Bridging the Gap between the Nash and Kalai-Smorodinsky Bargaining Solutions

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Abstract Bargaining solutions that satisfy weak Pareto optimality, symmetry, and independence of equivalent utility representations are called *standard*. The Nash (1950) solution is the unique independent standard solution and the Kalai-Smorodinsky (1975) solution is the unique monotonic standard solution. Every standard solution satisfies midpoint domination on triangles, or MDT for short. I introduce a formal axiom that captures the idea of a solution being "at least as independent as the Kalai-Smorodinsky solution." On the class of solutions that satisfy MDT and independence of non-individually-rational alternatives, this requirement implies that each player receives at least the minimum of the payoffs he would have received under the Nash and Kalai-Smorodinsky solutions. I refer to the latter property as *Kalai-Smorodinsky-Nash robustness.* I derive new axiomatizations of both solutions on its basis. Additional results concerning this robustness property, as well as alternative definitions of "at least as independent as the Kalai-Smorodinsky solution" are also studied.

Keywords: Bargaining; Kalai-Smorodinsky solution; Nash solution.

1. Introduction

Nash's (1950) bargaining problem is a fundamental problem in economics. Its formal description consists of two components: a feasible set of utility allocations, each of which can be achieved via cooperation, and one special utility allocation—the *disagreement point*—that prevails if the players do not cooperate. A solution is a function that picks a feasible utility allocation for every problem. The axiomatic approach to bargaining narrows down the set of "acceptable" solutions by imposing meaningful and desirable restrictions (axioms), to which the solution is required to adhere.

Weak and common restrictions are the following: *weak Pareto optimality*—the selected agreement should not be strictly dominated by another feasible agreement; *symmetry*—if the problem is symmetric with respect to the 45°-line then the players should enjoy identical payoffs; *independence of equivalent utility representations*—the selected agreement should be invariant under positive affine transformations of the problem. I will call a solution that satisfies these three restrictions *standard*.

Two additional restrictions that will be considered in the sequel are the following. *Midpoint domination* requires the solution to provide the players payoffs that are at least as large as the average of their best and worst payoffs. Every standard solution satisfies midpoint domination on triangular problems, which are the simplest kind of bargaining problems: each such problem is a convex hull of the disagreement point, the best point for player 1, and the best point for player 2. *Independence of* *non-individually-rational alternatives* requires the solution to depend only on those options that provide each player at least his disagreement utility; in other words, outcomes that clearly cannot be reached by voluntary behavior should not matter for bargaining.

Two other known principles are *independence* and *monotonicity*. Informally, the former says that if some options are deleted from a given problem but the chosen agreement of this problem remains feasible (it is not deleted) then this agreement should also be chosen in the problem that corresponds to the post-deletion situation; the latter says that if a problem "expands" in such a way that the set of feasible utilities for player i remains the same, but given every utility-payoff for i the maximum that player $j \neq i$ can now achieve is greater, then player j should not get hurt from this expansion. Within the class of standard solutions, these principles are incompatible: Nash (1950) showed that there exists a unique standard independent solution, while Kalai and Smorodinsky (1975) showed that a different solution is the unique standard monotonic one. The reconciliation of independence and monotonicity in bargaining, therefore, is a serious challenge.

One possible response to this challenge is to give up the restriction to standard solutions.[1] Here, however, I consider only standard solutions. When attention is restricted to standard solutions, the aforementioned challenge can, informally, be expressed in the form of the following question:

How can we "bridge the gap" between the Nash and Kalai-Smorodinsky solutions?

Motivated by this question, I introduce an axiom that formalizes the idea of a solution being "at least as independent as the Kalai-Smorodinsky solution."[2] I denote this axiom by $\succeq_{IIA} KS$.[3] I also consider the following requirement, which refers directly to both solutions: in every problem each player should receive at least the minimum of the payoffs he would have received under the Nash and Kalai-Smorodinsky solutions. I call this property *Kalai-Smorodinsky-Nash robustness*, or KSNR. On the class of solutions that satisfy midpoint domination on triangles and independence of non-individually-rational alternatives, $\succeq_{IIA} KS$ implies KSNR.

