The Irrational Behavior Proof Condition for Linear-Quadratic Discrete-time Dynamic Games with Nontransferable Payoffs*

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Abstract The paper considers linear-quadratic discrete-time dynamic games with nontransferable payoffs. Pareto-optimal solution is studied as optimality principle. The time consistency and irrational behavior proof condition of this solution are investigated. As an example, the government debt stabilization game is considered.

Keywords: linear-quadratic games, discrete-time games, games with nontransferable payoffs, Pareto-optimal solution, time consistency, PDP, irrational behavior proof condition.

1. Introduction

Consider N-person discrete-time dynamic game $\Gamma(k_0, x_0)$ which is described by the state equation

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{n} B_i(k)u_i(k), \tag{1}$$

$$k \geq k_0, \quad k_0 \in \mathcal{K}_+, \quad x(k_0) = x_0.$$

x is m-dimensional state of system, u_i is a r-dimensional control variable of player i, $x(k_0) = x_0$ is the arbitrarily chosen initial state of the system, $A(k), B_i(k) \in Z(\mathcal{K}_+)$ are matrices of appropriate dimensions, $\mathcal{K}_+$ is the set of nonnegative integers, $Z(\mathcal{K}_+)$ is the set of bounded real matrices. The payoff function of player $i \in N$ is

$$J_i = \sum_{k=k_0}^{\infty} w_i(k, x(k), u_i(k)), \quad \forall i = 1, \ldots, n, \tag{2}$$

$$w_i(k, x(k), u_i(k)) = x^T(k)P_i(k)x(k) + u_i^T(k)R_i(k)u_i(k),$$

$$P_i(k), R_i(k) \in Z(\mathcal{K}_+), \quad P_i(k) = P_i^T(k), \quad R_i(k) = R_i^T(k) \quad \forall i \in N.$$

Suppose that payoffs are nontransferable.

We will assume that the players use feedback strategies,

$$u_i(k, x) = M_i(k)x(k),$$

to control the system.

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Definition 1. A set of strategies

$$\{u_i(k,x) = M_i(k)x(k), \quad i = 1,\dots,n\} \tag{3}$$

is called permissible if the following conditions are satisfied:

1. $M_i(k) \in Z(\mathcal{K}_+) \quad \forall i = 1,\dots,n.$

2. The resulting system described by

$$x(k+1) = (A(k) + \sum_{i=1}^{n} B_i(k)M_i(k))x(k) \tag{4}$$

   is uniformly asymptotically stable (when $k \to \infty$).

Suppose that players agree to use a Pareto-optimal solution as optimality principle.

And suppose that players consent to use vector of weights

$$\alpha = (\alpha_1,\dots,\alpha_n): \quad \sum_{i=1}^{n} \alpha_i = 1, \quad 0 < \alpha_i < 1$$

on their payoffs to obtain a Pareto-optimal outcome.

Then the optimal cooperative strategies of players can be found by solving the following control problem (Engwerda, 2005)

$$\max_{(u_1,\dots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(k_0, x_0, u), \tag{5}$$

Let $u^\alpha(k) = (u_1^\alpha(k),\dots,u_n^\alpha(k))$ be the set of strategies solving this optimal control problem:

$$(u_1^\alpha,\dots,u_n^\alpha) = \arg \max_{(u_1,\dots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(k_0, x_0, u). \tag{6}$$

Assume $J^\alpha(k_0,x_0,u) = \sum_{i=1}^{n} \alpha_i J_i(k_0,x_0,u)$, $P^\alpha(k) = \sum_{i=1}^{n} \alpha_i P_i(k), \quad k \geq k_0,$

$$R^\alpha(k) = \begin{pmatrix} \alpha_1 R_1(k) & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{O} & \alpha_2 R_2(k) & \dots & \mathbb{O} \\ \dots & \dots & \dots & \dots \\ \mathbb{O} & \mathbb{O} & \dots & \alpha_n R_n(k) \end{pmatrix}, \quad k \geq k_0.$$

Then

$$J^\alpha(k_0,x_0,u) = \sum_{k=k_0}^{\infty} (x^T(k)P^\alpha(k)x(k) + u(k)R^\alpha(k)u(k)). \tag{7}$$

Finding of Pareto-optimal solution is reduced to linear-quadratic optimal control problem (1)-(7) with one control variable $u(k)$.

