Subgame Consistent Cooperative Solution of Stochastic Dynamic Game of Public Goods Provision ⋆

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Abstract The provision of public goods constitutes a classic case of market failure which calls for cooperative optimization. However, cooperation cannot be sustainable unless there is guarantee that the agreed-upon optimality principle can be maintained throughout the planning duration. This paper derives subgame consistent cooperative solutions for public goods provision by asymmetric agents with transferable payoffs in a stochastic discrete-time dynamic game framework. This is the first time that dynamic cooperative game in public goods provision is analysed.

Keywords: Public goods, stochastic dynamic games, dynamic cooperation, subgame consistency.

1. Introduction

Public goods, which are non-rival and non-excludable in consumption, are not uncommon in today's economy. Examples of public goods include clean environment, national security, scientific knowledge, accessible public capital, technical know-how and public information. The non-exclusiveness and positive externalities of public goods constitutes major factors for market failure in their provision. In many contexts, the provision and use of public goods are carried out in an intertemporal discrete time-period framework under uncertainty. Cooperation suggests the possibility of socially optimal solutions in public goods provision problem. A discrete-time game framework is developed for theoretical analysis and practical applications. Problems concerning private provision of public goods are studied in Bergstrom (1986). Static analysis on provision of public goods are found in Chamberlin (1974), McGuire (1974) and Gradstein and Nitzan (1989). In many contexts, the provision and use of public goods are carried out in an intertemporal framework. Fershtman and Nitzan (1991) and Wirl (1996) considered differential games of public goods provision with symmetric agents. Wang and Ewald (2010) introduced stochastic elements into these games. Dockner et al. (2000) presented a game model with two asymmetric agents in which knowledge is a public good. These studies on dynamic game analysis focus on the noncooperative equilibria and the collusive solution that maximizes the joint payoffs of all agents.

In dynamic cooperation, the solution scheme would offer a long-term solution only if there is guarantee that participants will always be better off throughout the entire cooperation duration and the agreed-upon optimality principle be

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maintained from the beginning to the end. To enable a cooperation scheme to be sustainable throughout the agreement period, a stringent condition is needed – that of subgame consistency. This condition requires that the optimality principle agreed upon at the outset must remain effective in any subgame starting at a later starting time with a realizable state brought about by prior optimal behaviour. Hence the players do not possess incentives to deviate from the cooperative scheme throughout the cooperative duration. The notion of subgame consistency in stochastic cooperative differential games was originated in Yeung and Petrosyan (2004) in which a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" (PDP) leading to subgame-consistent solutions has been developed. A discrete time version of the analysis is provided in Yeung and Petrosyan (2010). Yeung and Petrosyan (2013) presented subgame consistent cooperative solutions for public goods provision by asymmetric agents with transferable payoffs in a continuous-time stochastic differential game framework.

In this paper, an analytical framework entailing the essential features of public goods provision in a discrete-time stochastic dynamic paradigm is set up. The noncooperative game outcome is characterized and dynamic cooperation is considered. Group optimal strategies are derived and subgame consistent solutions are characterized. A ?payoff distribution procedure? leading to subgame-consistent solutions is derived. Illustrative examples are presented to demonstrate the derivation of subgame consistent solution for public goods provision game.

The paper is organized as follows. Section 2 provides the analytical framework and the non-cooperative outcome of public goods provision in a discrete-time stochastic dynamic framework. Details of a subgame consistent cooperative scheme are presented in Section 3. Illustrative examples are given in Section 4. Section 6 concludes the paper.

2. Analytical Framework and Non-cooperative Outcome

Consider the case of the provision of a public good in which a group of n agents carry out a project by making continuous contributions of some inputs or investments to build up a productive stock of a public good. The game horizon consists of T stages. We use K_t denote the level of the productive stock and I_t^i denote the contribution to the public capital or investment by agent i at stage $t \in \{1, 2, \cdots, T\}$. The stock accumulation dynamics is then

$$K_{t+1} = \sum_{j=1}^{n} I_t^j - \delta K_t + \vartheta_t, K_1 = K^0, \text{for } t \in \{1, 2, \cdots, T\}, \tag{2.1}$$

where ϑ_t is a sequence of statistically independent random variables and δ is the depreciation rate.

