# Playability Properties in Games of Deterrence and Evolution in the Replicator Dynamics

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**Abstract** Since the seminal work of John Maynard Smith (1982), a vast literature has developed on evolution analysis through game theoretic tools. Among the most popular evolutionary systems is the Replicator Dynamics, based in its classical version on the combination between a standard non cooperative matrix game and a dynamic system which evolution depends on the payoffs of the interacting species.
Despite its weaknesses, in particular the fact that it does not take into account emergence and development of species that did not initially exist, the Replicator Dynamics has the advantage of proposing a relatively simple model that analyzes and tests some core features of Darwinian evolution.
Nevertheless, the simplicity of the model reaches its limits when one needs to predict accurately the conditions for reaching evolutionary stability. The reason for it is quite obvious: it stems from the possible difficulties to find an analytical solution to the system of equations modelling the Replicator Dynamics.
An alternative approach has been developed, based on matrix games of a different kind, called Games of Deterrence. Matrix Games of Deterrence are qualitative binary games in which selection of strategic pairs results for each player in only two possible outcomes: acceptable (noted 1) and unacceptable (noted 0). It has been shown (Rudnianski, 1991) that each matrix Game of Deterrence can be associated in a one to one relation with a system of equations called the playability system, the solutions of which determine the playability properties of the players' strategies.
Likewise, it has been shown (Ellison and Rudnianski, 2009) that one could derive evolutionary stability properties of the Replicator Dynamics from the solutions of the playability system associated with a symmetric matrix Game of Deterrence on which the Replicator Dynamics is based.
Thus, it has been established that (Ellison and Rudnianski, 2009):

- To each symmetric solution of the playability system corresponds an evolutionarily stable equilibrium set (ESES)
- If a strategy is not playable in every solution of the playability system, the proportion of the corresponding species in the Replicator Dynamics vanishes with time in every solution of the dynamic system

**Keywords:** evolutionary games, Games of Deterrence, playability, Replicator Dynamics, species, strategies.

Based on these results, the proposed paper will first extend the analysis already undertaken and propose new results in terms of relations between the solutions of the Game of Deterrence playability system and the solutions of the dynamic system.

The paper will then provide a method for systematically modelling standard matrix games as Games of Deterrence, allowing the previous results to be extended to any standard matrix game. In particular, in certain situations where the standard methods for analyzing dynamic systems do not work, the above bridging between standard games and Games of Deterrence will enable to determine the systems' asymptotic behaviour.

More precisely, in a first part, after having briefly recalled the definition of the Replicator Dynamics, the paper will recall the definitions and basic properties of Games of Deterrence.

A second part will distinguish between three categories of strategies in the Game of Deterrence under consideration, and will associate specific evolutionary properties with each one.

The third and last part will then develop an algorithm associating a Game of Deterrence with any standard quantitative symmetric matrix game in a way that will enable to generalize the method to the analysis of quantitative evolutionary games.

## 1. Replicator Dynamics and Games of Deterrence

### 1.1. Replicator Dynamics

The Replicator Dynamics is a classical dynamic system describing the evolution of a population broken down into several species. The outcome of the interaction between two individuals is given by a symmetric matrix game $G$.
Moreover, if $\theta = (\theta_1, \theta_2, ..., \theta_n)$ represents the population's profile (i.e. $\theta_i$ is the proportion of species $i$ in the population), then the Replicator Dynamics associated with $G$ is the dynamical system $D(G)$ defined by $\theta_i' = \theta_i(u_i - u_T)$
where:

- $u_i = \sum_k \theta_k u_{ik}$ where $u_{ik}$ represents the payoff of species $i$ when interacting with species $k$
- $u_T = \sum_i \theta_i u_i$

$u_i$ defines the fitness of species $i$, and it then stems straightforwardly from the above system of differential equations that the evolution of the proportion of a species $i$ in the population depends on the relative fitness of $i$ with respect to the average fitness of the entire population.

The above classical representation of the Replicator Dynamics is equivalent to the following:

- Let $\Theta$ be the space of population profiles
- Let $f$ be a vector field on $\Theta$ such that $\theta' = f(\theta)$ with $f_i(\theta) = \theta_i(u_i - u_T)$

An equilibrium of the Replicator Dynamics is then defined as a fixed point of $f$.

In the following, we will always consider that all species are present in the initial state, i.e. $\forall i \in \{1, ..., n\}, \theta_i(0) \neq 0$

### 1.2. Games of Deterrence basic properties

Games of Deterrence consider only two possible states of the world:

- Those which are acceptable for the player under consideration (noted 1)
- Those which are unacceptable for that same player (noted 0)

Given that the players' objective is to be in an acceptable state of the world, Games of Deterrence analyze the strategies' playability.

For the sake of simplicity, in the following we shall only consider matrix games, but the definitions that will be introduced extend straightforwardly to N-player games.
Let $E$ and $R$ be two players with respective strategic sets $S_E$ (card $S_E = n$) and $S_R$ (card $S_R = p$).

We shall consider finite bi-matrix games $(S_E, S_R, U, V)$ in normal form where possible outcomes are taken from the set $\{0, 1\}$. More precisely, for any strategic pair $(i, k) \in S_E \times S_R, u_{ik}$ and $v_{ik}$ define the outcomes for player $E$ and $R$ respectively.

A strategy $i$ of $E$ is said to be *safe* iff $\forall k \in S_R, u_{ik} = 1$.
A non-safe strategy is said to be *dangerous*.

Let $J_E(i)$ be an index called *index of positive playability*, such that:
If $i$ is safe then $J_E(i) = 1$
If not, $J_E(i) = (1 - j_E)(1 - j_R) \prod_{k \in S_R} [1 - J_R(k)(1 - u_{ik})]$
With $j_E = \prod_{i \in S_E} (1 - J_E(i))$ ; and $j_R = \prod_{k \in S_R} (1 - J_R(k))$
If $J_E(i) = 1$, strategy $i \in S_E$ is said to be *positively playable*.
If there are no positively playable strategies in $S_E$, that is if $j_E = 1$, all strategies $i \in S_E$ are said to be *playable by default*.

Similar definitions apply by analogy to strategies $k$ of $S_R$.