KSNR, in turn, captures much of the essence of the Nash and Kalai-Smorodinsky solutions: the Nash solution is characterized by KSNR and independence and the Kalai-Smorodinsky solution—by KSNR and monotonicity. Both characterizations hold on the class of all solutions, not only the standard ones, not only the ones that satisfy midpoint domination on triangles.[4]

The rest of the paper is organized as follows. Section 2 describes the model. The axiom $\succeq_{IIA} KS$ is defined in Section 3. KSNR is introduced and discussed in Section 4. Section 5 elaborates on midpoint domination and its non-trivial connections to the Nash and Kalai-Smorodinsky solutions. Section 6 considers an alternative definition of "at least as independent as the Kalai-Smorodinsky solution." Section 7 concludes with a brief discussion.

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[1] For example, the *egalitarian solution* (due to Kalai (1977)) satisfies all the above mentioned requirements except independence of equivalent utility representations. A formal definition of this solution will be given in Section 4 below.

[2] The precise meaning of this term will be given in Section 3.

[3] The rationale behind this notation will be clarified in Sections 2 and 3.

[4] The class of solutions that satisfy midpoint domination on triangles contains the class of standard solutions.

2. Preliminaries

A *bargaining problem* is defined as a pair (S, d), where $S \subset \mathbb{R}^2$ is the *feasible set*, representing all possible (v-N.M) utility agreements between the two players, and $d \in S$, the *disagreement point*, is a point that specifies their utilities in case they do not reach a unanimous agreement on some point of S. The following assumptions are made on (S, d):

- S is compact and convex;
- $d < x$ for some $x \in S$.[5]

Denote by $\mathcal{B}$ the collection of all such pairs (S, d). A *solution* is any function $\mu\colon \mathcal{B} \to \mathbb{R}^2$ that satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. Given a feasible set S, the *weak Pareto frontier* of S is $WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$ and the *strict Pareto frontier* of S is $P(S) \equiv \{x \in S : y \gneqq x \Rightarrow y \notin S\}$. The best that player i can hope for in the problem (S, d), given that player j obtains at least d_j utility units, is $a_i(S, d) \equiv \max\{x_i : x \in S_d\}$, where $S_d \equiv \{x \in S : x \geq d\}$. The point $a(S, d) = (a_1(S, d), a_2(S, d))$ is the *ideal point* of the problem (S, d). The *Kalai-Smorodinsky solution*, KS, due to Kalai and Smorodinsky (1975), is defined by $KS(S, d) \equiv P(S) \cap [d; a(S, d)]$.[6] The *Nash solution*, N, due to Nash (1950), is defined to be the unique maximizer of $(x_1 - d_1) \times (x_2 - d_2)$ over S_d.

Nash (1950) showed that N is the unique solution that satisfies the following four axioms, in the statements of which (S, d) and (T, e) are arbitrary problems.

Weak Pareto Optimality (WPO): $\mu(S, d) \in WP(S)$.

Let F_A denote the set of positive affine transformations from $\mathbb{R}$ to itself.[7]

Independence of Equivalent Utility Representations (IEUR): $f = (f_1, f_2) \in F_A \times F_A \Rightarrow f \circ \mu(S, d) = \mu(f \circ S, f \circ d)$.[8]

Let $\pi(a, b) \equiv (b, a)$.

Symmetry (SY): $[\pi \circ S = S] \& [\pi \circ d = d] \Rightarrow \mu_1(S, d) = \mu_2(S, d)$.

Independence of Irrelevant Alternatives (IIA): $[S \subset T] \& [d = e] \& [\mu(T, e) \in S] \Rightarrow \mu(S, d) = \mu(T, e)$.