The payoff of agent i at stage t is

$$R^i(K_t) - C^i(I_t^i), i \in \{1, 2, \cdots, n\} = N, \tag{2.2}$$

where $R^i(K_t)$ is the revenue/payoff to agent i, $C^i(I_t^i)$ is the cost of investing $I_t^i \in X^i$.

The objective of agent $i \in N$ is to maximize its expected net revenue over the planning horizon, that is

$$E_{\vartheta_1, \vartheta_2, \cdots, \vartheta_T} \left\{ \sum_{s=1}^{T} [R^i(K_s) - C^i(I_s^i)](1+r)^{-(s-1)} + q^i(K_{T+1})(1+r)^{-T} \right\} \tag{2.3}$$

subject to the stock accumulation dynamics (2.1),

where r is the discount rate, and $q^i(K_{T+1}) \geqslant 0$ is an amount conditional on the productive stock that agent i would received at stage T.

Acting for individual interests, the agents are involved in a stochastic dynamic game. In such a framework, a feedback Nash equilibrium has to be sought. Let $\{\phi_s^i(K) \in I_s^i$, for $i \in N$ and $s \in \{1, 2, \cdots, T\}\}$ denote a set of feedback strategies that brings about a feedback Nash equilibrium of the game (2.1) and (2.3). Invoking the standard techniques for solving stochastic dynamic games, a feedback solution to the problem (2.1) and (2.3) can characterized by the following set of discrete-time Hamilton-Jacobi-Bellman equations (see Basar and Olsder 1995; Yeung and Petrosyan 2012):

$$V^i(t, K) = \max_{I_t^i} E_{\vartheta_t} \Big\{ \ [R^i(K) - C^i(I_t^i)](1+r)^{-(t-1)}$$

$$+V^i \Big[\ t+1, \sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_t^j(K) + I_t^i - \delta K + \vartheta_t \ \Big] \ \Big\}, \text{for } t \in \{1, 2, \cdots, T\}, \tag{2.4}$$

$$V^i(T+1, K) = q^i(K_{T+1})(1+r)^{-T}, \text{for } i \in N. \tag{2.5}$$

A Nash equilibrium non-cooperative outcome of public goods provision by the n agents is characterized by the solution of the system of equations $(2.4)-(2.5)$.

3. Subgame Consistent Cooperative Scheme

It is well-known problem that noncooperative provision of goods with externalities, in general, would lead to dynamic inefficiency. Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. Now consider the case when the agents agree to cooperate and extract gains from cooperation. In particular, they act cooperatively and agree to distribute the joint payoff among themselves according to an optimality principle. If any agent disagrees and deviates from the cooperation scheme, all agents will revert to the noncooperative framework to counteract the free-rider problem in public goods provision. In particular, free-riding would lead to a lower future payoff due to the loss of cooperative gains. Thus a credible threat is in place. In particular, group optimality, individual rationality and subgame consistency are three crucial properties that sustainable cooperative scheme has to satisfy.

3.1. Pareto Optimal Provision and Individual Rationality

To fulfill group optimality the agents would seek to maximize their expected joint payoff. To maximize their expected joint payoff the agents have to solve the stochastic dynamic programming problem

$$\max_{\{I_s^j \ for \ j \in N\}} E_{\vartheta_1, \vartheta_2, \cdots, \vartheta_T} \Big\{ \ \sum_{j=1}^{n} \sum_{s=1}^{T} [R^j(K_s) - C^i(I_s^j)](1+r)^{-(s-1)}$$

$$+ \sum_{j=1}^{n} q^j(K_{T+1})(1+r)^{-T} \ \Big\} \tag{3.1}$$

subject to the stock dynamics (2.1).

