An Axiomatization of the Myerson Value

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Abstract TU-games with communication structure are cooperative games with transferable utility where the cooperation between players is limited by a communication structure represented by a graph on the set of players. On this class of games, the Myerson value is one of the most well-known solutions and it is the Shapley value of the so-called restricted game. In this study we give another form of fairness axiom on the class of TU-games with communication structure so that the Myerson value is uniquely characterized by this fainess axiom with (component) efficiency, a kind of null player property and additivity. The combination is similar to the original characterization of the Shapley value.

Keywords: Cooperative TU-games, communication structure, Myerson value, Shapley value

Cooperative game theory describes situations of cooperation between players. A cooperative game with transferable utility, TU-game for short, expresses such situations by a finite set of players and a characteristic function that assigns a worth to any subset of players, a coalition. Players within a coalition can freely divide the worth of the cooperation among themselves. The main focuses of TU-games are investigating under which conditions the players cooperate to form the grand coalition of all players and how to divide the worth of this grand coalition into a payoff for each player.

A single-valued solution on a class of games assigns as an allocation a payoff vector to each game which belongs to the class. Shapley (1953) introduces one of the most well-known single-valued solution. The solution, the Shapley value, is the average of all marginal vectors of a TU-game, where a marginal vector corresponds to a payoff vector for a permutation on the player set. Each permutation can be seen as an ordering of the players joining to from the grand coalition, and in the marginal vector associated with a permutation each player gets as payoff the difference in worth of the set of players preceding him in the permutation with and without him. While being introduced, the Shapley value is characterized as the unique solution on the class of TU-games that satisfies efficiency, additivity, null player property and symmetry in Shapley (1953).

TU-games assume that any coalition can be formed to cooperate and gain its worth of their cooperation, but in many economic situations there exist restrictions which prevent some coalitions from cooperating. A TU-game with this kind of situation is firstly introduced by Myerson (1977) as a TU-game with communication structure. It arises when the restriction is represented by an undirected graph in which the vertices represent the players and a link between two players shows that these players can communicate and are able to cooperate by themselves.

One of the most well-known single-valued solutions on the class of TU-games with communication structure is the Myerson value (Myerson (1977)), defined as the Shapley value of the so-called Myerson restricted game. By Myerson (1977), the Myerson value is characterized by (component) efficiency and fairness, fair in the sense that if a link is deleted between two players, the Myerson value imposes the same loss on payoffs for each of these two players. Other characterizations of the Myerson value are given in Borm et al. (1992), Brink (2009) for the class of TU-games with cycle-free communication structure.

In this study we give an alternative axiomatization of the Myerson value for TU-games with communication structure. Our approach is to give another form of fainess axiom so that the Myerson value is characterized by (component) efficiency, a kind of null player property, additivity and a kind of fairness. The combination is similar to the original characterization of the Shapley value by Shapley (1953).

This paper is organized as follows. Section 2 introduces TU-games with communication structure and the Myerson value. In Section 3 an axiomatic characterization for the solution is given.

1. TU-games with communication structure and the Myerson value

A cooperative game with transferable utility, or a TU-game, is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set of n players and $v : 2^N \to R$ is a characteristic function with $v(\emptyset) = 0$. For a subset $S \in 2^N$, being the coalition consisting of all players in S, the real number $v(S)$ represents the worth of the coalition that can be maximially achived, and can be freely distributed among the players in S. Let $\mathcal{G}_N$ denote the class of TU-games with fixed player set N. We often identify a TU-game (N, v) by its characteristic function v.

A special class of TU-games is the class of unanimity games. For $T \in 2^N$, the unanimity game $(N, u_T) \in \mathcal{G}_N$ has characteristic function $u_T : 2^N \to R$ defined as

$$u_T(S) = \begin{cases} 1 & \textit{if } \ T \subseteq S, \\ 0 & \textit{otherwise.} \end{cases}$$

It is well-known that any TU-game can be uniquely expressed as a linear combination of unanimity games. Let $(N, \mathbf{0}) \in \mathcal{G}_N$ denote the zero game, i.e., $\mathbf{0}(S) = 0$ for all $S \in 2^N$.

A payoff vector $x = (x_1, ..., x_n) \in R^n$ is an n-dimentional vector and it assigns payoff x_i to player $i \in N$. A single-valued solution on $\mathcal{G}_N$ is a mapping $\xi : \mathcal{G}_N \to R^n$ which assigns to every TU-game (N, v) a payoff vector $\xi(N, v) \in R^n$.

