Symmetric Core of Cooperative Side Payments Game

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Abstract This paper concerns cooperative side payments games (with transferable utility and discrete) where at least two players are symmetric. The core and symmetric core properties are compared. The problem of symmetric core existence is considered.

Keywords: cooperative TU game, discrete game, core, symmetric core, balancedness.

1. Introduction

In many practical situations some of participants have the identical power (prestige, influence, resources, capitals). They are substitutes in associated cooperative game. Moreover, non-symmetric in the underlying problem agents may become substitutes in corresponding game. Player's status can also changes in the zero-normalization of a game. It seems reasonable to require that symmetric players should receive the same payoff. However, almost no set-valued solution concepts (including the core, core-based solutions, von Neumann-Morgenstern stable sets, the bargaining set) that satisfy the equal treatment property. It is not difficult to provide the examples of cooperative games, where the core allocations assign to symmetric players vastly different payoffs. Even multi-solutions based on a concept of egalitarianism cannot satisfy the equal treatment property (see for instance (Dutta and Ray, 1989)).

The symmetric core is a subset of core satisfying the equal treatment property. This notion has been introduced in (Norde et al., 2002) for TU games with special structure: the airport game, generalized airport game, maintenance cost game, infrastructure cost game. The symmetric core was used to get a minimal collection of conditions that are equivalent to balancedness. In (Hougaard et al., 2001) the symmetric core was used for calculation of Lorenz-solution of a production economy with a landowner and peasants. To the best of our knowledge, the symmetric core was not yet discussed for general TU game.

Next section recalls some definitions and notations. The core and the symmetric core properties are compared in section 3. It will be shown that symmetric core satisfies the most core axioms. The last section is devoted to the problem of symmetric core existence.

2. Preliminaries

A *cooperative game with transferable utility* (TU game) is given as $G_T = (N, \nu)$, where $N = \{1, ..., n\}$, $n \geqslant 2$, $\nu : 2^N \to \mathbf{R}$, $\nu(\emptyset) = 0$. So-called *discrete game* G_D differs from G_T that ν is integer-valued function and players payoffs must be integers (Azamkhuzhaev, 1991). In economic settings, the integer requirement reflects some forms of indivisibility. Both games summarizes the possible outcomes to a coalition by one number, i.e. side payments are allowed. G_T and G_D can be also described as a games with nontransferable utility (NTU games). Let $\mathcal{G}_T^N$ and $\mathcal{G}_D^N$ be the sets of n-person TU and discrete games respectively, $\mathcal{G}^N = \mathcal{G}_T^N \cup \mathcal{G}_D^N$. Denote by $\Omega = 2^N \setminus \{N, \oslash\}$ the family of proper coalitions. Given $x \in \mathbf{R}^N$ and $\emptyset \neq K \subseteq N$: $x(K) = \sum_{i \in K} x_i$, $x(\emptyset) = 0$. The cardinality of coalition $\emptyset \neq K \subseteq N$ is denoted by $|K|$. When there is no ambiguity, we write $\nu(i)$, $K \setminus i$ instead of $\nu(\{i\})$, $K \setminus \{i\}$ and so on.

Two players $i, j \in N$ are called *symmetric* (*substitutes*, *interchangeable*) in a game $G \in \mathcal{G}^N$ if

$$\nu(K \cup i) = \nu(K \cup j) \quad for\ every\ K \in N \setminus \{i, j\}. \tag{1}$$

Player $i \in N$ is *veto player* in a game $G \in \mathcal{G}^N$ if $\nu(K) = 0$ for all $K \not\ni i$. Denote by $veto(G)$ the set of veto players of $G \in \mathcal{G}^N$. A game G_T is called *convex* if $\nu(K) + \nu(H) \leqslant \nu(K \cup H) + \nu(K \cap H)$ for $K, H \subseteq N$. A game G_T is *integer* if $\nu : 2^N \to \mathbf{Z}$, where $\mathbf{Z}$ denotes the set of integer numbers. The operator $\Psi : \mathcal{G}_D^N \to \mathcal{G}_T^N$ will be used to compare TU and discrete game solutions, i.e. $\Psi(G_D)$ is an integer TU game corresponding to G_D.