Invoking the standard stochastic dynamic programming technique an optimal solution to the stochastic control problem (2.1) and (3.1) can characterized by the following set of equations (see Basar and Olsder (1995) and Yeung and petrosyan (2012)):

$$W(t,K) = \max_{\{I_t^j \ for\ j\in N,\}} E_{\vartheta_t}\Big\{ \sum_{j=1}^{n}[R^j(K) - C^i(I_t^j)](1+r)^{-(t-1)}$$

$$+W\Big[\ t+1, \sum_{j=1}^{n} I_t^j - \delta K + \vartheta_t\ \Big]\ \Big\}, \text{for } t\in\{1,2,\cdots,T\}, \tag{3.2}$$

$$W(T+1,K) = \sum_{j=1}^{n} q^j(K_{T+1})(1+r)^{-T}. \tag{3.3}$$

Let $\psi_s^*(K)$ =$\{\psi_s^{1*}(K)$,$\psi_s^{2*}(K),\cdots,\psi_s^{n*}(K)\}$, for $s\in\{1,2,\cdots,T\}$ denote a set of strategies that brings about an optimal cooperative solution. A group optimal solution of public goods provision by the n agents is characterized by the solution of the equation (3.2)-(3.3).

The optimal cooperative path can be derived as:

$$K_{t+1} = \sum_{j=1}^{n} \psi_t^{j*}(K_t) - \delta K_t + \vartheta_t, K_1 = K^0, \text{for } t\in\{1,2,\cdots,T\}, \tag{3.4}$$

We use X_s^* to denote the set of realizable values of K_s generated by (3.4) at stage s and use $K_s^*\in X_s^*$ to denote an element in the optimal set.

Let $\xi(\cdot,\cdot)$ denote the agreed-upon imputation vector guiding the distribution of the total cooperative payoff under the agreed-upon optimality principle along the cooperative trajectory $\{\ K_s^*\ \}_{s=1}^T$. At stage s and if the productive stock is K_s^*, the imputation vector according to $\xi(\cdot,\cdot)$ is

$$\xi(s,K_s^*) = [\xi^1(s,K_s^*), \xi^2(s,K_s^*),\cdots,\xi^n(s,K_s^*)], \text{for } s\in\{1,2,\cdots,T\}. \tag{3.5}$$

A variety of examples of imputations $\xi(s,K_s^*)$ can be found in Yeung and Petrosyan (2006 and 2012). For individual rationality to be maintained throughout all stages, it is required that:

$$\xi^i(s,K_s^*) \geq V^i(s,K_s^*), \text{for } i\in N \text{ and } s\in\{1,2,\cdots,T\}.$$

To satisfy group optimality, the imputation vector has to satisfy

$$W(s,K_s^*) = \sum_{j=1}^{n} \xi^i(s,K_s^*), \text{for } s\in\{1,2,\cdots,T\}.$$

3.2. Subgame Consistent Solutions and Payoff Distribution Procedure

Under a subgame consistent situation, an extension of the solution policy to a subgame starting at a later stage with a state brought about by previous optimal behaviour would remain optimal. For subgame consistency to be satisfied, the imputation $\xi(\cdot,\cdot)$ according to the original agreed-upon optimality principle in (3.5) has to be maintained along the cooperative trajectory $\{ K_s^* \}_{s=1}^{T}$.

Following the analysis of Yeung and Petrosyan (2010 and 2012), we formulate a Payoff Distribution Procedure so that the agreed-upon imputations (3.5) can be realized.

Let $B_k^i(K_k^*)$ denote the payment that agent i will received at stage k under the cooperative agreement if K_k^* is realized at stage $k \in \{1, 2, \cdots, T\}$.

The payment scheme involving $B_k^i(K_k^*)$ constitutes a PDP in the sense that if K_k^* is realized at stage $k$the imputation to agent i over the stages from k to T can be expressed as:

$$\xi^i(k, K_k^*) = B_k^i(K_k^*) \left(\frac{1}{1+r}\right)^{k-1}$$

$$+E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_\zeta} \left\{ \sum_{\zeta=k+1}^{T} B_\zeta^i(K_\zeta^*) \left(\frac{1}{1+r}\right)^{\zeta-1} + q^i(K_{T+1})(1+r)^{-T} \right\}, \quad (3.6)$$

for $i \in N$ and $k \in \kappa$.

Using (3.6) one can obtain

$$\xi^i(k+1, K_{k+1}^*) = B_{k+1}^i(K_{k+1}^*) \left(\frac{1}{1+r}\right)^{k}$$

$$+E_{\theta_{k+2},\theta_{k+3},\cdots,\theta_\zeta} \left\{ \sum_{\zeta=k+2}^{T} B_\zeta^i(K_\zeta^*) \left(\frac{1}{1+r}\right)^{\zeta-1} + q^i(K_{T+1})(1+r)^{-T} \right\}. \quad (3.7)$$

Upon substituting (3.7) into (3.6) yields

$$\xi^i(k, K_k^*) = B_k^i(K_k^*) \left(\frac{1}{1+r}\right)^{k-1}$$

$$+E_{\theta_k} \left(\xi^i[k+1, \sum_{j=1}^{n} \psi_t^{j*}(K_k^*) - \delta K_k^* + \vartheta_k] \right), \quad (3.8)$$

for $i \in N$ and $k \in \kappa$.