A strategy in $S_E \cup S_R$ is *playable* iff it is either positively playable or playable by default.
The system $P$ of all equations of $J_E(i), i \in S_E$, $J_R(k), k \in S_R$, $j_E$ and $j_R$ is called *the playability system* of the game.

$\{0, 1\}^{n+p+2}$ is called the *playability set* of $P$

The playability system $P$ may be considered as a dynamic system $J = \hat{f}(J)$ on the playability set.

A *solution* of the matrix Game of Deterrence is an element of the playability set which is a solution of $P$.

It has been shown in (Rudnianski, 1991) that any matrix Game of Deterrence has at least one solution, and that in the general case, there is no uniqueness of the solution.

Given a strategic pair $(i, k) \in S_E \times S_R$, $i$ is said to be a deterrent strategy vis-à-vis $k$ iff the three following conditions apply:

- $i$ is playable
- $v_{ik} = 0$
- $\exists k' \in S_R : J_R(k') = 1$

It has been shown (Rudnianski, 1991) that a strategy $k \in S_R$ is playable iff there is no strategy $i \in S_E$ deterrent vis-à-vis $k$. Thus, the study of deterrence properties amounts to analyzing the playability properties of the strategies.

A *symmetric Game of Deterrence* is a Game of Deterrence $(S_E, S_R, U, V)$ such that $S_E = S_R$ and $U = V^t$ (i.e. $\forall i, k, u_{ik} = v_{ki}$)

In the case of symmetric games, the strategic set will be noted $S$.
A *symmetric solution* is a solution in which $\forall i \in S, J_E(i) = J_R(i)$

It has been shown (Ellison and Rudnianski, 2009) that in a symmetric Game of Deterrence, $j_E = j_R$

### 1.3. Deterrence and evolution

It has been shown (Ellison and Rudnianski, 2009) that for a symmetric Game of Deterrence $G$ with playability system $P$ and Replicator Dynamics $D(G)$, if:

- $P$ has a symmetric solution for which no strategy is playable by default
- at $t = 0$, the proportion of each positively playable strategy is greater than the sum of the proportions of the non-playable strategies,

then, whatever the initial profile:

- The proportion of each non-playable strategy decreases exponentially towards zero
- The proportion of each playable strategy has a non-zero limit

This result can be interpreted as follows: each symmetric solution of the playability system is associated with an Evolutionarily Stable Equilibrium Set of the Replicator Dynamics, i.e. the union of the attraction basins of the equilibria is a neighbourhood of the equilibrium set.

## 2. Further properties of evolutionary Games of Deterrence

### 2.1. Equivalent strategies and evolution

**Definition 1.** Two strategies $i$ and $j$ are equivalent if $\forall k \in S, u_{ik} = u_{jk}$

**Lemma 1.** *If $i$ and $j$ are equivalent, then:*

- $\frac{\theta_i}{\theta_j}$ *is constant in every solution of the Replicator Dynamics*
- $i$ *and* $j$ *have the same playability in every solution of the playability system*

*Proof.* Since strategies $i$ and $j$ are equivalent, $u_i = u_j$
hence $(ln\frac{\theta_i}{\theta_j})' = (ln\theta_i)' - (ln\theta_j)' = (u_i - u_T) - (u_j - u_T) = 0$

**Definition 2.** Given a subset $X$ of the strategic set $S$, let $i, k \in S$,
$k$ is said to be $X$-dominant vis-à-vis $i$ if $\forall l \in X, u_{il} \leq u_{kl}$.
Likewise, $i$ and $k$ are said to be $X$-equivalent if $i$ is $X$-dominant vis-à-vis $k$ and $k$ is $X$-dominant vis-à-vis $i$.

$X$-dominance is a reflexive and transitive relation.

### 2.2. Categorization of playability system solutions

Let $G$ be a symmetric Game of Deterrence with playability system $P$.
Let $\Psi$ be a function which associates with any given solution $\sigma$ of $P$ a partition $(A, B, C)$ of the strategic set $S$ of $G$ such that:

- $A = \{i \in S|$ $i$ is positively playable for both players$\}$
- $B = \{i \in S|$ $i$ is either positively playable for exactly one player or playable by default for both players$\}$
- $C = \{i \in S|$ $i$ is non-playable for both players$\}$

**Proposition 1.** *If a partition $(A, B, C)$ of $S$ verifies:*

$$\left.\begin{array}{l} i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C) \\ i \in C \Leftrightarrow \exists k \in A : u_{ik} = 0 \end{array}\right\} (C1)$$

*then $(A, B, C) \in Im\Psi$*

*Conversely if $(A, B, C) \in Im\Psi$, then $(A, B, C)$ verifies:*

$$\left.\begin{array}{l} i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C) \\ \exists k \in A : u_{ik} = 0 \Rightarrow i \in C \end{array}\right\} (C2)$$

*Proof.* Let $(A, B, C)$ be a partition of $S$ verifying $(C1)$

-if $A \neq \emptyset$,
Let us consider the following element of the playability set defined by:

- $\forall i \in A, J_E(i) = J_R(i) = 1$
- $\forall i \in B, J_E(i) = 1$ and $J_R(i) = 0$
- $\forall i \in C, J_E(i) = J_R(i) = 0$
- $j_E = j_R = 0$

Let us now verify that this element is a solution of $P$:
It stems from $(C1)$ that:
$\forall i \in A, (1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 1$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 1$
$\forall i \in C, (1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 0$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 0$

It also stems from $(C1)$ that $\forall i \in B, \exists k \in B : u_{ik} = 0$.
Indeed, if $i \in B, i \notin A$ and $i \notin C$, so $\exists k \notin A \cup C : u_{ik} = 0$

Hence $\forall i \in B$,
$(1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 1$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 0$

Also $\prod_{k\in S}(1-J_R(k)) = 0$ and $\prod_{k\in S}(1-J_E(k)) = 0$
The chosen values indeed define a solution $\sigma$ of $P$, and $(A,B,C) = \Psi(\sigma)$

-if $A = \emptyset$,
it stems from the second part of $(C1)$ that $C = \emptyset$
Hence $B = S$

Also, it stems from the first part of $(C1)$ that no strategy in $S$ is safe.
Therefore, there is a solution $\sigma_0$ of $P$ in which all strategies are playable by default, and $(A,B,C) = (\emptyset, S, \emptyset) = \Psi(\sigma_0)$

Let $\tau$ be a solution of $P$ and $(A,B,C) = \Psi(\tau)$,
-if $j_E = j_R = 1$ in $\tau$,
then $(A,B,C) = (\emptyset, S, \emptyset)$
and since no strategy is safe, $\forall i \in S, \exists k \in S : u_{ik} = 0$
Hence $(A,B,C)$ verifies $(C1)$.