The unique control in class of admissible

$$\{u_i^\alpha(k) = M_i^\alpha(k)x, \quad i = 1,\dots,n\},$$

maximizing $J^\alpha(k_0,x_0,u)$ exists if and only if (Bertsekas, 2007) the following conditions are satisfied:

1. The system of matrix equations

$$\begin{cases} (A(k)+B(k)M^{\alpha}(k))^T \Theta^{\alpha}(k+1)(A(k)+B(k)M^{\alpha}(k)) - \Theta^{\alpha}(k) - \\ - P^{\alpha}(k) - M^{\alpha}(k)^T R^{\alpha}(k) M^{\alpha}(k) = 0, \\ M^{\alpha}(k) = -(-R^{\alpha}(k) + B^T(k)\Theta^{\alpha}(k+1)B(k))^{-1} B^T(k)\Theta^{\alpha} k+1)A(k), \\ k \geq k_0 \end{cases} \tag{8}$$

   has the solution $\{M^{\alpha}(k), \Theta^{\alpha}(k)\} \in Z(\mathcal{K}_+)$, with dimensions $rs \times m$ and $m \times m$ respectively, where $\Theta^{\alpha}(k)$ – is symmetric for all $k \geq k_0$.
2. The set of strategies

$$\{u_i^{\alpha}(k) = M_i^{\alpha}(k)x, \quad i = 1, \ldots, n\}, \tag{9}$$

   where $M_i^{\alpha}(k)$ – i-th block of the matrix $M^{\alpha}(k) = \begin{pmatrix} M_1^{\alpha}(k) \\ M_2^{\alpha}(k) \\ \ldots \\ M^{\alpha}(k) \end{pmatrix}$, is admissible.
3. $(-R^{\alpha}(k) + B^T(k)\Theta^{\alpha}(k+1)B(k))$ – positive definite matrices.

The cooperative state trajectory $x^{\alpha}(k)$ one can find by substituting the cooperative strategies $\{u_i^{\alpha}(k)\}$ in (1) and solving the system:

$$x(k+1) = A(k)x(k) + B(k)u^{\alpha}(k). \tag{10}$$

And payoffs of players are:

$$J_i^{\alpha}(k_0, x_0, u^{\alpha}) \quad = \quad \sum_{k=k_0}^{\infty} \Big((x^{\alpha}(k))^T P_i(k) x^{\alpha}(k) \; + \; (u_i^{\alpha}(k))^T R_i(k) u_i^{\alpha}(k) \Big). \tag{11}$$

Here $B(k) = \left(B_1(k)\ B_2(k)\ \ldots\ B_(k)\right)$.

2. Time-consistency

Suppose that there exists such α, that inequalities

$$J_i^{\alpha}(k_0, x_0, u^{\alpha}) \geq V_i(k_0, x_0), \quad i = 1, \ldots, n. \tag{12}$$

requiring for individual rationality in the cooperative game are satisfied at initial time. Here $V_i(k_0, x_0)$ – is Nash outcome of player i in game $\Gamma(k_0, x_0)$.

But if there exists $k > k_0$ such that for some i:

$$J_i^{\alpha}(k, x^{\alpha}(k), u^{\alpha}) < V_i(k, x^{\alpha}(k)),$$

then time-inconsistency of the individual rationality condition is appear.

To overcome the time inconsistency problem in the game with nontransferable payoffs the notion of Payoff Distribution Procedure (PDP) was introduced by L.A. Petrosyan (1997). In this paper the PDP and time-consistency of Pareto-optimal solution are detailed for linear-quadratic discrete-time dynamic games.

Definition 2. Vector $\beta(k) = (\beta_1(k), ..., \beta_n(k))$ is a PDP if

$$\sum_{k=k_0}^{\infty}\Big((x^\alpha(k))^T P_i(k)x^\alpha(k) + (u_i^\alpha(k))^T R_i(k)u_i^\alpha(k)\Big) = \sum_{k=k_0}^{\infty}\beta_i(k), \quad i = 1, \dots, n.$$

Definition 3. Pareto-optimal solution is called time-consistent if there exists a PDP such that the condition of individual rationality is satisfied

$$\sum_{k=l}^{\infty}\beta_i(k) \geq V_i(l, x^\alpha(l)), \quad \forall l \geq k_0, \quad i = 1, \dots, n, \tag{13}$$

where $V_i(l, x^\alpha(l))$ – is Nash outcome of player i in subgame $\Gamma(l, x^\alpha(l))$.