Theorem 3.1. *Given that the public capital stock is K_k^* in stage k a payment equalling*

$$B_k^i(K_k^*) = (1+r)^{k-1} \left\{ \xi^i(K, x_k^*) \right.$$

$$\left. -E_{\theta_k} \left(\xi^i[k+1, \sum_{j=1}^{n} \psi_t^{j*}(K_k^*) - \delta K_k^* + \vartheta_k] \right) \right\}, \quad (3.9)$$

for $i \in N$, be paid to agent i at stage $k \in \{1, 2, \cdots, T\}$ would lead to the realization of the imputation $\{\xi(k, K_k^*)$, for $k \in \{1, 2, \cdots, T\}\}$.

Proof. From (3.8), one can readily obtain (3.9). Theorem 4.1 can also be verified alternatively by showing that from (3.6)

$$\xi^i(k, K_k^*) = B_k^i(K_k^*)\left(\frac{1}{1+r}\right)^{k-1}$$

$$+E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_\zeta}\left\{ \sum_{\zeta=k+1}^{T} B_\zeta^i(K_\zeta^*)\left(\frac{1}{1+r}\right)^{\zeta-1} + q^i(K_{T+1})(1+r)^{-T} \right\}$$

$$= \left\{ \xi^i(k, K_k^*) - E_{\theta_k}\left(\xi^i[k+1, \sum_{j=1}^{n} \psi_t^{j*}(K_k^*) - \delta K_k^* + \vartheta_k] \right) \right\}$$

$$+ \sum_{\zeta=k+1}^{T} E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_\zeta}\left\{ \xi^i(\zeta, K_\zeta^*) - E_{\theta_\zeta}\left(\xi^i[\zeta+1, \sum_{j=1}^{n} \psi_t^{j*}(K_k^*) - \delta K_k^* + \vartheta_k] \right) \right\}$$

$$= \xi^i(k, K_k^*);$$

given that $\xi^i(T+1, K_{T+1}^*) = q^i(K_{T+1})(1+r)^{-T}$. □

Note that the payoff distribution procedure in Theorem 3.1 would give rise to the agreed-upon imputation in (3.5) and therefore subgame consistency is satisfied.

When all agents are using the cooperative strategies, the payoff that agent i will directly receive at stage s is

$$R^i(K_s^*) - C^i[\psi_s^{i*}(K_s^*)].$$

However, according to the agreed upon imputation, agent i is supposed to receive $B_s^i(K_s^*)$. Therefore a transfer payment (which could be positive or negative)

$$\varpi^i(s, K_s^*) = B_s^i(K_s^*) - \{R^i(K_s^*) - C^i[\psi_s^{i*}(K_s^*)]\} \tag{3.10}$$

will be allotted to agent $i \in N$ at stage s to yield the cooperative imputation $\xi^i(k, K_k^*)$.

4. An Illustration

In this section, we provide an illustration with an application in the build-up of public capital by multiple asymmetric agents which is a discrete time counter-part of example in Yeung and Petrosyan (2013). Consider an economic region with n asymmetric agents. These agents receive benefits from an existing public capital stock $K(s)$. The accumulation dynamics of the public capital stock is governed by

$$K_{t+1} = \sum_{j=1}^{n} I_t^j - \delta K_t + \vartheta_t, K_1 = K^0, \text{for } t \in \{1, 2, \cdots, T\}, \tag{4.1}$$

where δ is the depreciation rate of the public capital, I_t^i is the investment made by the ith agent in the public capital in stage t, and ϑ_t is an independent random variable with non-negative range $\{\vartheta_t^1, \vartheta_t^2, \cdots, \vartheta_t^{\omega_t}\}$ and corresponding probabilities $\{\lambda_t^1, \lambda_t^2, \cdots, \lambda_t^{\omega_t}\}$. Moreover $\sum_{h=1}^{\omega_t} \lambda_t^h \vartheta_t^h = \bar{\vartheta}_t > 0$.