-if $j_E = j_R = 0$,
$A = \{i \in S | J_E(i) = J_R(i) = 1\} = \{i \in S | \prod_{k\in S}(1-J_R(k)(1-u_{ik})) = \prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 1\}$
$= \{i \in S | u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0\} = \{i \in S | u_{ik} = 0 \Rightarrow k \in C\}$

similarly, $C = \{i \in S | \exists k \in A : u_{ik} = 0\}$
Hence $(A,B,C)$ verifies $(C1)$.

Let $\sigma$ be a solution of $P$ and $(A,B,C) = \Psi(\sigma)$
let $i \in S$,

$i \in A \Leftrightarrow J_E(i) = J_R(i) = 1$
$\Leftrightarrow i$ is safe or $(1-j_E)(1-j_R)\prod_{k\in S}[1-J_R(k)(1-u_{ik})] = (1-j_E)(1-j_R)\prod_{k\in S}[1-J_E(k)(1-u_{ik})] = 1$

$\Leftrightarrow i$ is safe or $(j_e = j_R = 0$ and $(u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0))$

Yet $i$ is safe $\Rightarrow (j_e = j_R = 0$ and $(u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0))$
so $i \in A \Leftrightarrow (j_e = j_R = 0$ and $(u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0))$
$i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C)$

If $\exists k \in A : u_{ik} = 0$,
then $k$ is deterrent vis-à-vis $i$ for both players.
Hence $i \in C$

### 2.3. Categorization of the solutions of the Replicator Dynamics

Let $G$ be a symmetric Game of Deterrence and $D(G)$ its Replicator Dynamics.
Let $\Gamma$ be a function which associates with any given solution $\sigma$ of $D(G)$ a partition $(A', B', C')$ of the strategic set $S$ of $G$ such that:

- $A' = \{i \in S |\ \theta_i$ does not have a zero limit$\}$
- $B' = \{i \in S |\ lim\ \theta_i = 0$ and $\theta(i)$ is not integrable$\}$
- $C' = \{i \in S |\ \theta_i$ is integrable$\}$

**Proposition 2.** *If a solution $\sigma$ of $D(G)$ verifies $\int_0^\infty 1 - u_T < \infty$, then $(A', B', C') = \Gamma(\sigma)$ verifies:*

$$\left.\begin{array}{l} A' \neq \emptyset \\ i \in A' \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C') \\ \exists k \notin C' : k \textit{ is } (A' \cup B')\textit{-dominant vis-à-vis } i \textit{ and } u_{ik} < u_{kk} \Rightarrow i \in C' \\ \exists k \in C' : k \textit{ is } (A' \cup B')\textit{-dominant vis-à-vis } i \Rightarrow i \in C' \end{array}\right\} (C3)$$

*Proof.* Let $\sigma$ be a solution of $D(G)$ such that $\int_0^\infty 1 - u_T < \infty$,
and let $(A', B', C') = \Gamma(\sigma)$.

$A' \neq \emptyset$ because $\sum_{i \in S} \theta_i = 1$

Let $i \in S$,
$\frac{\theta_i'}{\theta_i} = u_i - u_T = (1 - u_T) - (1 - u_i)$
hence $\theta_i(t) = \theta_i(0) e^{\int_0^t 1 - u_T} e^{-\int_0^t 1 - u_i}$

$\theta_i(0) e^{\int_0^t 1 - u_T}$ has a non-zero finite limit,
and $e^{-\int_0^t 1 - u_i}$ has a finite limit, since it is positive and decreasing
so $\theta_i$ has a limit.
This being true for all $i \in S$, the solution $\sigma$ converges towards an equilibrium.

Also $lim\ \theta_i = 0 \Leftrightarrow lim \int_0^t 1 - u_i = +\infty$
$1 - u_i = 1 - \sum_{k \in S} \theta_k u_{ik} = \sum_{k \in S} \theta_k (1 - u_{ik}) = \sum_{k | u_{ik} = 0} \theta_k$
hence $lim\ \theta_i = 0 \Leftrightarrow \exists k \in S : u_{ik} = 0$ and $\theta_k$ is not integrable
$i \in A' \Leftrightarrow (\forall k \in S, u_{ik} = 0 \Rightarrow k \in C')$

Let $i, k \in S$ such that $k$ is $(A' \cup B')$-dominant vis-à-vis $i$,

let $\theta_{C'} = \sum_{c \in C'} \theta_c$,
By definition of $C'$, $\theta_{C'}$ is integrable.

$u_i - u_k = \sum_{l \in S} \theta_l(u_{il} - u_{kl}) = \sum_{l \in C'} \theta_l(u_{il} - u_{kl}) + \sum_{l \notin C'} \theta_l(u_{il} - u_{kl}) \leq \theta_{C'}$
$\frac{\theta_i}{\theta_k}(t) = \frac{\theta_i}{\theta_k}(0)e^{\int_0^t u_i - u_k} \leq \frac{\theta_i}{\theta_k}(0)e^{\int_0^t \theta_{C'}} \leq \frac{\theta_i}{\theta_k}(0)e^{\int_0^\infty \theta_{C'}} < +\infty$
$\frac{\theta_i}{\theta_k}$ is upper-bounded.
Hence, if $k \in C'$, then $i \in C'$