Let for some Pareto-optimal solution the condition (12) is satisfied. Then there exist such functions $\eta_i(k) \geq 0$, that

$$J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0) = \sum_{k=k_0}^{\infty}\eta_i(k). \tag{14}$$

In (Petrosyan, 1997) the formula for PDP, which guarantees a time-consistency in cooperative differential game with nontransferable payoffs, is considered. The following theorem gives an analog of this formula.

Theorem 1. *Let inequalities*

$$J_i^\alpha(k_0, x_0, u^\alpha) \geq V_i(k_0, x_0), \quad i = 1, \dots, n,$$

are satisfied for some Pareto-optimal solution. Then PDP $\beta(k)$ computed by formula

$$\beta_i(k) = \eta_i(k) - V_i(k+1, x^\alpha(k+1)) + V_i(k, x^\alpha(k)) \qquad i = 1, \dots, n, \quad k > k_0 \tag{15}$$

guarantees time-consistency of this Pareto-optimal solution along the cooperative trajectory $x^\alpha(k)$ for $k > k_0$. Here $\eta_i(k) \geq 0$ – are functions satisfying (14).

Proof. Show that $\beta(k)$ is a PDP:

$$\sum_{k=k_0}^{\infty}\beta_i(k) = \sum_{k=k_0}^{\infty}\eta_i(k) - V_i(\infty, x^\alpha(\infty)) + V_i(k_0, x_0) = \\ = J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0) + V_i(k_0, x_0) = J_i^\alpha(k_0, x_0, u^\alpha). \tag{16}$$

Here $V_i(\infty, x^\alpha(\infty)) = \lim_{k\to\infty} V_i(k, x^\alpha(k)) = 0$. So $\beta(k)$ satisfies definition 2.

Now show that the condition of individual rationality is satisfied. Using (15) we obtain

$$\sum_{k=l}^{\infty}\beta_i(k) = \sum_{k=l}^{\infty}\eta_i(k) - V_i(\infty, x^\alpha(\infty)) + V_i(l, x^\alpha(l)) = \\ = \sum_{k=l}^{\infty}\eta_i(k) + V_i(l, x^\alpha(l)) \geq V_i(l, x^\alpha(l)). \tag{17}$$

□

2.1. Irrational Behavior Proof Condition

The condition under which even if irrational behaviors appear later in the game the concerned player would still be performing better under the cooperative scheme was considered in (Yeung, 2006). The irrational behavior proof condition for differential games with nontransferable payoffs is proposed in (Belitskaia, 2012). In this paper the irrational behavior proof condition is concretized for linear-quadratic discrete-time dynamic games with nontransferable payoffs.

Definition 4. Pareto-optimal solution $(J_1^\alpha(k_0, x_0, u^\alpha), \ldots, J_n^\alpha(k_0, x_0, u^\alpha))$ satisfies the irrational behavior proof condition (Yeung, 2006) in the game $\Gamma(k_0, x_0)$, if the following inequalities hold

$$\sum_{k=k_0}^{l} \beta_i(k) + V_i(l+1, x^\alpha(l+1)) \geq V_i(k_0, x_0), \quad i = 1, \ldots, n \tag{18}$$

for all $l \geq k_0$, where $\beta(k) = (\beta_1(k), \ldots, \beta_n(k))$ is time-consistent PDP of $(J_1^\alpha(k_0, x_0, u^\alpha), \ldots, J_n^\alpha(k_0, x_0, u^\alpha))$.

So if for all $i = 1, \ldots, n$ the following inequalities holds

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) \geq 0, \quad k \geq k_0,$$

then the Pareto-optimal solution satisfies the irrational behavior proof condition.

Rewrite these inequalities using (8)

$$\beta_i(k) + (x^\alpha(k))^T \Bigg((A(k) + B(k)M^\alpha(k))^T \Theta_i(k+1)(A(k) + B(k)M^\alpha(k)) - \Theta_i(k) \Bigg) x^\alpha(k) \geq 0, \quad k \geq k_0 \tag{19}$$

If we use formala (15), then

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) = \eta_i(k), \quad k \geq k_0,$$

where $\eta_i(k) \geq 0$ for all $k \geq k_0$. It means that conditions (19) are always satisfied in this case.

Let's formulate these results.