The most well-known single-valued solution on the class of TU-games is the Shapley value, see Shapley (1953). It is the average of the marginal vectors induced from the collection of all permutations of players. Let $\Pi(N)$ be the collection of all permutations on N. Given a permutation $\sigma \in \Pi(N)$, the set of predecessors of any element $i \in N$ in σ is defined as

$$P_\sigma(i) = \{h \in N | \ \sigma^{-1}(h) < \sigma^{-1}(i)\}.$$

Given a TU-game $(N, v) \in \mathcal{G}_N$, for a permutation σ in $\Pi(N)$ the marginal vector $m^\sigma(N, v)$ assigns payoff

$$m_i^\sigma(N, v) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))$$

to agent $i = \sigma(k)$, $k = 1, \ldots, n$. The Shapley value of (N, v), $Sh(N, v)$, is the average of all $n!$ marginal vectors, i.e.,

$$Sh(N, v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v).$$

A graph on N is a pair (N, L) where $N = \{1, \ldots, n\}$ is a set of vertices and $L \subseteq L_N^c$, where $L_N^c = \{\{i, j\} \mid i, j \in N, i \neq j\}$ is the complete set of undirected links without loops on N and an unordered pair $\{i, j\} \in L$ is called an edge in (N, L). A subset $S \in 2^N$ is connected in (N, L) if for any $i \in S$ and $j \in S$, $j \neq i$, there is a sequence of vertices $(i_1, i_2, \ldots, i_k)$ in S such that $i_1 = i$, $i_k = j$ and $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. The collection of all connected coalitions in (N, L) is denoted $C^L(N)$. By definition, the empty set $\emptyset$ and every singleton $\{i\}$, $i \in N$, are connected in (N, L). For $S \in 2^N$, the subset of edges $L(S) \subseteq L$ is defined as $L(S) = \{\{i, j\} \in L \mid i, j \in S\}$, being the subset of L of edges that can be established within S. The graph $(S, L(S))$ is a subgraph of (N, L). A component of a subgraph $(S, L(S))$ of (N, L) is a maximally connected coalition in $(S, L(S))$ and the collection of components of $(S, L(S))$ is denoted $\widehat{C}^L(S)$. For a graph (N, L), if $\{i, j\} \in L$, then i is called a neighbor of j and vice versa. Given (N, L) and $i \in N$, the collection of neighbors of i is denoted by D_i^L, that is, $D_i^L = \{j \in N \setminus \{i\} \mid \{i, j\} \in L\}$. The collection of neighbors of $S \in 2^N$ is defined similarly as $D_S^L = \{j \in N \setminus S \mid \exists i \in S : \{i, j\} \in L\}$.

The combination of a TU-game and an (undirected) graph on the player set is a TU-game with communication structure, introduced by Myerson (1977) and denoted by a triple (N, v, L) where (N, v) is a TU-game and (N, L) is a graph on N. A link between two players has as interpretation that the two players are able to communicate and it is assumed that only a connected set of players in the graph is able to cooperate to obtain its worth to freely transfer as payoff among the players in the coalition. Let $\mathcal{G}_N^{cs}$ denote the class of TU-games with communication structure and fixed player set N. A single-valued solution on $\mathcal{G}_N^{cs}$ is a mapping $\xi : \mathcal{G}_N^{cs} \to R^n$ which assigns to every TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ a payoff vector $\xi(N, v, L) \in R^n$.

The most well-known single-valued solution on the class of TU-games with communication structure is the Myerson value, see Myerson (1977). It is the Shapley value of the so-called Myerson restricted game. Following Myerson (1977), the restricted characteristic function $v^L : 2^N \to R$ of (N, v, L) is defined as

$$v^L(S) = \sum_{K \in \widehat{C}^L(S)} v(K), \; S \in 2^N.$$

The pair (N, v^L) is a TU-game and is called the Myerson restricted game of (N, v, L), and the Myerson value of a game $(N, v, L) \in \mathcal{G}_N^{cs}$ is defined as

$$\mu(N, v, L) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v^L).$$

2. An axiomatic characterization of the Myerson value

Most of the single-valued solutions proposed in the literature are characterized by axioms which state desirable properties a solution possesses. The most well-known characterization of the Shapley value for TU-games is given by Shapley (1953) as the unique solution on the class of TU-games that satisfies efficiency, additivity, the null player property and symmetry. Other characterizations of the Shapley value are proposed in for example Young (1985) and Brink (2002). While introducing the class of TU-games with communication structure, Myerson (1977) characterizes the Myerson value by component efficiency and fairness axioms.