Now if $k \notin C'$ and $u_{ik} < u_{kk}$,
$u_i - u_k \leq \theta_{C'} + (u_{ik} - u_{kk})\theta_k = \theta_{C'} - \theta_k$
so $(\frac{\theta_i}{\theta_k})' = \frac{\theta_i}{\theta_k}(u_i - u_k) \leq \frac{\theta_i}{\theta_k}(\theta_{C'} - \theta_k) = \frac{\theta_i}{\theta_k}\theta_{C'} - \theta_i$
$0 \leq \frac{\theta_i}{\theta_k}(t) \leq \frac{\theta_i}{\theta_k}(0) + \int_0^t \frac{\theta_i}{\theta_k}\theta_{C'} - \int_0^t \theta_i$
hence $\int_0^t \theta_i \leq \frac{\theta_i}{\theta_k}(0) + \int_0^t \frac{\theta_i}{\theta_k}\theta_{C'}$

Since $\frac{\theta_i}{\theta_k}$ is upper-bounded and $\theta_{C'}$ is integrable, $\frac{\theta_i}{\theta_k}\theta_{C'}$ is integrable
hence $\theta_i$ is integrable, and $i \in C'$

**Corollary 1.** *For any solution $\sigma$ of $D(G)$, let $\Gamma_\sigma : S \to \{A', B', C'\}$ be such that $\forall i \in S, i \in \Gamma_\sigma(i)$ in the partition $\Gamma(\sigma)$. Let us equip the set $\{A', B', C'\}$ with the alphebetical order: $A' \geq B' \geq C'$.*
*Let $(i,k) \in S^2$. If $k$ is $(A' \cup B')$-dominant vis-à-vis $i$, then $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$*
*Also if $i$ and $k$ are $(A' \cup B')$-equivalent, then $\Gamma_\sigma(i) = \Gamma_\sigma(k)$*

*Proof.* Let $k$ be $(A' \cup B')$-dominant vis-à-vis $i$,
If $k \in C'$, then it stems from proposition 2 that $i \in C'$

If $k \in B'$, then $k \notin A'$, hence $\exists l \notin C' : u_{kl} = 0$
and since $u_{il} \leq u_{kl}$, $u_{il} = 0$
whence $i \notin A'$

If $k \in A'$, then $A' \geq \Gamma_\sigma(i)$
Hence $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$

If $i$ and $k$ are $(A' \cup B')$-equivalent,
then $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$ and $\Gamma_\sigma(i) \geq \Gamma_\sigma(k)$
Hence $\Gamma_\sigma(i) = \Gamma_\sigma(k)$

## 3. Bridging binary and quantitative games

In a first part, the present section will proceed to a classical analysis of the Replicator Dynamics associated with an elementary example of 2x2 standard game. In a second part, an alternative approach based on the transformation of the standard game into a Game of Deterrence will be developed. The third part will generalize the new approach, which will be applied in the fourth part to a case which the standard approach cannot solve comprehensively.

### 3.1. Example 1: the standard approach

Let us consider the following symmetric matrix game $G$ in which $0 < a < 1$:

$G$

| | $i$ | $k$ |
|---|---|---|
| $i$ | $(1, 1)$ | $(1, a)$ |
| $k$ | $(a, 1)$ | $(0, 0)$ |

Let $\theta = (\theta_i, \theta_k) \in \Theta$ be the profile of the population.

The average payoffs of the two species are:
$u_i = 1$
$u_k = a\theta_i$
and $u_T = \theta_i + a\theta_i\theta_k$

Hence $\theta' = f(\theta) = (\theta_i(1 - \theta_i - a\theta_i\theta_k), \theta_k(a\theta_i - \theta_i - a\theta_i\theta_k)$

It can be seen by the classical analysis of the Replicator Dynamics that in every solution of $D(G)$, $\theta_k$ decreases exponentially, leading to the equilibrium $\theta = (1, 0)$. Indeed, in this simple example, the classical approach enables to completely determine the trajectories, and the equilibria.

### 3.2. Alternative approach

Let us now introduce the following alternative approach the rationale of which will be justified later.

A possible interpretation of player Column receiving payoff $a$ when the strategic pair $(i, k)$ is selected, is that species $i$ can be divided into two sub-species $i_1$ and $i_2$, such that player Column, when playing species $k$, gets a payoff of 1 against species $i_1$, and 0 against species $i_2$, provided that the proportion in species $i$ of $i_1$ and $i_2$ is given by $(a, 1 - a)$.

This in turn implies that the dynamics associated with $G$ may be considered equivalent to the dynamics of the following game $G'$ when the ratio of the two sub-species equals $\frac{a}{1-a}$.

$G'$

| | $i_1$ | $i_2$ | $k$ |
|---|---|---|---|
| $i_1$ | $(1, 1)$ | $(1, 1)$ | $(1, 1)$ |
| $i_2$ | $(1, 1)$ | $(1, 1)$ | $(1, 0)$ |
| $k$ | $(1, 1)$ | $(0, 1)$ | $(0, 0)$ |

Let $\zeta = (\zeta_{i_1}, \zeta_{i_2}, \zeta_k)$ be the profile of the population.

The average payoffs of the three species are:
$v_{i_1} = 1$
$v_{i_2} = 1$
$v_k = \zeta_{i_1}$
and $v_T = \zeta_{i_1} + \zeta_{i_2} + \zeta_{i_1}\zeta_k$

Hence the Replicator Dynamics $\zeta' = g(\zeta)$ is such that:
$\zeta'_{i_1} = \zeta_{i_1}(1 - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$
$\zeta'_{i_2} = \zeta_{i_2}(1 - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$
$\zeta'_k = \zeta_k(\zeta_{i_1} - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$

As it stems from the matrix of $G'$ that strategies $i_1$ and $i_2$ are equivalent, $\frac{\zeta_{i_1}}{\zeta_{i_2}}$ is constant (lemma 1).