Theorem 2. *If in linear-quadratic discrete-time dynamic games with nontransferable payoffs for some Pareto-optimal solutions and its PDP the following inequalities hold*

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) \geq 0, \quad k \geq k_0 \quad i = 1, \ldots, n.$$

where $V_i(l, x^\alpha(l))$ – is Nash outcome of player i in subgame $\Gamma(l, x^\alpha(l))$, then the irrational behavior proof condition for this Pareto-optimal solutions is satisfied.

Proposition 1. *If the PDP $\beta(k)$ of Pareto-optimal solution in linear-quadratic discrete-time dynamic games with nontransferable payoffs is calculated using formula (15), then the irrational behavior proof condition for this Pareto-optimal solutions is satisfied.*

3. Example

As an example consider the government debt stabilization game(van Aarle, Bovenberg and Raith, 1995). Pareto solution of this game is considered in (Engwerda, 2005). This paper shows the discrete-time case of this problem and time-consistency of cooperative solution.

Assume that government debt accumulation, $d(k)$, is the sum of interest payments on government debt, $rd(k)$, and primary fiscal deficits, $f(k)$, minus the seignorage (i.e. the issue of base money) $m(k)$. So,

$$d(k+1) = rd(k) + f(t) - m(t), \qquad d(0) = d_0,$$

The objective of the fiscal authority is to minimize a sum of time profiles of the primary fiscal deficit, base-money growth and government debt

$$J_1 = \sum_{k=0}^{\infty} \left(\frac{1}{1+\rho}\right)^k ((f(k) - \overline{f})^2 + \eta(m(k) - \overline{m})^2 + \lambda(d(k) - \overline{d})^2).$$

The monetary authorities are assumed to choose the growth of base money such that a sum of time profiles of base-money growth and government debt is minimized. That is

$$J_2 = \sum_{k=0}^{\infty} \left(\frac{1}{1+\rho}\right)^k ((m(k) - \overline{m})^2 + \gamma(d(k) - \overline{d})^2).$$

Let

$$x_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (d(k) - \overline{d}),$$

$$x_2(k) = (\overline{f} - \overline{m} + (r-1)\overline{d}) \left(\frac{1}{1+\rho}\right)^{\frac{k+1}{2}},$$

$$u_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (f(k) - \overline{f}),$$

$$u_2(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (m(k) - \overline{m})$$

Then our system can be rewritten as

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{2} B_i(k)u_i(k)$$

$$A = \begin{pmatrix} r\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} & 1 \\ 0 & \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix}, B_2 = \begin{pmatrix} \left(-\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix},$$

The payoff function of player i

$$J_i = \sum_{k=k_0}^{\infty} (x^T(k)P_i(k)x(k) + \sum_{j=1}^{2} u_j^T(k)R_{ij}(k)u_i(k)), \quad \forall i = 1, 2$$

$$P_1 = \begin{pmatrix} \lambda \; 0 \\ 0 \; 0 \end{pmatrix}, P_2 = \begin{pmatrix} \gamma \; 0 \\ 0 \; 0 \end{pmatrix}, \quad R_{11} = 1, \quad R_{12} = \eta, \quad R_{21} = 0, \quad R_{22} = 1.$$

Following (Basar and Olsder, 1999) to find the Nash equilibrium we solve the system

$$\begin{cases} (A(k) + \sum\limits_{i=1}^{2} B_i(k) M_i^{NE}(k))^T \Theta_i(k+1)(A(k) + \sum\limits_{i=1}^{2} B_i(k) M_i^{NE}(k)) - \\ - \Theta_i(k) + P_i(k) + M_j^{NE}(k)^T R_{ij}(k) M_j^{NE}(k) + M_i^{NE}(k)^T R_{ii}(k) M_i^{NE}(k) = 0, \\ M_i^{NE}(k) = -(R_{ii}(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1} B_i^T(k)\Theta_i(k+1) \times \\ \times (A(k) + B_j(k) M_j^{NE}(k)), \quad i = 1,2, \quad j \neq i. \end{cases}$$