The *set of feasible payoff vectors* $X^*(G_T)$ and *pre-imputation set* $X(G_T)$ of TU game G_T are defined by

$$X^*(G_T) = \{x \in \mathbf{R}^N | x(N) \leqslant \nu(N)\},\ X(G_T) = \{x \in \mathbf{R}^N | x(N) = \nu(N)\}.$$

The related sets of discrete game G_D are

$$X^*(G_D) = X^*(\Psi(G_D)) \cap \mathbf{Z}^N,\ X(G_D) = X(\Psi(G_D)) \cap \mathbf{Z}^N.$$

For any set $\tilde{\mathcal{G}^N} \subseteq \mathcal{G}^N$ a *set-valued solution* (or multisolution) on $\tilde{\mathcal{G}^N}$ is a mapping $\varphi : \tilde{\mathcal{G}^N} \to\to \mathbf{R}^N$ which assigns to every $G \in \tilde{\mathcal{G}^N}$ a set of payoff vectors $\varphi(G) \subseteq X^*(G)$. Notice that the solution set $\varphi(G)$ is allowed to be empty. A *value* of game G is a function $f : \tilde{\mathcal{G}^N} \to X(G)$. The *core of TU game* and *core of discrete game* are the sets

$$C(G_T) = \{x \in X(G_T) | x(K) \geqslant \nu(K), K \in \Omega\},\ C(G_D) = C(\Psi(G_D)) \cap \mathbf{Z}^N.$$

The formulas to obtain the *CIS-value*, *ENSC-value*, *Shapey value* and *equal division solution* of a game G_T are

$$CIS_i(G_T) = \nu(i) + \frac{\nu(N) - \sum_{j \in N} \nu(j)}{n},$$

$$ENSC_i(G_T) = \nu^*(i) + \frac{\nu(N) - \sum_{j \in N} \nu^*(j)}{n},$$

$$Sh_i(G_T) = \sum_{K \not\ni i} \frac{|K|!(n - |K| - 1)!}{n!}(\nu(K \cup i) - \nu(K)),\ ED_i(G_T) = \frac{\nu(N)}{n},$$

where $i \in N$, $\nu^*(K) = \nu(N) - \nu(N \setminus K)$, $K \subseteq N$. The CIS-value is also called the *equal surplus division solution.* Notice that CIS-value, ENSC-value and equal division solution assign to every player some initial payoff and distribute the remainder of $\nu(N)$ equally among all players. For CIS-value (the center of gravity of imputation set $I(G_T) = \{x \in X(G_T) | x_i \geqslant \nu(i), i \in N\}$) initial payoff to player $i \in N$ is equal to its individual worth $\nu(i)$. For ED-value and ENSC-value the initial payoffs are equal to zero and player's marginal contribution $\nu^*(i)$ to grand coalition N, respectively. Thus, the ENSC-value assigns to any game G_T the CIS-value of dual game (N, ν^*).

3. Symmetric core properties

For a game $G \in \mathcal{G}^N$ denote by $\Im(G)$ the family of coalitions each of which contains only symmetric players

$$\Im(G) = \{K \in 2^N | |K| \geqslant 2,\ every\ i,j \in K,\ i \neq j,\ are\ symmetric\ in\ G\}.$$

Definition 1. A game $G \in \mathcal{G}^N$ is called *semi-symmetric* if at least two players are symmetric in G, i.e. $\Im(G) \neq \emptyset$. A game $G \in \mathcal{G}^N$ is (totally) *symmetric* if $\Im(G) = \{\{N\}\}$. A game $G \in \mathcal{G}^N$ is *non-symmetric* if $\Im(G) = \emptyset$.

Let $\mathcal{SG}^N = \mathcal{SG}^N_T \cup \mathcal{SG}^N_D$ be the set of semi-symmetric games $G \in \mathcal{G}^N$.

Definition 2. The *symmetric core* $SC(G)$ of a game $G \in \mathcal{G}^N$ is the set of core allocations for which the payoffs of symmetric players are equal

$$SC(G) = \{x \in C(G) | x_i = x_j\ for\ all\ i,j \in K,\ i \neq j,\ K \in \Im(G)\}.$$

Example 1. Let $U^H = (N, u^H)$ be n-person ($n \geqslant 3$) *unanimity game* for a coalition $H \in \Omega$: $u^H(K) = 1$ for $K \supseteq H$, $u^H(K) = 0$ otherwise. Since

$$\Im(U^H) = \begin{cases} \{H\} & if\ \ |H| = n-1, \\ \{N \setminus H\} & if\ \ |H| = 1, \\ \{H, N \setminus H\} & else, \end{cases}$$

then the game U^H is semi-symmetric. Well known that any unanimity game is convex and $C(U^H) = \{x \in \mathbf{R}^N | x_i = 0,\ i \in N \setminus H,\ x(H) = 1\}$. Therefore, the symmetric core $SC(U^H)$ consists of one point which is the Shapey value: $SC(U^H) = \{Sh(U^H)\}$, where $Sh_i(U^H) = \frac{1}{|H|}$ for $i \in H$, $Sh_i(U^H) = 0$ otherwise.