Let $H$ be the subset of the set of profiles of $D(G')$ such that $(1-a)\zeta_{i_1} = a\zeta_{i_2}$.
Since the ratio is constant, $H$ is stable under the dynamics $D(G')$.
Let us then denote by $D_H(G')$ the restriction of $D(G')$ to $H$

let us then define the splitting maps $h$ and $\tilde{h}$ as follows:
$h : \Theta \to H$
$(\theta_i, \theta_k) \mapsto (a\theta_i, (1-a)\theta_i, \theta_k)$

and $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^3$
$(x, y) \mapsto (ax, (1-a)x, y)$

It can be easily seen from the above that $\tilde{h} \circ f = g \circ h$

$h$ generates the breakdown of species $i$ into $i_1$ and $i_2$ on the set of profiles, while $\tilde{h}$ does the same on the tangent space of $\Theta$

This relation translates in terms of flows as follows:
Let $\phi^t_f$ and $\phi^t_g$ be the flows associated with $f$ and $g$.
$h \circ \phi^t_f(\theta) = h(\theta + \int_0^t f(\theta)) = h(\theta) + \int_0^t \tilde{h} \circ f(\theta) = h(\theta) + \int_0^t g \circ h(\theta) = \phi^t_g(h(\theta))$
$h \circ \phi^t_f = \phi^t_g \circ h$

Hence, since $h$ is bijective, $D(G)$ and $D_H(G')$ are topologically conjugate.
In other words, the dynamics of $G$ is equivalent to the dynamics of $G'$ restricted to $H$.

The playability system $P'$ of $G'$ has a unique solution in which strategies $i_1$ and $i_2$ are positively playable while $k$ is not playable for both players. Indeed, strategies $i_1$ and $i_2$ are safe and $i_2$ is deterrent vis-à-vis $k$.

It then stems from (Ellison and Rudnianski, 2009) that whatever the initial profile:
$\zeta_{1_1}$ and $\zeta_{1_2}$ have a non-zero limit
$\zeta_2$ has a zero limit

Since $f$ and $g|_H$ are topologically conjugate, whatever the initial profile $\theta(0)$ in $G$:
$\theta_1$ has a limit equal to 1
$\theta_2$ has a zero-limit

These conclusions match exactly those drawn from the standard approach.

### 3.3. Generalization

Let $\widetilde{G}$ be a standard symmetric matrix game,
Let $M = max\ u_{ik}$ and $m = min\ u_{ik}$
Through replacing all the payoffs $u_{ik}$ by their images via the affinity $x \mapsto \frac{x-m}{M-m}$, we obtain a game $G$ with payoffs comprised between 0 and 1.

It is well known (Weibull, 1995) that the Replicator Dynamics is invariant under positive affine transformation of payoffs. In this case, it is accelerated by a factor $\frac{1}{M-m}$. If $\tilde{f}$ and $f$ denote the vector fields of $D(\widetilde{G})$ and $D(G)$ respectively, the associated flows satisfy the following relation:
$\phi^t_{\tilde{f}} = \phi^{(M-m)t}_f$

**Proposition 3.** *Given a standard symmetric game $G$ with payoffs comprised between 0 and 1, there is a binary symmetric matrix game $G'$ and a subset $H$ of its set of profiles such that the restriction $D_H(G')$ of $D(G')$ to $H$ and $D(G)$ are topologically conjugate.*

*Proof.* This demonstration will use an algorithmic construction of the game $G'$.
Let $G$ be a standard symmetric matrix game with strategic set $S = \{1, ..., n\}$

Let $i \in S$,
let $p = card(\{u_{ki}, k \in S\} \cup \{0, 1\}) - 1$,
let $(a_0, ..., a_p)$ be such that:
$0 = a_0 < a_1 < ... < a_p = 1$ and $\{a_0, ..., a_p\} = \{u_{ki}, k \in S\} \cup \{0, 1\}$

Let $G_i$ be the game obtained from $G$ by replacing strategy $i$ with $p$ equivalent strategies $i_1, ..., i_p$ and by setting the following payoffs:
$v_{kl} = u_{kl}$, for $k, l \in S - \{i\}$
$v_{i_m l} = u_{il}$, for $1 \leq m \leq p, l \in S - \{i\}$
$v_{ki_m} = 1$ if $m \leq r$, where $r$ is such that $u_{ki} = a_r$ ; and $v_{ki_m} = 0$ otherwise, for $k \in S - \{i\}, 1 \leq m \leq p$
$v_{i_m i_{m'}} = 1$ if $m' \leq r$, where $r$ is such that $u_{ii} = a_r$ ; and $v_{i_m i_{m'}} = 0$ otherwise, for $1 \leq m, m' \leq p$

Let $H_i$ be the subset of the set of profiles $\Theta_i$ of $G_i$ defined by the following equations:
$\forall 1 \leq m \leq p, \theta_{i_m} = (a_m - a_{m-1}) \sum\limits_{m'=1}^{p} \theta_{i_{m'}}$

The strategies $i_1, ..., i_p$ are equivalent.
Hence, it stems from lemma 1 that $H_i$ is stable under the dynamics $D(G_i)$

Let $h_i$ be the splitting map:
$h_i : \Theta \to H_i$
$\theta \mapsto (\theta_1, ..., \theta_{i-1}, (a_1 - a_0)\theta_i, ..., (a_p - a_{p-1})\theta_i, \theta_{i+1}, ..., \theta_n)$

and $\tilde{h}_i : \mathbb{R}^n \to \mathbb{R}^{n+p-1}$
$(x_1, ..., x_n) \mapsto (x_1, ..., x_{i-1}, (a_1 - a_0)x_i, ..., (a_p - a_{p-1})x_i, x_{i+1}, ..., x_n)$

Let $\theta \in \Theta$ and $k \in S - \{i\}$,
$v_k(h_i(\theta)) = v_k(\theta_1, ..., \theta_{i-1}, (a_1 - a_0)\theta_i, ..., (a_p - a_{p-1})\theta_i, \theta_{i+1}, ..., \theta_n)$
$= \sum\limits_{l \neq i} \theta_l v_{kl} + \sum\limits_{m=1}^{p} (a_m - a_{m-1})\theta_i v_{ki_m}$
$= \sum\limits_{l \neq i} \theta_l u_{kl} + \sum\limits_{m=1}^{r} (a_m - a_{m-1})\theta_i$ where $r$ is such that $a_r = u_{ki}$
$= \sum\limits_{l \neq i} \theta_l u_{kl} + a_r \theta_i$
$= u_k(\theta)$
hence $\forall k \in S - \{i\}, v_k \circ h_i = u_k$

Similarly, for $k \in \{i_1, ..., i_p\}, v_k \circ h_i = u_i$

hence, by linearity $v_T \circ h_i = u_T$
and if $f$ and $f_i$ denote the vector fields of the Replicator Dynamics of $G$ and $G_i$ respectively,
$\tilde{h}_i \circ f = f_i \circ h_i$

Hence the flows are conjugate via $h_i$, i.e. $h_i \circ \phi_f^t = \phi_{f_i}^t \circ h_i$

And since $h_i$ is a one-to-one correspondance between $\Theta$ and $H_i$,
$D(G)$ and $D_{H_i}(G_i)$ are topologically conjugate via $h_i$.