Let $\lambda = \frac{1}{2}, \eta = 1, \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} = \frac{1}{4}, s = 2, \gamma = 1$. Then

$$u_1^{NE}(k,x) = \begin{pmatrix} -0.073193 \; -0.166311 \end{pmatrix} x(k),$$

$$u_2^{NE}(k,x) = \begin{pmatrix} 0.142083 \; 0.318188 \end{pmatrix} x(k),$$

$$J_1 = x_0^T \begin{pmatrix} 0.656174 \; 0.354202 \\ 0.354202 \; 0.844156 \end{pmatrix} x_0,$$

$$J_2 = x_0^T \begin{pmatrix} 1.273766 \; 0.613087 \\ 0.613087 \; 1.444844 \end{pmatrix} x_0.$$

$$V(1, x(k)) = x^T(k) \begin{pmatrix} 0.656174 \; 0.354202 \\ 0.354202 \; 0.844156 \end{pmatrix} x(k),$$

$$V(2, x(k)) = x^T(k) \begin{pmatrix} 1.273766 \; 0.613087 \\ 0.613087 \; 1.444843 \end{pmatrix} x(k),$$

According to (8) to find the Pareto Solution we solve the system

$$\begin{cases} (A(k) + B_1 M_1^\alpha + B_2 M_2^\alpha)^T \Theta^\alpha(k+1)(A(k) + B_1 M_1^\alpha + B_2 M_2^\alpha) - \\ - \Theta^\alpha(k) + P^\alpha(k) + M^\alpha(k)^T R^\alpha(k) M^\alpha(k) = 0, \\ M^\alpha(k) = -(R^\alpha(k) + B^T(k)\Theta^\alpha(k+1)B(k))^{-1} \times \\ \times B^T(k)\Theta^\alpha(k+1)A(k). \end{cases}$$

Where $P^\alpha(k) = \alpha P_1(k) + (1-\alpha)P_2(k), \quad R^\alpha(k) = \begin{pmatrix} \alpha R_{11} & \mathbb{O} \\ \mathbb{O} & \alpha R_{21} + (1-\alpha)R_{22} \end{pmatrix},$

$$B(k) = \begin{pmatrix} B_1(k) \; B_2(k) \end{pmatrix}.$$

For $\alpha = 0,45$

$$M_1^\alpha = (-0.2272618408 \quad -0.5075099515)$$

$$M_2^\alpha = (0.1022678284 \quad 0.2283794781)$$

$$J_1(u^\alpha) = x_0^T \begin{pmatrix} 0.6808499028 \; 0.4139353163 \\ 0.4139353163 \; 0.9409769084 \end{pmatrix} x_0$$

$$J_2(u^\alpha) = x_0^T \begin{pmatrix} 1.223914910 & 0.4917964794 \\ 0.4917964794 & 1.139011465 \end{pmatrix} x_0$$

If, for example, $x_0 = \begin{pmatrix} -3 & 2 \end{pmatrix}$, then

$$J_1^\alpha(k_0, x_0, u^\alpha) - V_1(k_0, x_0) = -0.107435164999999722$$

$$J_2^\alpha(k_0, x_0, u^\alpha) - V_2(k_0, x_0) = -0.216497664600000528$$

So, conditions (12) are satisfied (we consider the minimization problem, that is why we have an opposite sign in (12)).

But on the next step we have

$$J_1^\alpha(k_1, x_1, u^\alpha) - V_1(k_1, x_1) = 0.0504046297943969643$$

It means, that time-inconsistency of the individual rationality condition is appear. To avoid this problem, use PDP, calculated by formula (15)

$$\begin{aligned} \beta_1(k) &= \frac{-0.107435164999999722}{k(k+1)} + x^{\alpha T}(k) \begin{pmatrix} 0.537430998 & 0.0789600736 \\ 0.0789600736 & 0.16307449 \end{pmatrix} x^\alpha(k), \\ \beta_2(k) &= \frac{-0.216497664600000528}{k(k+1)} + x^{\alpha T}(k) \begin{pmatrix} 1.060309389 & 0.144646529 \\ 0.144646529 & 0.35798954 \end{pmatrix} x^\alpha(k). \end{aligned} \tag{20}$$

Note, that $\eta_i(k) < 0$, because we consider the minimization problem now.

Sufficient condition for realization of irrational behavior proof condition has form:

$$\beta_1(k) - x^{\alpha T}(k) \begin{pmatrix} 0.537430998 & 0.0789600736 \\ 0.0789600736 & 0.16307449 \end{pmatrix} x^\alpha(k) \le 0$$

$$\beta_2(k) - x^{\alpha T}(k) \begin{pmatrix} 1.060309389 & 0.144646529 \\ 0.144646529 & 0.35798954 \end{pmatrix} x^\alpha(k) \le 0.$$

And they are satisfied for $\beta(k)$, computed by formula (20).

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