Whereas the first three axioms are widely accepted, criticism has been raised regarding IIA. The idea behind a typical such criticism is that the bargaining solution could, or even should, depend on the shape of the feasible set. In particular, Kalai and Smorodinsky (1975) noted that when the feasible set expands in such a way that for every feasible payoff for player 1 the maximal feasible payoff for player 2 increases, it may be the case that player 2 loses from this expansion under the Nash

[5] Vector inequalities: xRy if and only if x_iRy_i for both $i \in \{1, 2\}$, $R \in \{>, \geq\}$; $x \gneqq y$ if and only if $x \geq y$ & $x \neq y$.

[6] Given two vectors x and y, the segment connecting them is denoted $[x; y]$.

[7] i.e., the set of functions f of the form $f(x) = \alpha x + \beta$, where $\alpha > 0$.

[8] If $f_i : \mathbb{R} \to \mathbb{R}$ for each $i = 1, 2$, $x \in \mathbb{R}^2$, and $A \subset \mathbb{R}^2$, then: $(f_1, f_2) \circ x \equiv (f_1(x_1), f_2(x_2))$ and $(f_1, f_2) \circ A \equiv \{(f_1, f_2) \circ a : a \in A\}$.

solution. Given $x \in S_d$, let $g_i^S(x_j)$ be the maximal possible payoff for i in S given that j's payoff is x_j, where $\{i, j\} = \{1, 2\}$. What Kalai and Smorodinsky noted, is that N violates the following axiom, in the statement of which (S, d) and (T, d) are arbitrary problems with a common disagreement point.

Individual Monotonicity (IM):

$$[a_j(S, d) = a_j(T, d)]\&[g_i^S(x_j) \leq g_i^T(x_j)\ \forall x \in S_d \cap T_d] \Rightarrow \mu_i(S, d) \leq \mu_i(T, d).$$

Furthermore, they showed that when IIA is deleted from the list of Nash's axioms and replaced by IM, a characterization of KS obtains.[9] Following Trockel (2009), a solution that satisfies the common axioms—namely WPO, SY, and IEUR—will be referred to in the sequel as *standard*.

Most solutions from the bargaining literature (standard or not) also satisfy the following axiom, in the statement of which (S, d) is an arbitrary problem.

Independence of Non-Individually-Rational Alternatives (INIR): $\mu(S, d) = \mu(S_d, d)$.[10]

3. Relative independence

Consider the following partial order on the plane, $\preceq$. Given $x, y \in \mathbb{R}^2$, write $x \preceq y$ if $x_1 \leq y_1$ and $x_2 \geq y_2$. That is, $x \preceq y$ means that x is (weakly) to the north-west of y.

Let μ be a solution and let $(S, d) \in \mathcal{B}$ be a problem such that $\mu(S, d) = KS(S, d)$. Say that *μ is at least as independent as KS given* (S, d) if the following is true for every $(Q, e) \in \mathcal{B}$ such that $e = d$, $Q \subset S$, and $KS(S, d) \in Q$:

1. $KS(Q, e) \preceq KS(S, d) \Rightarrow KS(Q, e) \preceq \mu(Q, e) \preceq KS(S, d)$, and
2. $KS(S, d) \preceq KS(Q, e) \Rightarrow KS(S, d) \preceq \mu(Q, e) \preceq KS(Q, e)$.

That is, μ is at least as independent as KS given (S, d) if in every relevant "sub-problem" the solution point according to μ is between the solution point of KS and the solution point "of IIA." A solution, μ, is *at least as independent as KS* if it is at least as independent as KS given (S, d), for every (S, d) such that $\mu(S, d) = KS(S, d)$. Denote this property (or axiom) by $\succeq_{IIA} KS$.

There is no shortage of standard solutions satisfying this property. Obviously, KS is such a solution and every standard IIA-satisfying solution is such a solution. However, there are many others. For describing such a solution, the following notation will be useful (it will also turn out handy in the next Section). For each $(S, d) \in \mathcal{B}$ and each i, let:

$$m_i(S, d) \equiv \min\{N_i(S, d), KS_i(S, d)\}.$$

[9] In many places throughout the paper I refer to "individual monotonicity" and "independence of irrelevant alternatives" simply as "monotonicity" and "independence." (see, e.g., the Introduction). The longer names for these axioms, and their respective abbreviations IM and IIA, are presented here in order to distinguish them from other monotonicity and independence axioms from the literature.