Also, $\{v_{ki_m}, k \in S - \{i\} \cup \{i_1, ..., i_p\}, 1 \leq m \leq p\} \subset \{0, 1\}$
Hence, the splitting of species $i$ reduces by 1 the number of species which, when selected by one player, may generate a non-binary payoff for the other player, unless strategy $i$ already verifies that property, in which case the algorithm does not modify the game.

Let $G' = G_{1_{2_{\dots n}}}$ be the game obtained from $G$ by successively applying the above transformation for each strategy of $S$,
and let $H$ be the corresponding subset of the set of profiles $\Theta'$ of $G'$,

$G'$ is a binary matrix game and $D(G)$ and $D_H(G')$ are topologically conjugate.

**Consequence:** the asymptotic properties of $G$ can be analyzed through $G'$ and its playability system.

As the algorithm is applied to $G$, each strategy is split into up to $n$ equivalent strategies. Hence, $G'$ may have up to $n^2$ strategies which can be grouped into $n$

sets of equivalent strategies. Now, it is generally useful to reduce the size of the playabily system. In the case of $G'$, the fact that equivalent strategies have the same playability in every solution (cf. lemma 1) allows us to reduce the playability system. Indeed:

**Proposition 4.** *Let $G'$ be a symmetric Game of Deterrence with strategic set $S' = \{1, ..., i-1,$
$i_1, ..., i_p, i+1, ..., n\}$, where $i_1, ..., i_p$ are equivalent strategies, and let $G''$ be the game obtained from $G'$ by replacing strategies $i_1, ..., i_p$ by a strategy $i_0$ and by setting:*
$w_{kl} = v_{kl}, \forall k, l \neq i_1, ..., i_p$
$w_{i_0 k} = v_{i_1 k} \forall k \neq i_1, ..., i_p$
$w_{k i_0} = \prod_{m=1}^{p} v_{k i_m}, \forall k \neq i_1, ..., i_p$
$w_{i_0 i_0} = \prod_{m=1}^{p} v_{i_1 i_m}$
*Let $H' = \{(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p),$
$J_R(i+1), ..., J_R(n), j_E, j_R) | J_E(i_1) = ... = J_E(i_p)$ and $J_R(i_1) = ... = J_R(i_p)\} \subset \{0,1\}^{2n+2p}$,*

*let $P'$ and $P''$ be the playability systems associated with $G'$ and $G''$ respectively,*

*$H'$ is stable under $P'$, and the restriction $P'_{H'}$ of $P'$ to $H'$ is topologically conjugate to $P''$.*

*Proof.* Let $\hat{f} : \{0,1\}^{2n+2p} \rightarrow \{0,1\}^{2n+2p}$ and $\hat{\hat{f}} : \{0,1\}^{2n+2} \rightarrow \{0,1\}^{2n+2}$ be the playability systems $P'$ and $P''$ respectively.

Since, $i_1, ..., i_p$ are equivalent, the components of $\hat{f}$ corresponding to $J_E(i_1), ..., J_E(i_p)$ are equal, as are those corresponding to $J_R(i_1), ..., J_R(i_p)$.
Hence $H'$ is stable under $\hat{f}$. (In fact, $Im\hat{f} \subset H'$.)
So $P'$ can be restricted to $H'$.

Let $h_i : H' \rightarrow \{0,1\}^{2n+2}$ be such that:
$h_i : (J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1),$
$J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R) \mapsto (J_E(1), ..., J_E(i-1), J_E(i_1), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1),$
$J_R(i_1), J_R(i+1), ..., J_R(n), j_E, j_R)$
$h_i$ is a bijection.

In order to prove the topological conjugacy, we must verify that $h_i \circ \hat{f}|_{H'} = \hat{\hat{f}} \circ h_i$

Let $(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1),$
$J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R) \in H'$, let $k \neq i_1, ..., i_p$,
It stems from the construction of $G''$ that $k$ is safe in $G''$ iff it is safe in $G'$
So if $k$ is safe, the components of $h_i \circ \hat{f}|_{H'}$ and $\hat{\hat{f}} \circ h_i$ corresponding to $J_E(k)$ and $J_R(k)$ are all equal to 1.

Similarly, $i_0$ is safe iff $i_1, ..., i_p$ are all safe.
Let us now suppose that strategy $k$ is dangerous.
The component of $\hat{\hat{f}}(J_E(1), ..., J_E(i-1), J_E(i_1), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1),$
$J_R(i+1), ..., J_R(n), j_E, j_R)$ corresponding to $J_E(k)$ is:

$$(1 - j_E)(1 - j_R) \prod_{l \neq i_0} (1 - J_R(l)(1 - w_{kl})) \times (1 - J_R(i_1)(1 - w_{ki_0}))$$
$$= (1 - j_E)(1 - j_R) \prod_{l \neq i_0} (1 - J_R(l)(1 - v_{kl})) \times (1 - J_R(i_1)(1 - \prod_{m=1}^{p} v_{ki_m}))$$
$$= (1 - j_E)(1 - j_R) \prod_{l \neq i_0} (1 - J_R(l)(1 - v_{kl})) \times \prod_{m=1}^{p} 1 - J_R(i_1)(1 - vki_m)$$
$$= (1 - j_E)(1 - j_R) \prod_{l \in S'} (1 - J_R(l)(1 - v_{kl})),$$

which is exactly the same component of $h_i \circ \hat{f}(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p),$
$J_E(i+1), ..., J_E(n),$
$J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R)$

Similarly, all other components match.
Hence $P'_{H'}$ and $P''$ are topologically conjugate.

