseteq N, \\ S \ni j, S \not\ni i}} \sum_{S' \in H(S)} v(S').$$

Hence

$$\Phi'_i(N,v) - \Phi'_j(N,v) = \left(\sum_{\substack{S \subseteq N, \\ S \ni i, S \not\ni j}} - \sum_{\substack{S \subseteq N, \\ S \ni j, S \not\ni i}} \right) (p^n_{s-1} + p^n_s) \cdot b^n_s \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') = 0.$$

This proves the generalized symmetry. Next we show that Φ' satisfies the generalized efficiency: for any $v \in \mathcal{G}'$,

$$\begin{aligned}
\sum_{i \in N} \Phi'_i(N,v) &= \sum_{i \in N} \left(\sum_{\substack{S \subseteq N, \\ S \ni i}} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{\substack{S \subseteq N, \\ S \not\ni i}} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \right) \\
&= \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} \left(\sum_{i \in S} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{i \notin S} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \right) \\
&= \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} s \cdot (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} (n-s) \cdot (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \\
&= \frac{1}{n!} \sum_{N' \in H(N)} v(N').
\end{aligned}$$

This completes the sufficient proof. Now we show the uniqueness. Suppose there is another value ϕ on $\mathcal{G}'$ satisfying the generalized efficiency, linearity and the generalize symmetry. With every ordered coalition $T' \in \Omega$, $T' \neq \emptyset$, there is an associated zero-one game $\langle N, e_{T'} \rangle$ defined by $e_{T'}(T') = 1$ and $e_{T'}(S') = 0$ for all $S' \neq T'$, $S' \in \Omega$. Since $v(S') = \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot e_{T'}(S')$ for all $S' \in \Omega$, all $v \in \mathcal{G}'_N$, by linearity we have

$$\phi_i(N, v) = \phi_i(N, \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot e_{T'}) = \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot \phi_i(N, e_{T'}),$$

for all $i \in N$. Next we determine $\phi_i(N, e_{T'})$. Fix the coalition $T' \in \Omega$, by symmetry we know that players in T' as well as players outside T' get the fixed payoff respectively, which only depend on the size of T'. Then by efficiency, (3.14) is derived.

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