[10] To the best of my knowledge, the earliest paper that utilizes this axiom is Peters (1986).

Now consider the following solution, μ^*:

$$\mu^*(S,d) \equiv \begin{cases} m(S,d) & \text{if } N(S,d) = KS(S,d) \\ KS(\{x \in S : x \geq m(S,d)\}, m(S,d)) & \text{otherwise} \end{cases}$$

It is easy to see that μ^* is a standard solution which is at least as independent as KS, it is different from KS, and it violates IIA.

The property $\succeq_{IIA} KS$ is, of course, not the only possible formalization of the idea "at least as independent as KS." Moreover, it is not immune to criticism. I will discuss one of its drawback and propose an alternative formal definition for "at least as independent as KS" later in the paper, in Section 6.

Finally, it is worth noting that one may very well question the validity of taking *the* monotonic solution and employing it as the "measuring stick" for the extant to which independence can be violated. Why not prefer the analogous criterion, where N is taken to be the measuring stick for the degree to which monotonicity can be compromised? In principle, this alternative approach expresses a sensible consideration, but in practice it is problematic. To see this, consider $S \equiv \text{conv hull}\{\mathbf{0}, (0,1), (1,1), (2,0)\}$ and $T \equiv \{x \in \mathbb{R}^2_+ : x \leq (2,1)\}$.[11] Note that when we move from S to T the feasible set "stretches" in the direction of coordinate 1 and hence, by IM, player 1 should not get hurt from this expansion. Accordingly, an "at least monotonic as N" relation would naturally impose that player 1's benefit from the change $(S,\mathbf{0}) \mapsto (T,\mathbf{0})$ would be at least as large as the one he would have obtained under N. However, even KS fails this test.

4. Kalai-Smorodinsky-Nash robustness

Consider the following axiom, in the statement of which (S,d) is an arbitrary problem.

Midpoint Domination (MD): $\mu(S,d) \geq \frac{1}{2}d + \frac{1}{2}a(S,d)$.

This axiom is due to Sobel (1981), who also proved that N satisfies it. The idea behind it is that a "good" solution should always assign payoff that Pareto-dominate "randomized dictatorship" payoffs. Anbarci (1998) considered a weakening of this axiom, where the the requirement $\mu(S,d) \geq \frac{1}{2}d + \frac{1}{2}a(S,d)$ is applied only to problems (S,d) for which $S = S_d$ and is a triangle. I will refer to this weaker axiom as *midpoint domination on triangles*, or MDT.[12] It is easy to see that every standard solution satisfies MDT. Next, consider the following axiom, in the statement of which (S,d) is an arbitrary problem.

Kalai-Smorodinsky-Nash Robustness (KSNR): $\mu(S,d) \geq m(S,d)$.

KSNR implies MD since both N and KS satisfy MD.[13] Therefore, the following implications hold:

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[11] $\mathbf{0} \equiv (0,0)$.

[12] Anbarci calls it *midpoint outcome on a linear frontier*.

[13] It is straightforward that KS satisfies MD; as mentioned above, the fact that N satisfies it was proved by Sobel (1981).

$$\text{KSNR} \Rightarrow \text{MD} \Rightarrow \text{MDT}. \tag{1}$$

Now recall that the big-picture goal in this paper is to offer a reconciliation of monotonicity and independence within the class of standard solutions; this class, in turn, is a subclass of the MDT-satisfying solutions, so MDT essentially expresses no loss of generality for our purpose. With the additional (weak) restriction of INIR, the following theorem says that the aforementioned reconciliation, as expressed by $\succeq_{IIA} KS$, implies KSNR.