**Corollary 2.** *Let $G'$ be a symmetric Game of Deterrence with a strategic set containing several subsets of equivalent strategies.*
*Let $G''$ be the game obtained by replacing each subset of equivalent strategies by a single strategy as in proposition 4.*
*Let $H'$ be the subset of the playability set of elements such that any two equivalent strategies have the same playability for both players.*
*Then, using the notations of proposition 4, $P'_{H'}$ and and $P''$ are topologically conjugate.*

*Proof.* The result stems straightforwardly from the application of proposition 4 to each subset of equivalent strategies.

**Remark 1:** Since $Im\hat{f} \subset H'$, all the solutions of the playability system $P'$ are in $H'$, and restricting $P'$ to $H'$ does not reduce the number of solutions. Thus, solving $P''$ is equivalent to solving $P'$.

**Remark 2:** The above simplification of the playability system also works in the case of non symmetric Games of Deterrence, when either player E or player R has equivalent strategies.

**Remark 3:** Let $G$ be a symmetric game with payoffs comprised between 0 and 1, and let $G''$ be the game obtained by first transforming $G$ into $G'$ as in proposition 3, then transforming $G'$ into $G''$ as in proposiiton 4. If $G$ does not have equivalent strategies in its strategic set, then the strategic set of $G''$ contains the same number of strategies as that of $G$. Indeed, each strategy is first replaced by a set of equivalent strategies, which is in turn replaced by a single strategy. If there are equivalent strategies in the strategic set of $G$, we will choose not to regroup those strategies when building $G''$, so as to maintain the number of strategies.

### 3.4. Example 2

Let us consider the following example deriving from the one developped in (Ellison and Rudnianski, 2009), in which individuals may adopt one of three possible behaviours:

- $A$: aggressive
- $D$: defensive
- $N$: neutral

Furthermore, let us assume that:

- when two individuals of the same type interact, the outcome for each one is 1, which means that an aggressive individual will not try to attack another aggressive individual (maybe because of the fear of the outcome)
- a defensive type, when encoutering an aggressive individual, will respond by inflicting damages, represented by a payoff $0 \leq x < 1$ for the aggressor, and will get a 0
- when meeting a defensive or a neutral type, the defensive type does not attack, and the outcome pair is $(1, 1)$
- a neutral type never responds agressively, and receives a payoff $0 \leq y < 1$ when attacked.

$G$

| | $A$ | $D$ | $N$ |
|---|---|---|---|
| $A$ | $(1,1)$ | $(x,0)$ | $(1,y)$ |
| $D$ | $(0,x)$ | $(1,1)$ | $(1,1)$ |
| $N$ | $(y,1)$ | $(1,1)$ | $(1,1)$ |

It has been shown (Ellison and Rudnianski, 2009) that in the extreme case where $x = y = 0$, the profile $(1, 0, 0)$, which corresponds to the whole population being aggressive, is an evolutionarily stable equilibrium, and the set of profiles $\{(0, t, 1 - t), 0 < t < 1\}$, which are not individually evolutionarily stable, is an evolutionarily stable equilibrium set.

Let us now consider the case where $0 < x, y < 1$.

Let $G'$ and $G''$ be the following matrix games:

$G'$

| | $A_1$ | $A_2$ | $D_1$ | $D_2$ | $N$ |
|---|---|---|---|---|---|
| $A_1$ | $(1,1)$ | $(1,1)$ | $(1,0)$ | $(0,0)$ | $(1,1)$ |
| $A_2$ | $(1,1)$ | $(1,1)$ | $(1,0)$ | $(0,0)$ | $(1,0)$ |
| $D_1$ | $(0,1)$ | $(0,1)$ | $(1,1)$ | $(1,1)$ | $(1,1)$ |
| $D_2$ | $(0,0)$ | $(0,0)$ | $(1,1)$ | $(1,1)$ | $(1,1)$ |
| $N$ | $(1,1)$ | $(0,1)$ | $(1,1)$ | $(1,1)$ | $(1,1)$ |

$G''$

| | $A_0$ | $D_0$ | $N$ |
|---|---|---|---|
| $A_0$ | $(1,1)$ | $(0,0)$ | $(1,0)$ |
| $D_0$ | $(0,0)$ | $(1,1)$ | $(1,1)$ |
| $N$ | $(0,1)$ | $(1,1)$ | $(1,1)$ |

Let $H$ be the set of profiles in $G'$ such that $(1-x)\theta_{A_1} = x\theta_{A_2}$ and $(1-y)\theta_{D_1} = y\theta_{D_2}$. By proposition 3, $D(G)$ and $D_H(G')$ are topologically conjugate.

Let $H'$ be the subset of the playability set of $G'$ comprised of elements such that $A_1$ and $A_2$ on one hand, and $D_1$ and $D_2$ on the other hand, have the same playability for both players.
By proposition 4, $P'_{H'}$ and $P''$ are topologically conjugate.

It can be easily seen from the matrix of $G''$ that $P''$ has three solutions:

- $(1,0,0,1,0,0,0,0)$ ($A_0$ is positively playable while $D_0$ and $N$ are not playable for both players)
- $(0,1,1,0,1,1,0,0)$ ($D_0$ and $N$ are positively playable while $A_0$ is not playable for both players)
- $(0,0,0,0,0,0,1,1)$ (all the strategies are playable by default for both players)

Hence, it stems from the topological conjugacy that $P'$ also has three solutions:

- $(1,1,0,0,0,1,1,0,0,0,0,0)$ ($A_1$ and $A_2$ are positively playable while $D_1$, $D_2$ and $N$ are not playable)
- $(0,0,1,1,1,0,0,1,1,1,0,0)$ ($D_1$, $D_2$ and $N$ are positively playable while $A_1$ and $A_2$ are not playable)
- $(0,0,0,0,0,0,0,0,0,0,1,1)$ (all the strategies are playable by default for both players)

The first two of these solutions satisfy the conditions described in section 1.3. Hence $D(G')$ has two evolutionarily stable equilibrium sets:

- $ESES_1 = \{(t, 1-t, 0, 0, 0), 0 < t < 1\}$ where only species $A_1$ and $A_2$ remain
- $ESES_2 = \{(0, 0, t_1t_2, (1-t_1)t_2, 1-t_2), 0 < t_1, t_2 < 1\}$ where only species $D_1$,$D_2$ and $N$ remain

Hence $ESES_1 \cap H$ and $ESES_2 \cap H$ are asymptotically stable equilibrium sets in $D_H(G')$
$ESES_1 \cap H = \{(x, 1-x, 0, 0, 0)\}$ and $ASES_2 \cap H = \{(0, 0, yt_2, (1-y)t_2, 1-t_2), 0 < t_2 < 1\}$

Now $D_H(G')$ is topologically equivalent to $D(G)$,
hence $(1,0,0)$ is an evolutionarily stable equilibium and $\{(0, t, 1-t), 0 < t < 1\}$ is an evolutionarily stable equilibrium set in $D(G)$.