Definition 1. A solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies *component efficiency* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ it holds that $\sum_{i \in Q} \xi_i(N, v, L) = v(Q)$ for all $Q \in \widehat{C}^L(N)$.

A solution on the class of TU-games with communication structure satisfies component efficiency if the solution allocates to each component as the sum of payoff among its members the worth of the component.

Definition 2. A solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies *fairness* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ and $\{i, j\} \in L$ it holds that

$$\xi_i(N, v, L) - \xi_i(N, v, L \setminus \{i, j\}) = \xi_j(N, v, L) - \xi_j(N, v, L \setminus \{i, j\}).$$

A solution on the class of TU-games with communication structure satisfies fairness if the deletion of an edge from the game results in the same payoff change for the two players who own the edge.

Theorem 1. ***(Myerson, 1977)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cs}$ that satisfies component efficiency and fairness.*

For the class of TU-games with cycle-free communication structure, which is a subclass of TU-games with communication structure, other characterizations of the Myerson value are given by Borm et al. (1992) and Brink (2009). The axioms we propose in this study are modified versions of the four axioms used in Shapley (1953), i.e., an efficiency axiom (component efficiency), an additivity axiom, a null player property and a fairness axiom.

For any two TU-games v and w in $\mathcal{G}_N$, the game $v + w$ is well defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in 2^N$.

Definition 3. A solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies *additivity* if for any (N, v, L), $(N, w, L) \in \mathcal{G}_N^{cs}$ it holds that $\xi(N, v + w, L) = \xi(N, v, L) + \xi(N, w, L)$.

Additivity of a solution means that if there are two TU-games with the same communication structure, the resulting payoff vectors coincide when applying the solution to each of the two games and adding the two vectors and when applying the solution to the game which is the sum of the two games.

A player $i \in N$ is a *restricted null player* in a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ if this player never contributes whenever he joins to form a connected coalition, that is, $v(S \cup \{i\}) - \sum_{K \in \widehat{C}^L(S)} v(K) = 0$ for all $S \in 2^N$ such that $i \notin S$ and $S \cup \{i\} \in C^L(N)$. The restricted null player property says that this player must get zero payoff.

Definition 4. A solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies the *restricted null player property* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ and restricted null player $i \in N$ in (N, v, L) it holds that $\xi_i(N, v, L) = 0$.

Note that a restricted null player of a TU-game with communication structure is a null player of its Myerson restricted game. The last axiom replaces symmetry.

Definition 5. A solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies *coalitional fairness* if for any two TU-games $(N, v, L), (N, v', L) \in \mathcal{G}_N^{cs}$ and $Q \in 2^N$ it holds that $\xi_i(N, v, L) - \xi_i(N, v', L) = \xi_j(N, v, L) - \xi_j(N, v', L)$ for all $i, j \in Q$ whenever $v(S) = v'(S)$ for all $S \in 2^N$, $S \neq Q$.

Coalitional fairness of a solution implies that given a TU-game with communication structure, if the worth of a single coalition changes, then the payoff change should be equal among all players in that coalition. From additivity and the restricted null player property we have the following lemma.

Lemma 1. *Let a solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfy additivity and the restricted null player property. Then for any two TU-games with the same communication structure (N, v, L), $(N, v', L) \in \mathcal{G}_N^{cs}$ it holds that $\xi(N, v, L) = \xi(N, v', L)$ whenever $v(S) = v'(S)$ for all $S \in C^L(N)$.*

Proof. Consider the game (N, w, L) where $w = v - v'$. Then every player is a restricted null player in this game because $w(S) = 0$ for all $S \in C^L(N)$. Therefore every player must receive zero payoff, that is, $\xi(N, w, L) = 0$. From additivity and $v = w + v'$ it follows that $\xi(N, v, L) = \xi(N, w, L) + \xi(N, v', L) = 0 + \xi(N, v', L) = \xi(N, v', L)$. □

This lemma says that the worth of an unconnected coalition does not affect the outcome of a solution that satisfies additivity and the restricted null player property, which leads to the following corollary.