Example 2. Consider situation with four investors having the endowments 80, 60, 50, 50 units of money (m.u. for short). Assume the following investment projects are available: a bank deposit that yields 10 interest rate whatever the outlay, two production processes that require an initial investment of 100 ore 200 m.u. and yields 15 ore 20 rate of return, respectively. The related four-person investment game (de Waegenaere et al., 2005) $G_T \in \mathcal{G}^N_T$ is given by

$$\left.\begin{array}{l} N = \{1,2,3,4\}, \quad \nu(N) = 284, \\ \nu(1) = 88, \quad \nu(2) = 66, \quad \nu(3) = \nu(4) = 55, \\ \nu(1,2) = 159, \quad \nu(1,3) = \nu(1,4) = 148, \\ \nu(2,3) = \nu(2,4) = 126, \quad \nu(3,4) = 115, \\ \nu(1,2,3) = \nu(1,2,4) = 214, \quad \nu(1,3,4) = 203, \quad \nu(2,3,4) = 181. \end{array}\right\}$$

We obtain non-convex ($\nu(2,4)+\nu(3,4) > \nu(4)+\nu(2,3,4)$) balanced semi-symmetric game with symmetric players 3 and 4, $\Im(G_T) = \{\{3,4\}\}$. The core of game G_T has 16 extreme points whereas symmetric core is the convex hull of 4 points

$$SC(G_T) = co\{x^1, x^2, x^3, x^4\},$$

$$\begin{array}{ll} x^1 = (100\frac{1}{2}, 68\frac{1}{2}, 57\frac{1}{2}, 57\frac{1}{2}), & x^2 = (90\frac{1}{2}, 78\frac{1}{2}, 57\frac{1}{2}, 57\frac{1}{2}), \\ x^3 = (98, 66, 60, 60), & x^4 = (88, 76, 60, 60). \end{array}$$

Denote by $G_T^0 = (N, \nu^0)$, where

$$\nu^0(K) = \begin{cases} 5 & if \ \ |K| \in \{2,3\}, \\ 20 & if \ \ K = N, \\ 0 & else, \end{cases}$$

the zero-normalization of game G_T. All players are substitutes in G_T^0, $\Im(G_T^0) = \{\{1,2,3,4\}\}$. The symmetric core of game G_T^0 consists of one point

$$SC(G_T^0) = \{x^0\}, \ x^0 = (5,5,5,5) = Sh(G_T^0) = CIS(G_T^0) = ENSC(G_T^0) = ED(G_T^0).$$

The payoff vector x^0 corresponds to symmetric core allocation $x^6 = (93, 71, 60, 60)$ of original game G_T. Notice, that $x^6 = \frac{x^3+x^4}{2}$, it is equal the Shapey value $Sh(G_T)$ of original game, but does not coincide with the barycenter $(94\frac{1}{4}, 72\frac{1}{4}, 58\frac{3}{4}, 58\frac{3}{4})$ of the symmetric core of game G_T.

In game theory literature there exist two (equivalent) versions of *TU game balancedness*: a game $G_T \in \mathcal{G}_T^N$ is called balanced if it has a nonempty core ore if it satisfies the Bondareva-Shapley condition

$$\sum_{K \in \Omega} \lambda_K \nu(K) \leqslant \nu(N), \ \lambda : \Omega \to \mathbf{R}_+, \ \sum_{K \in \Omega, \ K \ni i} \lambda_K = 1, \ i \in N, \tag{2}$$

see (Bondareva, 1963) and (Shapley, 1967). Since (2) is necessary but not sufficient condition for the nonemptiness of core of discrete game, the unified definition is required.

Definition 3. A game $G \in \mathcal{G}^N$ with nonempty core is called *balanced*.

We need the following axiom to be satisfied by solution φ.

Axiom 3.1 *(equal treatment). For all* $G \in \tilde{\mathcal{G}^N}$, *all* $x \in \varphi(G)$ *and every symmetric players* i, j *in* G*:* $x_i = x_j$*.*

Known that $Sh(G_T)$, $CIS(G_T)$, $ENSC(G_T)$ and $ED(G_T)$ satisfy *equal treatment*.

From above definitions it straightforwardly follows that:

- the symmetric core of a game $G \in \mathcal{G}^N$ may be empty;
- the symmetric core of TU game G_T is a convex subset of its core;
- the symmetric core of non-symmetric game $G \in \mathcal{G}^N$ coincides with its core, therefore, apart from their different definitions the real difference is exposed for semi-symmetric balanced games;
- the symmetric core of balanced symmetric TU game consists of one point which is the equal division solution $SC(G_T) = \{ED(G_T)\}$;
- the symmetric core of balanced semi-symmetric TU game contains all core selectors satisfying *equal treatment*, in particular, the nucleolus that realizes a fairness principle based on lexicographic minimization of maximum excess for all coalitions;
- if the Shapley value of semi-symmetric TU game satisfies the core inequalities then it belongs to symmetric core, the Shapley value is always symmetric core allocation on the domain of convex TU games;
- the CIS-value, the ENSC-value, the equal division solution which "have some egalitarian flavour" (Brink and Funaki, 2009) and any convex combination of these

solutions cannot belong to symmetric core of balanced semi-symmetric TU game.