Theorem 1. *Let μ be a solution that satisfies independence of non-individually-rational alternatives and midpoint domination on triangles. Suppose further that it is at least as independent as the Kalai-Smorodinsky solution. Then μ satisfies Kalai-Smorodinsky-Nash robustness.*

Proof. Let μ satisfy INIR, MDT and $\succeq_{IIA} KS$. Let (S,d) be an arbitrary problem. By INIR we can assume $S = S_d$. Let $x \equiv \mu(S,d)$, $k \equiv KS(S,d)$, and $n \equiv N(S,d)$. We need to prove that $x_i \geq \min\{n_i, k_i\}$. Let $T = \text{conv hull}\{d, (2n_1 - d_1, d_2), (d_1, 2n_2 - d_2)\}$. By MDT, $\mu(T,d) = KS(T,d) = N(T,d) = n$. If $k = n$, then by $\succeq_{IIA} KS$ it follows that $x = k = n$. Now consider $k \neq n$; wlog, $k_1 < n_1$. In this case $\succeq_{IIA} KS$ dictates that $k \preceq x \preceq n$, which implies that $x_i \geq \min\{n_i, k_i\}$.

Let $\mathcal{A} = \{\text{INIR, MDT}, \succeq_{IIA} KS, \text{KSNR}\}$ and let A denote a generic axiom in $\mathcal{A}$. Theorem 1 says that $A =$KSNR is implied by $\mathcal{A}\setminus\{A\}$. From (1) we see that the same is trivially true for $A =$MDT. On the other hand, for $A =\succeq_{IIA} KS$ or $A =$INIR, an analogous conclusion cannot be drawn.

Consider first $A =\succeq_{IIA} KS$. Let $Q \equiv \text{conv hull}\{\mathbf{0}, (0,1), (1,1), (2,0)\}$, $Q' \equiv \{x \in Q : x_1 \leq KS_1(Q,\mathbf{0})\}$, and consider the following solution, μ^{**}. For (S,d) such that $S = S_d = S_\mathbf{0}$, $\mu^{**}(S,d) = N(Q',\mathbf{0})(= (1,1))$ if $S = Q'$ and $\mu^{**}(S,d) = KS(S,d)$ otherwise; for other problems the solution point is obtained by a translation of d to the origin and deletion of non-individually-rational alternatives. It is obvious that μ^{**} satisfies MDT, INIR, and KSNR. However, in the move from $(Q,\mathbf{0})$ to $(Q',\mathbf{0})$, μ^{**} violates $\succeq_{IIA} KS$.

Regarding $A =$INIR, consider the following solution, μ^{***}:

$$\mu^{***}(S,d) \equiv \begin{cases} N(S,d) & \text{if } \{x \in S : x < d\} \neq \emptyset \\ KS(S,d) & \text{otherwise} \end{cases}$$

It is immediate to see that μ^{***} satisfies KSNR and MDT, and it is also not hard to check that it also satisfies $\succeq_{IIA} KS$.

Finally, $A =$KSNR is not implied by any strict subset of $\mathcal{A}\setminus\{A\}$. The *egalitarian solution* (due to Kalai (1977)), E, which is defined by $E(S,d) \equiv WP(S_d) \cap \{d + (x,x) : x \geq 0\}$, satisfies IIA (and therefore$\succeq_{IIA} KS$) as well as INIR, but does not satisfy MDT or KSNR. The *Perles-Maschler solution* (Perles and Maschler (1981)), PM, is an example of a solution that satisfies MDT and INIR, but not $\succeq_{IIA} KS$ or KSNR. For (S,d) with $d = \mathbf{0}$ and $P(S) = WP(S)$, it is defined to be the point $u \in P(S)$ such that $\int_{(0,a_2)}^{u} \sqrt{-dxdy} = \int_{u}^{(a_1,0)} \sqrt{-dxdy}$, where $a = a(S,d)$; for other problems it is extended by IEUR and continuity is an obvious fashion. This is a standard solution, and therefore it satisfies MDT; in fact, it actually satisfies MD.[14] It is easy to see that it violates $\succeq_{IIA}$ KS; to see that it violates KSNR, look at $S^* = \text{conv}\{\mathbf{0}, (0,1), (\frac{3}{4}, 0), (\frac{3}{4}, \frac{1}{4})\}$: it is easily verified that $PM_1(S^*, \mathbf{0}) = \frac{3}{8} < \frac{3}{7} = KS_1(S^*, \mathbf{0}) < \frac{3}{6} = N_1(S^*, \mathbf{0})$. Finally, the following solution, μ^{****}, satisfies $\succeq_{IIA}$ KS and MDT, but not INIR or KSNR.[15]