Corollary 1. *If a solution $\xi : \mathcal{G}_N^{cs} \to R^n$ satisfies additivity and the restricted null player property, then $\xi(N, v, L) = \xi(N, v^L, L)$ for any $(N, v, L) \in \mathcal{G}_N^{cs}$.*

To prove that on the class of TU-games with communication structure the axioms above uniquely define the Myerson value, we consider Myerson restricted unanimity games. Given a unanimity game with communication structure $(N, u_T, L) \in \mathcal{G}_N^{cs}$ with $T \in 2^N$, the Myerson restricted unanimity game $(N, u_T^L) \in \mathcal{G}_N$ is given by

$$u_T^L(S) = \begin{cases} 1 & if \ \ \exists\, K \in \widehat{C}^L(S), \ \ T \subseteq K, \\ 0 & otherwise. \end{cases}$$

Given a graph (N, L) and $S \in 2^N$, let $\overline{C}^L(S)$ denote the collection of connected coalitions which minimally contain S, that is,

$$\overline{C}^L(S) = \{K \in C^L(N) \mid S \subseteq K, \ K \setminus \{i\} \notin C^L(N) \ \ \forall \ \ i \in K \setminus S\}.$$

Lemma 2. *For a unanimity TU-game with communication structure $(N, u_T, L) \in \mathcal{G}_N^{cs}$ with $T \in 2^N$, it holds that*

$$u_T^L = \begin{cases} \displaystyle\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} u_{\cup_{j \in J} Q_j} & if \ \ \overline{C}^L(T) = \{Q_1, \ldots, Q_k\}, \\ 0 & if \ \ \overline{C}^L(T) = \emptyset. \end{cases}$$

Proof. First consider the case when $\overline{C}^L(T) = \emptyset$. This implies that there exists no $K \in \widehat{C}^L(N)$ which contains T, and from the definition of u_T^L it follows that $u_T^L(S) = 0$ for all $S \in 2^N$. Next, let $v = \sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} u_{\cup_{j\in J} Q_j}$ when $\overline{C}^L(T) \neq \emptyset$. If $T \in C^L(N)$, then $\overline{C}^L(T) = \{T\}$ and therefore it holds that $v = u_T = u_T^L$. Suppose $T \notin C^L(N)$. It is to show that $v(S) = u_T^L(S)$ holds for every $S \in 2^N$. First take $S \in 2^N$ such that there is no $K \in \widehat{C}^L(S)$ satisfying $T \subseteq K$. This implies that $Q \not\subseteq S$ for any $Q \in \overline{C}^L(T)$, and thus we have $u_{\cup_{j\in J} Q_j}(S) = 0$ for all $J \subseteq \{1,\ldots,k\}$, which results in $v(S) = 0 = u_T^L(S)$. Next, take any $S \in 2^N$ such that there exists $K \in \widehat{C}^L(S)$ satisfying $T \subseteq K$. This K is unique and let $M \subseteq \{1,\ldots,k\}$ be such that $Q_j \subseteq K$ for all $j \in M$ and $Q_j \not\subseteq K$ for all $j \notin M$. Among all $J \subseteq \{1,\ldots,k\}$, it holds that $u_{\cup_{j\in J} Q_j}(S) = 1$ only when $J \subseteq M$, and otherwise $u_{\cup_{j\in J} Q_j}(S) = 0$. Let $|M| = m$. Then $v(S) = \sum_{J\subseteq M} (-1)^{|J|+1} u_{\cup_{j\in J} Q_j}(S) = \sum_{k=1}^{k=m} (-1)^{k+1} \binom{m}{k} = 1 = u_T^L(S)$, since it is known from the binominal theorem that $\sum_{k=0}^{k=m} (-1)^k \binom{m}{k} = 0$ and therefore $\sum_{k=1}^{k=m} (-1)^{k+1} \binom{m}{k} = -\sum_{k=1}^{k=m} (-1)^k \binom{m}{k} = \binom{m}{0} = 1$. □

Note that for any $J \subseteq \{1,\ldots,k\}$, it holds that $\cup_{j\in J} Q_j$ is connected, since for each $j \in J$, the set Q_j itself is connected and it also contains T. This lemma shows that any restricted unanimity TU-game with communication structure can be uniquely expressed as a linear combination of unanimity TU-games with the same communication structure for connected coalitions.

On the class of unanimity TU-games with communication structure, we have the following expression, which is well known and we present without proof.