The results previously established for the game $G$ in the case where $x = y = 0$ have been extended to all $0 < x, y < 1$. The bridging between binary and quantitative games allows us to establish asymptotic properties of evolutionary quantitative games via playability properties of associated Games of Deterrence.

Also, if a solution $\sigma$ of $D(G)$ tends towards the equilibrium $(1,0,0)$, then $\theta_D$ and $\theta_N$ decrease exponentially. So $\Gamma(\sigma) = (\{A\}, \emptyset, \{D, N\})$.

And if $\sigma$ tends towards $\{(0, t, 1-t), 0 < t < 1\}$, then $\theta_A$ decreases exponentially. So $\Gamma(\sigma) = (\{D, N\}, \emptyset, \{A\})$.

It can be easily seen from the matrix of $G$ that these two partitions are the only ones which verify condition $(C3)$. In this case, $Im\Gamma$ is exactly the set of partitions of $S$ which verify $(C3)$.

### 3.5. Shortcut

**Proposition 5.** *Let $\widetilde{G}$ be a symmetric matrix game.*
*Let $M$ and $m$ be the maximal and minimal payoffs in $\widetilde{G}$.*
*Let $G$ be the game obtained by applying the affinity $x \mapsto \frac{x-m}{M-m}$ to all the payoffs of $\widetilde{G}$. Let $G'$ be defined as in proposition 3, and $G''$ as in proposition 4.*
*Then $G''$ is the game obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$.*

*Proof.* Using the previous notations ($u_{ik}, v_{ik}$ and $w_{ik}$ represent the payoffs in the games $G, G'$ and $G''$ respectively), we have:

$$w_{ki_0} = \prod_{m=1}^{p} v_{ki_m}, \forall k \neq i_1, ..., i_p$$

$$w_{i_0 i_0} = \prod_{m=1}^{p} v_{i_1 i_m}$$

and:
$v_{ki_m} = 1$ if $m \leq r$, where $r$ is such that $u_{ki} = a_r$ ; and $v_{ki_m} = 0$ otherwise, for $k \in S - \{i\}, 1 \leq m \leq p$
$v_{i_m i_{m'}} = 1$ if $m' \leq r$, where $r$ is such that $u_{ii} = a_r$ ; and $v_{i_m i_{m'}} = 0$ otherwise, for $1 \leq m, m' \leq p$

Hence:
$w_{ki_0} = 1$ if $u_{ki} = 1$ and $w_{ki_0} = 0$ otherwise
$w_{i_0 i_0} = 1$ if $u_{ii} = 1$ and $w_{i_0 i_0} = 0$ otherwise

As payoff 1 in game $G$ is the image of payoff $M$ in game $\widetilde{G}$, it follows that $G''$ is obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$.

**Proposition 6.** *Let $\widetilde{G}$ be a symmetric matrix game, and let $G''$ be the game obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$. Let $\sigma$ be a solution of $D(G)$. If:*

- *the playability system $P''$ of $G''$ has a symmetric solution for which no strategy is playable by default*
- *$\sigma$ is such that at $t = 0$, the proportion of each strategy of $\widetilde{G}$ corresponding to a positively playable strategy in $G''$ is greater than the sum of the proportions of the strategies of $\widetilde{G}$ corresponding to non-playable strategies in $G''$,*

*then:*

- *The proportion of each strategy of $\widetilde{G}$ corresponding to a non-playable strategy in $G''$ decreases exponentially towards zero*

- *The proportion of each strategy of $\widetilde{G}$ corresponding to a playable strategy in $G''$ has a non-zero limit*

*Proof.* Let $M$ and $m$ be the maximal and minimal payoffs in $\widetilde{G}$.
Let $G$ be the game obtained by applying the affinity $x \mapsto \frac{x-m}{M-m}$ to all the payoffs of $\widetilde{G}$. Let $G'$ and $H$ be defined as in proposition 3, and $G''$ and $H'$ as in proposition 4.

$P''$ is topologically conjugate to $P'_{H'}$, so the symmetric solution of $P''$ is conjugate to a solution $\tau$ of $P'$, which is also symmetric.
By applying the result of section 1.3 to $\tau$, we obtain that if at $t = 0$, the proportion of each strategy which is positively playable in $\tau$ is greater than the sum of the proportions of the non-playable strategies, then the proportion of each positively playable strategy has a non-zero limit, and the proportion of each non-playable strategy decreases exponentially towards zero.

Then, the conclusions about $\widetilde{G}$ follow from the topological conjugacy between $D(G)$ and $D_H(G')$ and the invariance by affine transformation linking $D(\widetilde{G})$ and $D(G)$.

## 4. Conclusion

Starting from a symmetric quantitative game $\widetilde{G}$, we have established the following construction:

$$\widetilde{G} \longrightarrow G \longrightarrow G' \longrightarrow G''$$

such that:

- the payoffs of $G$ are comprised between 0 and 1 and $\phi^t_{\tilde{f}} = \phi_f^{(M-m)t}$
- $G'$ is binary and $D_H(G')$ and $D(G)$ are topologically conjugate
- $G''$ has the same size as $\widetilde{G}$ and $P'_{H'}$ and $P''$ are topologically conjugate

Now, $G''$ can be constructed directly from $\widetilde{G}$ without computing $G$ and $G'$.

The results obtained in the previous sections thus enable to:

1. overcome the possible difficulties of solving analytically the Replicator Dynamics
2. establish asymptotic properties of solutions of the Replicator Dynamics associated with any standard symmetric matrix game
3. bridge standard quantitative games with Games of Deterrence, thus paving the way for a treatment of optimality issues through acceptability analysis.

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