A nonempty core of NTU game (even 3-person) may contains no equal treatment outcomes (Aumann, 1987). The following two propositions show that balancedness of TU game is the necessary and sufficient condition for nonemptiness of its symmetric core, but the same is not true for balanced discrete game.

Proposition 1. *Let $G_T \in \mathcal{SG}_T^N$. Then $SC(G_T) \neq \emptyset$ iff $C(G_T) \neq \emptyset$.*

Proof. If $SC(G_T) \neq \emptyset$ then $C(G_T) \neq \emptyset$ by inclusion $SC(G_T) \subseteq C(G_T)$. Assume now that $C(G_T) \neq \emptyset$ and take $x^1 \in C(G_T)$. If $x^1 \in SC(G_T)$ then $SC(G_T) \neq \emptyset$. Otherwise, there exist a coalition $K \in \Im(G_T)$ and players $i, j \in K$ such that $x_i^1 < x_j^1$. Construct $x^2 \in \mathbf{R}^N$ as follows: $x_i^2 = x_j^1$, $x_j^2 = x_i^1$, $x_l^2 = x_l^1$ for $l \in N \setminus \{i, j\}$. Using (1) we see that $x^2 \in C(G_T)$. By core convexity, $x^3 = \frac{x^1+x^2}{2} \in C(G_T)$. So, we get the core allocation x^3 satisfying $x_i^3 = x_j^3$, $x_l^3 = x_l^1$ for $l \in N \setminus \{i, j\}$. If $x^3 \notin SC(G_T)$ then by repeated application of above procedure one obtains the payoff vector belonging to $SC(G_T)$. □

Proposition 2. *There exist discrete games $G_D \in \mathcal{SG}_D^N$ such that $C(G_D) \neq \emptyset$ but $SC(G_D) = \emptyset$.*

Proof. Consider discrete games G_D^s, defined by set function ν^s on N: $\nu^s(K) \in \{0, 1\}$ for $K \subset N$ and $\nu^s(N) = 1$. The associated TU game $\Psi(G_D^s) = (N, \nu^s)$ is *simple*. Assume $|veto(\Psi(G_D^s))| \geqslant 2$. Then $C(\Psi(G_D^s)) = co\{e^i \in \mathbf{Z}^N | i \in veto(\Psi(G_D^s))\}$ and $C(G_D^s) = \{e^i \in \mathbf{Z}^N | i \in veto(\Psi(G_D^s))$, where $e_j^i = 0$ for $i \neq j$, $e_i^i = 1$. Obviously, veto players are substitutes in games $\Psi(G_D^s)$ and G_D^s. However $x_i \neq x_j$ for all $x \in C(G_D^s)$ and every $(i, j) \in veto(G_D^s)$. Thus $SC(G_D^s) = \emptyset$. □

The core of TU game has been intensely studied and axiomatized. We shall formulate some convenient properties of a solution concept φ on $\tilde{\mathcal{G}^N} \subseteq \mathcal{G}^N$ which has been employed in the well-known core axiomatizations. The axiomatic characterizations of discrete game solutions are not yet provided.

Axiom 3.2 *(efficiency). $x(N) = \nu(N)$ for all $x \in \varphi(G)$ and all $G \in \tilde{\mathcal{G}^N}$.*

Axiom 3.3 *(symmetry). For all $G \in \tilde{\mathcal{G}^N}$ and every symmetric players i, j in G: if $x \in \varphi(G)$ then there exists $y \in \varphi(G)$ such that $x_i = y_j$, $x_j = y_i$ and $x_p = y_p$ for $p \in N \setminus \{i, j\}$.*

Axiom 3.4 *(modularity). For any modular game $G \in \tilde{\mathcal{G}^N}$ generated by the vector $x \in R^N$: $\varphi(G) = \{x\}$.*

Axiom 3.5 *(antimonotonicity). For any pair of games $G^1, G^2 \in \tilde{\mathcal{G}^N}$ defined by set functions ν^1, ν^2 on N such that $\nu^1(N) = \nu^2(N)$ and $\nu^1(K) \leqslant \nu^2(K)$ for all $K \subset N$, it holds that $\varphi(G^2) \subseteq \varphi(G^1)$.*