Lemma 3. *For any TU-game with communication structure $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ with $T \in C^L(N)$, $T \neq \emptyset$, and $c \in R$, it holds that*

$$\mu_j(N, cu_T, L) = \begin{cases} c/|T| & if \quad j \in T, \\ 0 & if \quad j \notin T. \end{cases}$$

This lemma says that the Myerson value of a unanimity TU-game with communication structure with a connected coalition assigns the allocation which gives zero payoffs to the players who do not belong to the connected coalition and the worth of the connected coalition is shared equally among those who belong to it. Next, we give a characterization of the Myerson value in the following theorem.

Theorem 2. *The Myerson value is the unique solution on $\mathcal{G}_N^{cs}$ that satisfies component efficiency, additivity, the restricted null player property, and coalitional fainess.*

Proof. First, we show that the Myerson value satisfies all properties. Component efficiency follows from the fact that all marginal vectors are component efficient by construction. Since all marginal vectors of a TU-game with communication structure are linear in the worths of the connected coalitions and the Myerson value is the average of these vectors, the Myerson value satisfies additivity. If a player is a restricted null player, this player has marginal contribution equal to zero at any permutation and therefore the average is also zero. Finally, suppose there are two TU-games with the same communication structure $(N, v, L), (N, v', L) \in \mathcal{G}_N^{cs}$ and $Q \in C^L(N)$ such that $v(S) = v'(S)$ for all $S \in C^L(N)$, $S \neq Q$, and take any $i \in Q$. It holds that $m_i^\sigma(N, v, L) = m_i^\sigma(N, v', L)$ for any $\sigma \in \Pi(N)$ unless $P_\sigma(i) = Q \setminus \{i\}$.

There are $(|Q|-1)!(n-|Q|)!$ permutations σ such that $P_\sigma(i) = Q \setminus \{i\}$ and for each such σ the marginal contribution of i changes by $m_i^\sigma(N,v,L) - m_i^\sigma(N,v',L) = (v^L(Q) - v^L(Q \setminus \{i\})) - (v'^L(Q) - v^L(Q \setminus \{i\})) = v^L(Q) - v'^L(Q)$, which is independent of i. Therefore every player in Q receives the same change the same number of times and so the change in the Myerson value is the same among all players in Q.

Second, let $\xi : \mathcal{G}_N^{cs} \to R^n$ be a solution which satisfies all four axioms. Since ξ satisfies additivity and the restricted null player property, with Corollary 1 and Lemma 2, it suffices to show that for any graph (N,L) it holds that $\xi(N, cu_T, L) = \mu(N, cu_T, L)$ for any $T \in C^L(N)$ and $c \in R$. Let (N,L) be any graph on N. First consider the zero game $(N, \mathbf{0}, L) \in \mathcal{G}_N^{cs}$. In this game all players are restricted null players and therefore it follows from the restricted null player property that $\xi_i(N, \mathbf{0}, L) = 0 = \mu_i(N, \mathbf{0}, L)$ for all $i \in N$. Next consider the game $(N, cu_N, L) \in \mathcal{G}_N^{cs}$ with $N \in C^L(N)$. Between the games (N, cu_N, L) and $(N, \mathbf{0}, L)$ it holds that $c_N(N) = c$ and $cu_N(K) = \mathbf{0}(K) = 0$ for all $K \in 2^N$, $K \neq N$. From efficiency, coalitional fairness, and Lemma 3, we have

$$\xi_i(N, cu_N, L) = \frac{c}{n} = \mu_i(N, cu_N, L) \quad \forall\, i \in N.$$

Now consider a game $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ with $T \in C^L(N)$, $|T| = n-1$. It follows from the restricted null player property that player $i \notin T$ receives zero payoff, since this player yields zero marginal contribution when joining to any set of players to form a connected coalition. For the games (N, cu_T, L) and (N, cu_N, L), it holds that $cu_T(K) = cu_N(K)$ for all $K \in 2^N$, $K \neq T$. Coalitional fairness then implies that

$$\xi_i(N, cu_T, L) - \xi_i(N, cu_N, L) = \xi_j(N, cu_T, L) - \xi_j(N, cu_N, L) \ \ \forall\, i, j \in T,$$