Each agent gains from the existing level of public capital and the ith agent seeks to maximize its expected stream of monetary gains:

$$E_{\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \sum_{s=1}^{T}[\alpha^i K_s - c^i (I_s^i)^2](1+r)^{-(s-1)} + (q_1^i K_{T+1} + q_2^i)(1+r)^{-T} \right\}, \quad (4.2)$$

subject to (4.1);

where α^i, c^i, q_1^i and q_2^i are positive constants.

In particular, α^i gives the gain that agent i derives from the public capital, $c^i(I_s^i(s))^2$ is the cost of investing I_s^i in the public capital, and $(q_1^i K_{T+1} + q_2^i)$ is the terminal valuation of the public capital at stage $T+1$. The noncooperative market outcome of the industry will be explored in the next subsection.

4.1. Noncooperative Market Outcome

Invoking the analysis in (2.1)-(2.5) in section 2 we obtain the corresponding Hamilton-Jacobi-Bellman equations

$$V^i(t,K) = \max_{I_t^i} E_{\vartheta_t}\Big\{ [\alpha^i K - c^i (I_t^i)^2](1+r)^{-(t-1)}$$

$$+V^i\Big[\ t+1, \sum_{\substack{j=1\\ j\neq i}}^{n} \phi_t^j(K) + I_t^i - \delta K + \vartheta_t\ \Big]\ \Big\}, \text{for } t\in\{1,2,\cdots,T\}, \quad (4.3)$$

$$V^i(T+1,K) = (q_1^i K_{T+1} + q_2^i)(1+r)^{-T}, \text{for } i\in N. \quad (4.4)$$

Performing the maximization operator in (4.3) yields:

$$\phi_t^i(K) = \sum_{h=1}^{\omega_t} \lambda_t^h \frac{1}{2c^i} V^i_{K_{t+1}}\,[\,t+1, \sum_{j=1}^{n}\phi_t^j(K) - \delta K + \vartheta_t^h\,]\,(1+r)^{(t-1)}, \text{for } i\in N. \quad (4.5)$$

To solve the game (4.1)-(4.2) we first obtain the value functions as follows.

Proposition 4.1. *The value function of agent i can be obtained as:*

$$V^i(t,K) = (A_t^i K + C_t^i)(1+r)^{-(t-1)}, \quad (4.6)$$

for $t\in\{1,2,\cdots,T+1\}$ and $i\in N$;

where $A_{T+1}^i = q_1^i$ and $C_{T+1}^i = q_2^i$,

$A_t^i = (\alpha^i - A_{t+1}^i\delta)$ and $C_t^i = -\frac{(A_{t+1}^i)^2}{4c^i} + A_{t+1}^i\left(\sum_{j=1}^n \frac{A_{t+1}^j}{2c^j} + \bar{\vartheta}_t^h \right) + C_{t+1}^i$,

for $t\in\{1,2,\cdots,T\}$.

Proof. See Appendix A. □

Using Proposition 4.1 and (4.5) the game equilibrium strategies can be obtained to characterize the market equilibrium. The asymmetry of agents brings about different payoffs and investment levels in public capital investments.

4.2. Cooperative Provision of Public Capital

Now we consider the case when the agents agree to act cooperatively and seek higher gains. They agree to maximize their expected joint gain and distribute the cooperative gain proportional to their non-cooperative expected gains. To maximize their expected joint gains the agents maximize

$$E_{\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\Big\{\sum_{j=1}^{n}\sum_{s=1}^{T}[\alpha^j K_s - c^j (I_s^j)^2](1+r)^{-(s-1)}$$
$$+\sum_{j=1}^{n}(q_1^j K_{T+1}+q_2^j)(1+r)^{-T}\Big\}, \tag{4.7}$$

subject to dynamics (4.1).
Following the analysis in (3.2)-(3.3) in Section 3, the corresponding stochastic dynamic programming equation can be obtained as:

$$W(t,K) = \max_{\{I_t^j \ for\ j\in N\}} E_{\vartheta_t}\Big\{\sum_{j=1}^{n}[\alpha^j K - c^j (I_t^j)^2](1+r)^{-(t-1)}$$
$$+W\Big[\ t+1, \sum_{\ell=1}^{n} I_t^{\ell} - \delta K + \vartheta_t\ \Big]\ \Big\}, \text{for } t\in\{1,2,\cdots,T\}, \tag{4.8}$$

$$W(T+1,K) = \sum_{j=1}^{n}(q_1^j K_{T+1}+q_2^j)(1+r)^{-T}. \tag{4.9}$$

Performing the maximization operator in (4.8) yields:

$$\psi_t^i(K) = \sum_{h=1}^{\omega_t}\lambda_t^h \frac{1}{2c^i} W_{K_{t+1}}\,[\,t+1, \sum_{j=1}^{n}\psi_t^j(K) - \delta K + \vartheta_t^h\,]\,(1+r)^{(t-1)}, \text{for } i\in N. \tag{4.10}$$

Proposition 4.2. *The value function $W(t,K)$ can be obtained as*

$$W(t,K) = (A_t K + C_t)(1+r)^{-(t-1)}, \tag{4.11}$$

for $t\in\{1,2,\cdots,T+1\}$;
where $A_{T+1}=\sum_{j=1}^{n} q_1^j$ and $C_{T+1}=\sum_{j=1}^{n} q_2^j$,
$A_t=\sum_{j=1}^{n}\alpha^j - A_{t+1}\delta$ and $C_t=\sum_{j=1}^{n}\frac{(A_{t+1})^2}{4c^j}+A_{t+1}\bar{\vartheta}_t^h+C_{t+1}$,
for $t\in\{1,2,\cdots,T\}$.

Proof. Follow the proof of Proposition 4.1. □

Using (4.10) and Proposition 4.2 the optimal investment strategy of public capital stock can be obtained as:

$$\psi_t^i(K) = \frac{A_{t+1}}{2c^i}, \text{for } i\in N \ \text{ and } t\in\{1,2,\cdots,T\}. \tag{4.12}$$

Using (4.1) and (4.12) the optimal trajectory of public capital stock can be expressed as:

$$K_{t+1} = [\ \sum_{j=1}^{n} \frac{A_{t+1}}{2c^j} - \delta K_t\ ] + \vartheta_t, K_1 = K^0, \text{for } t \in \{1, 2, \cdots, T\}, \tag{4.13}$$

We use X_s^* to denote the set of realizable values of K_s generated by (4.13) at stage s. The term $K_s^* \in X_s^*$ is used to denote and element in X_s^*.

4.3. Subgame Consistent Payoff Distribution

Next, we will derive the payoff distribution procedure that leads to a subgame consistent solution. With the agents agreeing to distribute their gains proportional to their non-cooperative gains, the imputation vector becomes

$$\xi^i(s, K_s^*) = \frac{V^i(s, K_s^*)}{\sum_{j=1}^n V^j(s, K_s^*)} W(s, K_s^*)$$

$$= \frac{A_s^i K_s^* + C_s^i}{\sum_{j=1}^n (A_s^j K_s^* + C_s^j)} (A_s K_s^* + C_s)(1+r)^{-(s-1)}, \tag{4.14}$$

for $i \in N$ and $s \in \{1, 2, \cdots, T\}$ if the public capital stock is $K_s^* \in X_s^*$.

To guarantee dynamical stability in a dynamic cooperation scheme, the solution has to satisfy the property of subgame consistency which requires the satisfaction of (4.14) at all stages $s \in \{1, 2, \cdots, T\}$. Invoking Theorem 3.1 we can obtain:

Proposition 4.3. *A PDP which would lead to the realization of the imputation $\xi(s, K_s^*)$ in (4.14) includes a terminal payment $(q_1^i K_{T+1}^* + q_2^i)$ to agent $i \in N$ at stage $T+1$ and an payment at stage $s \in \{1, 2, \cdots, T\}$:*

$$B_s^i(K_s^*) = \frac{A_s^i K_s^* + C_s^i}{\sum_{j=1}^n (A_s^j K_s^* + C_s^j)} (A_s K_s^* + C_s)$$

$$- \sum_{h=1}^{\omega_s} \lambda_s^h \frac{A_{s+1}^i K_{s+1}^*(\vartheta_s^h) + C_{s+1}^i}{\sum_{j=1}^n [A_{s+1}^j K_{s+1}^*(\vartheta_s^h) + C_{s+1}^j]} [A_{s+1} K_{s+1}^*(\vartheta_s^h) + C_{s+1}](1+r)^{-1}, \textit{for } i \in N, \tag{4.15}$$

where $K_{s+1}^*(\vartheta_s^h) = [\ \sum_{j=1}^n \frac{A_{s+1}}{2c^j} - \delta K_s^*\ ] + \vartheta_s^h$. □