Axiom 3.6 *(reasonableness (from above)). For all $G \in \tilde{\mathcal{G}^N}$, all $x \in \varphi(G)$ and every $i \in N$: $x_i \leqslant \max\limits_{K \subseteq N \setminus i} \{\nu(K \cup i) - \nu(K)\}$.*

Axiom 3.7 *(covariance). For any pair of games $G^1, G^2 \in \tilde{\mathcal{G}^N}$ defined by set functions ν^1, ν^2 such that $\nu^2 = \alpha\nu^1 + \beta$ for some α >0 and some $\beta \in R^N$ it holds that $\varphi(G^2) = \alpha\varphi(G^1) + \beta$.*

Axiom 3.8 *(projection consistency (or reduced game property)). Let* $G \in \tilde{\mathcal{G}^N}$, $\emptyset \neq H \subset N$ *and* $x \in \varphi(G)$, *then* $R_x^H = (H, r_x^H) \in \tilde{\mathcal{G}^H}$ *and* $x_H \in \varphi(R_x^H)$, *where* $x_H = (x_i)_{i \in H} \in \mathbf{R}^H$ *and*

$$r_x^H(K) = \begin{cases} 0 & if\ K = \emptyset, \\ \nu(K) & if\ \emptyset \neq K \subset H, \\ \nu(N) - x(N \setminus H) & if\ K = H, \end{cases}$$

is the projected reduced game with respect to H *and* x.

Known (Llerena and Carles, 2005) that the core is the only solution on $\mathcal{G}_T^N$ satisfying *projection consistency*, *reasonableness (from above)*, *antimonotonicity* and *modularity*. Notice that *projection consistency* is one of the fundamental principle used in this field. By summarizing the statements formulated above we can say that the symmetric core of balanced semi-symmetric TU and discrete games satisfies *equal treatment*, *efficiency*, *symmetry*, *modularity*, *reasonableness (from above)* and many other core axioms based on only the original game. Theorem 1 (below) shows that for the class of balanced semi-symmetric games the symmetric core is in conflict with *antimonotonicity*, *covariance* and *projection consistency*. All these properties involve the pairs of games.

Lemma 1. *Let* $G \in \mathcal{SG}^N$ *is a balanced game and* G^0 *is its zero-normalization. Then* $G^0 \in \mathcal{SG}^N$, $SC(G^0) \subseteq SC(G)$ *and there exist games* $G \in \mathcal{SG}^N$ *such that* $SC(G^0) \neq SC(G)$.

Proof. The zero-normalization G^0 of any game $G \in \mathcal{G}^N$ is uniquely determined by set function ν^0 on N, where

$$\nu^0(K) = \nu(K) - \sum_{l \in K} \nu(l), \quad \emptyset \neq K \subseteq N. \tag{3}$$

Obviously, $G^0 \in \mathcal{SG}^N$. Let $i, j \in N$, $i \neq j$, are symmetric players in G. The formulas (1) and (3) imply that $\nu^0(K \cup i) = \nu^0(K \cup j)$ for all $K \subseteq N \setminus \{i, j\}$. Thus, symmetric players in G remain symmetric in G^0. Example 2 shows that non-symmetric in G players can become symmetric in G^0. If $G = G_T$ then a linear system defining $SC(G_T^0)$ contains the one for $SC(G_T)$ and, perhaps, additional equality constraints. So $SC(G_T^0) \subseteq SC(G_T)$. In view of Example 2 this inclusion can be strict. For discrete game $G = G_D$ the final part of lemma is proved analogously. □

Theorem 1. *Let* $G \in \mathcal{SG}^N$ *is a balanced game. Then* $SC(G)$ *does not satisfy*
(i) Axiom 3.5;
(ii) Axiom 3.7 even for $\alpha = 1$ *and* $\beta = (\nu(1), ..., \nu(n))$*;*
(iii) Axiom 3.8.

Proof. (i) Consider two balanced four-person TU games G_T^1, G_T^2 defined by set functions ν^1, ν^2 such that

$$\nu^1(K) = \begin{cases} 2 & |K| = 2, \\ 4 & |K| = 3, \\ 6 & K = N, \\ 0 & else, \end{cases} \qquad \nu^2(K) = \begin{cases} \nu^1(K) + 1 = 5, & K = \{1, 3, 4\}, \\ \nu^1(K), & else. \end{cases}$$

The games G_T^1 and G_T^2 are symmetric and semi-symmetric, respectively. $\Im(G_T^1) = \{\{1,2,3,4\}\}$, $\Im(G_T^2) = \{\{3,4\}\}$, $\nu^1(N) = \nu^2(N)$ and $\nu^1(K) \leqslant \nu^2(K)$ for all $K \subset N$. It holds that

$$SC(G_T^2) = co\{(2,1,1\frac{1}{2},1\frac{1}{2}),(2,0,2,2),(1,1,2,2)\} \not\subset SC(G_T^1) = \{(1\frac{1}{2},1\frac{1}{2},1\frac{1}{2},1\frac{1}{2})\}.$$

Consider now discrete games G_D^1, G_D^2 corresponding to given TU games. We have

$$SC(G_D^2) = \{(2,0,2,2),(1,1,2,2)\} \not\subset SC(G_T^1) = \emptyset.$$

Thus, *antimonotonicity* is violated by $SC(G)$.