which, with efficiency and Lemma 3, results in

$$\xi_i(N, cu_T, L) = \frac{c}{|T|} = \mu_i(N, cu_T, L) \ \ \forall\, i \in T.$$

Next, suppose $\xi(N, cu_T, L) = \mu(N, cu_T, L)$ holds for all $T \in C^L(N)$, $|T| > m > 1$. Consider $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ with $T \in C^L(N)$, $|T| = m$. For $i \notin T$, it follows from the restricted null player property that $\xi_i(N, cu_T, L) = 0$. By comparing (N, cu_T, L) and (N, v, L) with $v = \sum_{\ell \in D_T^L} cu_{T \cup \{\ell\}} - (k-1)cu_N$ where $k = |D_T^L|$ is the number of neighbors of T in (N,L), it holds that $cu_T(S) = v(S)$ for all $S \in 2^N$, $S \neq T$, and $cu_T(T) = c$ while $v(T) = 0$. Then coalitional fairness implies

$$\xi_i(N, cu_T, L) - \xi_i(N, v, L) = \xi_j(N, cu_T, L) - \xi_j(N, v, L) \ \ \forall\, i, j \in T.$$

From additivity and the supposition that $\xi(N, cu_S, L) = \mu(N, cu_S, L)$ for all connected S with $|S| > m$, it follows that

$$\begin{aligned}
\xi_i(N, v, L) = \sum_{\ell \in D_T^L} \xi_i(N, cu_{T \cup \{\ell\}}, L) - (k-1)\xi_i(N, cu_N, L) = \\
\sum_{\ell \in D_T^L} \mu_i(N, cu_{T \cup \{\ell\}}, L) - (k-1)\mu_i(N, cu_N, L) = \\
\sum_{\ell \in D_T^L} \mu_j(N, cu_{T \cup \{\ell\}}, L) - (k-1)\mu_j(N, cu_N, L) = \xi_j(N, v, L)
\end{aligned}$$

for all $i, j \in T$, and therefore

$$\xi_i(N, cu_T, L) = \xi_j(N, cu_T, L) \ \ \forall \ i, j \in T.$$

By efficiency it holds that $\xi_i(N, cu_T, L) = c/|T|$ for all $i \in T$, which implies $\xi(N, cu_T, L) = \mu(N, cu_T, L)$. When $|T| = 1$, efficiency and the restricted null player property imply that ξ allocates the Myerson value to $(N, cu_T, L) \in \mathcal{G}_N^{cs}$. Therefore for a multiple of any unanimity TU-game with communication structure for a connected coalition, the four axioms uniquely give the allocation of the Myerson value. Since ξ satisfies additivity and the restricted null player property, it follows from Corollary 1 that $\xi(N, v, L) = \xi(N, v^L, L)$ for any $(N, v, L) \in \mathcal{G}_N^{cs}$. By Lemma 2 it holds that v^L can be expressed as a unique linear combination of unanimity games for connected coalitions. That is, given any $(N, v, L) \in \mathcal{G}_N^{cs}$ there exist unique numbers $c_T \in R$ for $T \in C^L(N)$, $T \neq \emptyset$, such that $v^L = \sum_T c_T u_T$. The proof is completed since for any $(N, v, L) \in \mathcal{G}_N^{cs}$ it holds from additivity that

$$\xi(N, v, L) = \xi(N, v^L, L) = \xi(N, \sum_{T \in C^L(N), T \neq \emptyset} c_T u_T, L) =$$

$$\sum_{T \in C^L(N), T \neq \emptyset} \xi(N, c_T u_T, L) = \sum_{T \in C^L(N), T \neq \emptyset} \mu(N, c_T u_T, L) = \mu(N, v, L).$$

□

To show the independence of the four axioms, consider the linear solution $\xi(N, v, L) = \sum_{T \in C^L(N)} f(N, c_T u_T, L)$ where $v = \sum_{T \in C^L(N)} c_T u_T$ and $f(N, c_T u_T, L)$ allocates c to the player in T who has the smallest index and 0 to any other player. It only fails coalitional fairness. Next, consider the solution $\xi(N, v, L)$ that allocates payoff vector $\xi(N, v, L)$ as follows. When $N = \{1, 2\}$, $L = \{1, 2\}$, $v^L(S) \neq 0$ for all $S \in C^L(N)$, and further $v(S) \neq v(T)$ for all distinct $T, S \in 2^N$, then it gives $\xi_j(N, v, L) = v(N)/2$, and in any other case it gives $\xi(N, v, L) = \mu(N, v, L)$. This solution satisfies all axioms except additivity. The equal sharing solution, where each agent receives $v(N)/n$, satisfies every axiom except the restricted null player property. Finally, the solution where each agent receives zero payoff only fails efficiency.

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