Finally, when all agents are using the cooperative strategies, the payoff that agent i will directly receive at stage s is

$$\alpha^j K_s^* - \frac{(A_{S+1})^2}{4c^j}.$$

However, according to the agreed upon imputation, agent i is to receive $B_s^i(K_s^*)$ in Proposition 4.3. Therefore a transfer payment (which can be positive or negative) equalling

$$\varpi_i^i(s, K_s^*) = B_s^i(K_s^*) - \left[\alpha^j K_s^* - \frac{(A_{S+1})^2}{4c^j}\right] \tag{4.16}$$

will be imputed to agent $i \in N$ at stage $s \in \{1, 2, \cdots, T\}$.

5. Concluding Remarks

This paper presented subgame consistent cooperative solutions for stochastic discrete-time dynamic games in public goods provision. The solution scheme guarantees that the agreed-upon optimality principle can be maintained in any subgame and provides the basis for sustainable cooperation. A "payoff distribution procedure" (PDP) leading to subgame-consistent solutions is developed. Illustrative examples are presented to demonstrate the derivation of subgame consistent solution for public goods provision game. This is the first time that subgame consistent cooperative provision of public goods is analysed in discrete time. Various further research and applications, especially in the field of operations research, are expected.

Appendix A. Proof of Proposition 4.1.

Using the value functions in Proposition 4.1 the optimal strategies in (4.5) becomes:

$$\phi_t^i(K) = \frac{A_{t+1}^i}{2c^i}, \text{for } i \in N \text{ and } t \in \{1, 2, \cdots, T\}. \tag{A.1}$$

Using (A.1) the Hamilton-Jacobi-Bellman equations (4.4)-(4.5) reduces to:

$$A_t^i K + C_t^i = \alpha^i K - \frac{(A_{t+1}^i)^2}{4c^i} + \sum_{h=1}^{\omega_t} \lambda_t^h \left[A_{t+1}^i \left(\sum_{j=1}^{n} \frac{A_{t+1}^j}{2c^j} - \delta K + \vartheta_t^h \right) + C_{t+1}^i \right], \tag{A.2}$$

for $i \in N$ and $t \in \{1, 2, \cdots, T\}$,

$$A_{T+1}^i K + C_{T+1}^i = q_1^i K + q_2^i, \text{for } i \in N. \tag{A.3}$$

For (A.3) to hold it requires

$$A_{T+1}^i = q_1^i \text{ and } C_{T+1}^i = q_2^i. \tag{A.4}$$

Re-arranging terms in (A.2) yields:

$$A_t^i K + C_t^i = (\alpha^i - A_{t+1}^i \delta) K - \frac{(A_{t+1}^i)^2}{4c^i} + A_{t+1}^i \left(\sum_{j=1}^{n} \frac{A_{t+1}^j}{2c^j} + \bar{\vartheta}_t^h \right) + C_{t+1}^i, \tag{A.5}$$

for $i \in N$ and $t \in \{1, 2, \cdots, T\}$.

For (A.5) to hold it requires

$$A_t^i = (\alpha^i - A_{t+1}^i \delta) \text{ and } C_t^i = -\frac{(A_{t+1}^i)^2}{4c^i} + A_{t+1}^i \left(\sum_{j=1}^{n} \frac{A_{t+1}^j}{2c^j} + \bar{\vartheta}_t^h \right) + C_{t+1}^i. \tag{A.6}$$

Note that A_t^i and C_t^i depend on the model parameters and the succeeding values of A_{t+1}^i andC_{t+1}^i. Using (A.4) all A_t^i and C_t^i, for $i \in N$ and $t \in \{1, 2, \cdots, T\}$, are explicitly obtained.

Hence Proposition 4.1 follows. Q.E.D.

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