(ii) This statement follows from lemma 1.

(iii) In four-person TU game G_T defined by

$$\left.\begin{array}{l} N = \{1,2,3,4\}, \quad \nu(N) = 8, \quad \nu(i) = 0, \quad i \in N, \\ \nu(1,2) = \nu(1,3) = \nu(1,4) = \nu(2,3) = \nu(2,4) = 2, \quad \nu(3,4) = 3, \\ \nu(1,2,3) = \nu(1,2,4) = 6, \quad \nu(1,3,4) = 5, \quad \nu(2,3,4) = 4 \end{array}\right\}$$

players 3 and 4 are symmetric, $\Im(G_T) = \{\{3,4\}\}$. The symmetric core is the convex hull of four points $SC(G_T) = co\{x^1,x^2,x^3,x^4\}$, where $x^1 = (4,0,2,2)$, $x^2 = (4,1,1\frac{1}{2},1\frac{1}{2})$, $x^3 = (1,3,2,2)$ and $x^4 = (2,3,1\frac{1}{2},1\frac{1}{2})$. The projected reduced game $R_{x^2}^H = (H, r_{x^2}^H)$ relative to $H = \{1,2,3\}$ at x^2 is defined by:

$r_{x^2}^H(1,2,3) = 6\frac{1}{2}$, $\quad r_{x^2}^H(i) = 0$, $i \in H$, $\quad r_{x^2}^H(1,2) = r_{x^2}^H(1,3) = r_{x^2}^H(2,3) = 2$.

The reduced game is symmetric. Its symmetric core consists of one point $(2\frac{1}{6},2\frac{1}{6},2\frac{1}{6})$. The restriction of x^2 to H, $x_H^2 = (4,1,1\frac{1}{2})$, does not belong to the symmetric core of reduced game. For discrete game G_D corresponding to last TU game G_T we have $SC(G_D) = \{x^1,x^3,x^5,x^6\}$, where $x^5 = (3,1,2,2)$, $x^6 = (2,2,2,2)$. The projected reduced game $R_{x^1}^H$ relative to $H = \{1,2,3\}$ at x^1 is defined by:

$r_{x^1}^H(1,2,3) = 6$, $\quad r_{x^1}^H(i) = 0$, $i \in H$, $\quad r_{x^1}^H(1,2) = r_{x^1}^H(1,3) = r_{x^1}^H(2,3) = 2$.

Since the reduced game is symmetric $SC(R_{x^1}^H) = \{(2,2,2)\}$. The restriction of x^1 to H, $x_H^1 = (4,0,2)$, does not belong to $SC(R_{x^1}^H)$. So $SC(G)$ does not provide *projection consistency*. □

It has been interesting to study the interrelation between the symmetric core of a game $G \in \mathcal{SG}^N$ and strongly egalitarian core allocations.

Definition 4. Let $G \in \mathcal{G}^N$, $x \in C(G)$ and $\overline{x} \in \mathbf{R}^N$ is obtained from x by permuting its coordinates in a non-decreasing order: $\overline{x}_1 \leqslant \overline{x}_2 \leqslant ... \leqslant \overline{x}_n$. A core allocation x is *Lorenz allocation* (*Lorenz maximal*, *strongly egalitarian*) iff it is undominated in the sense of Lorenz, i.e. there does not exist $y \in C(G)$ such that $\sum_{i=1}^p \overline{y}_i \geqslant \sum_{i=1}^p \overline{x}_i$ for all $p \in \{1,...,n-1\}$ with at least one strict inequality.

For a game $G \in \mathcal{G}^N$ denote by $LA(G)$ the set of its Lorenz allocations.

Example 3. Consider balanced four-player TU game G_T defined by

$$\nu(K) = \begin{cases} 7 & if \quad (K = \{1,2\}) \vee (K = \{1,3\}), \\ 12 & if \quad K = N, \\ 0 & else. \end{cases}$$

In was proved (Arin et al., 2008) that the set of Lorenz allocations is of the form

$$LA(G_T) = \{x \in C(G_T)|\ x = (7-\mu,\mu,\mu,5-\mu),\ 2\frac{1}{2} \leqslant \mu \leqslant 3\frac{1)}{2}\}.$$

Taking $\mu = 3\frac{1}{2}$, $\mu = 2\frac{1}{2}$ and $\mu = 3$ yield the lexmax solution $Lmax(G_T) = (3\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}, 1\frac{1}{2})$, the lexmin solution $Lmin(G_T) = (4\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2})$ and least squares solution $LS(G_T) = (4, 3, 3, 2)$, respectively ((Arin et al., 2008, p.571)). By the formulas in section 2 one obtains $Sh(G_T) = (4\frac{1}{6}, 3, 3, 1\frac{5}{6}) \notin LA(G_T)$, $CIS(G_T) = ENSC(G_T) = ED(G_T) = (3, 3, 3, 3) \notin LA(G_T)$.

The next theorem states that the symmetric core of balanced semi-symmetric TU game contains all Lorenz allocations. Besides, $SC(G_T)$ is externally stabile with respect to Lorenz domination, but internal stability does not hold.

Theorem 2. *Let $G_T \in \mathcal{SG}_T^N$ is a balanced game. Then*
(i) $LA(G_T) \subseteq SC(G_T)$ and the inclusion can be strict;
(ii) $SC(G_T)$ Lorenz dominates every other core allocation.

Proof. (i) $LA(G_T)$ satisfies *equal treatment* and $LA(G_T) \subseteq C(G_T)$. Therefore, $LA(G_T) \subseteq SC(G_T)$. The four-person TU game in Example 3 is semi-symmetric $\Im(G_T) = \{\{2, 3\}\}$,

$$LA(G_T) = co\{(3\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}, 1\frac{1}{2}), (4\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2})\}$$
$$\subset SC(G_T) = co\{(2, 5, 5, 0), (7, 0, 0, 5), (12, 0, 0, 0)\}.$$

(ii) If $C(G_T) = SC(G_T)$ then the statement is straightforward. Let $C(G_T) \neq SC(G_T)$ and take $x^0 \in C(G_T)\backslash SC(G_T)$. Then there exists $K \in \Im(G_T)$ and $i, j \in K$ such that $x_j^0 > x_i^0$. By *symmetry* there is $y \in C(G_T)$ with $y_i = x_j^0$, $y_j = x_i^0$, $y_l = x_l^0$ for $l \in N \setminus \{i, j\}$. Consider $x^1 = \frac{x^0+y}{2}$. By core convexity $x^1 \in C(G_T)$. Vector x^1 Lorenz dominates x^0 ($x^1 \succ_L x^0$) since $x_j^1 = x_i^1 = x_j^0 - \delta = x_i^0 + \delta$, $x_l^1 = x_l^0$ for $l \in N\backslash\{i, j\}$, $\delta > 0$. Repetition of this procedure gets the sequence $x^0, x^1, ..., x^p$ core allocations, where $x^k \succ_L x^{k-1}$ for all $k \in \{1, ..., p\}$, $x^0 \notin SC(G_T)$, $x^p \in SC(G_T)$. The transitive property of Lorenz domination completes the proof. □

4. Existence conditions

The balancedness condition (2) is derived by means of dual linear programming problems associated with a game $G_T \in \mathcal{G}_T^N$

$$f(x) = \sum_{i \in N} x_i \to min, \qquad \sum_{i \in K} x_i \geqslant \nu(K), \quad K \in \Omega, \tag{4}$$

$$g(\lambda) = \sum_{K \in \Omega} \nu(K)\lambda_K \to max, \quad \sum_{K \in \Omega, i \in K} \lambda_K = 1, \quad i \in N, \ \lambda \in \mathbf{R}_+^{2^n-2}. \tag{5}$$

The condition (2) can be as well written as

$$\sum_{K \in \Omega} \lambda_K \nu(K) \leqslant \nu(N), \quad \lambda \in ext(M^n),$$

where $ext(M^n)$ is the set of extreme points of problem (5) constraint set M^n. The number of extreme points and their explicit representation known only for small n

$$|ext(M^3)| = 5, \quad |ext(M^4)| = 41, \quad |ext(M^5)| = 1291, \quad |ext(M^6)| = 200213.$$

We concentrate now on n-person non-negative semi-symmetric TU games in zero-normal form $(\mathcal{SG}_T^N)_+^0$. The following example illustrates how the problem (4) is modified by replacing the core by symmetric core.

Example 4. Consider two four-person games $(G_T^0)^1, (G_T^0)^2 \in (\mathcal{SG}_T^N)_+^0$ with two and three symmetric players, $\Im((G_T^0)^1) = \{\{3,4\}\}$, $\Im((G_T^0)^2) = \{\{2,3,4\}\}$. The explicit representations of (4) and modified problems given in table 1. It is remarkable that the number of extreme points of modified dual problems constraint sets M_s^4, where s is the number of symmetric players, decreases as s increases: $|ext(M_2^4)| = 21$, $|ext(M_3^4)| = 6$.

Table 1.

Original problem	Modified problem 1, $\Im((G_T^0)^1) = \{\{3,4\}\}$	Modified problem 2, $\Im((G_T^0)^2) = \{\{2,3,4\}\}$
$f(x) = x_1 + x_2 + x_3 + x_4 \to min$	$f(x) = x_1 + x_2 + 2x_3 \to min$	$f(x) = x_1 + 3x_2 \to min$
$x_i \geqslant 0,\ i \in \{1,2,3,4\}$	$x_i \geqslant 0,\ i \in \{1,2,3\}$	$x_i \geqslant 0,\ i \in \{1,2\}$
$x_1 + x_2 \geqslant \nu(1,2)$	$x_1 + x_2 \geqslant \nu(1,2)$	$x_1 + x_2 \geqslant \nu(1,2)$
$x_1 + x_3 \geqslant \nu(1,3)$	$x_1 + x_3 \geqslant \nu(1,3)$	
$x_1 + x_4 \geqslant \nu(1,4)$		
$x_2 + x_3 \geqslant \nu(2,3)$	$x_2 + x_3 \geqslant \nu(2,3)$	$2x_2 \geqslant \nu(2,3)$
$x_2 + x_4 \geqslant \nu(2,4)$		
$x_3 + x_4 \geqslant \nu(3,4)$	$2x_3 \geqslant \nu(3,4)$	
$x_1 + x_2 + x_3 \geqslant \nu(1,2,3)$	$x_1 + x_2 + x_3 \geqslant \nu(1,2,3)$	$x_1 + 2x_2 \geqslant \nu(1,2,3)$
$x_1 + x_2 + x_4 \geqslant \nu(1,2,4)$		
$x_1 + x_3 + x_4 \geqslant \nu(1,3,4)$	$x_1 + 2x_3 \geqslant \nu(1,3,4)$	
$x_2 + x_3 + x_4 \geqslant \nu(2,3,4)$	$x_2 + 2x_3 \geqslant \nu(2,3,4)$	$3x_2 \geqslant \nu(2,3,4)$

The symmetry of all players makes a game especially easy to handle. The criterion for existence of its core (and, by Proposition 1, for symmetric core too) contains $(n-1)$ inequalities only

$$\frac{\nu(K)}{|K|} \leqslant \frac{\nu(N)}{n} \quad for\ all\ K \in \Omega.$$

It is then natural to focus the attention on games with $(n-1)$ symmetric players. Notice that any such game is determined by $2(n-2)$ numbers $\nu(K)$, $K \in \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{\{2,3\},\{2,3,4\},...,\{2,...,n\}\}, \quad \Omega_2 = \{\{1,2\},\{1,2,3\},...,\{1,...,n-1\}\}.$$

A few of their applications:

- market with one seller and symmetric buyers;
- games with a landlord and landless workers;
- weighted majority game with one large party and $(n-1)$ equal sized smaller parties;
- patent licensing game with the firms each producing an identical commodity and a licensor of a patented technology (Watanabe and Muto, 2008);
- subclass of games related information collecting situations under uncertainty (Branzei et al., 2000) where an action taker can obtain more information from other agents;
- big boss games (Muto et al.,1988) with symmetric powerless players.

The characterization of such games and the sufficient conditions under which the symmetric core is a singleton have been provided in (Zinchenko, 2012). Let

$G_T^0 \in (\mathcal{SG}_T^N)_+^0$, $\Im(G_T^0) = \{\{2,...,n\}\}$ and $n \geqslant 3$. The symmetric core of game G_T^0 is nonempty iff the system

$$\nu^0(T) + \frac{n-|T|}{|H|}\nu^0(H) \leqslant \nu^0(N),\ \frac{n-1}{|H|}\nu^0(H) \leqslant \nu^0(N),\ H \in \Omega_1,\ T \in \Omega_2$$

is consistent. Notice that system consists of $(n-1)(n-2)$ inequalities. If $G_T^0 \in (\mathcal{SG}_T^N)_+^0$ is a balanced game, $\Im(G_T^0) = \{\{2,...,n\}\}$, $n \geqslant 4$ and ν^0 satisfies at least one of three equalities

$$\frac{n-1}{n-2}\nu^0(N \setminus \{1,n\}) = \nu^0(N),\quad \frac{n-2}{n-1}\nu^0(N \setminus 1) + \nu^0(1,2) = \nu^0(N),$$

$$\frac{\nu^0(N \setminus 1)}{n-1} + \nu^0(N \setminus n) = \nu^0(N)$$

then $SC(G_T^0)$ consists of a unique allocation.

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