$$\mu^{****}(S,d) \equiv \begin{cases} KS(S,d) \text{ if } d \notin intS \\ \frac{1}{2}K(S,d) \text{ otherwise} \end{cases}$$

KSNR captures much of the essence of the Nash and Kalai-Smorodinsky solutions. This is expressed in the following theorems.

Theorem 2. *The Nash solution is the unique solution that satisfies Kalai-Smorodinsky-Nash robustness and independence of irrelevant alternatives.*

Theorem 3. *The Kalai-Smorodinsky solution is the unique solution that satisfies Kalai-Smorodinsky-Nash robustness and individual monotonicity.*

The proofs of Theorems 2 and 3 follow from the combination of two results from the existing literature. Before we turn to these results, one more axiom needs to be introduced. This axiom, which is due to Anbarci (1998), has a similar flavor to that of MD, but the two are not logically comparable.

Balanced Focal Point (BFP): If $S = d + \text{conv hull}\{\mathbf{0}, (a,b), (\lambda a, 0), (0, \lambda b)\}$ for some $\lambda \in [1,2]$, then $\mu(S,d) = d + (a,b)$.[16]

The justification for this axiom is that the equal areas to the north-west and south-east of the focal point $d+(a,b)$ can be viewed as representing equivalent concessions. Similarly to MD, BFP is implied by KSNR and implies MDT:

$$\text{KSNR} \Rightarrow \text{BFP} \Rightarrow \text{MDT}. \tag{2}$$

The first implication is due to the fact that every standard solution satisfies BFP, and both N and KS are standard. The second implication follows from setting $\lambda = 2$ in BFP's definition.

Anbarci (1998) showed that KS is characterized by IM and BFP. His work was inspired by that of Moulin (1983), who in what is probably the simplest and most elegant axiomatization of N, proved that it is the unique solution that satisfies IIA and MD. Combining the results of Moulin (1983), Anbarci (1998), and the implication KSNR⇒ [MD, BFP], one obtains a proof for the theorems.

It is worth noting that whereas the implication KSNR⇒ [MD, BFP] is true, not only the converse is not true, but, moreover, even the combination of MD and "standardness"(and, therefore, the combination of MD and BFP) does not imply KSNR. The Perles-Maschler solution, PM, is an example.

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[14] See, e.g., Salonen (1985).

[15] In the definition of this solution, *int* stands for "interior."

[16] Anbarci assumes the normalization $d = \mathbf{0}$; the version above is the natural adaptation of his axiom to a model with an arbitrary d.

5. Midpoint domination

Both N and KS are related to MD. Regarding N, we already encountered the results of Moulin (1983) and Sobel (1981). Additionally, a related result has been obtained by de Clippel (2007), who characterized N by MD and one more axiom—*disagreement convexity*.[17] As for KS, Anbarci (1998) characterized it by BFP and IM, and we already noted the relation between BFP and MD—each can be viewed as an alternative strengthening of MDT which is also a weakening of KSNR. More recently, I characterized KS by MD and three additional axioms (see Rachmilevitch (2013)). Finally, Chun (1990) characterized the *Kalai-Rosenthal* (1978) solution—which is closely related to KS—on the basis of several axioms, one of which is a variant of MD.

Below is an example for another non-trivial link between MD and N/KS. In it, KSNR takes center stage.

The following family of standard solutions is due to Sobel (2001). For each number $a \leq 1$ corresponds a solution, which is defined on *normalized problems*—(S, d) for which $d = \mathbf{0}$ and $a(S, d) = (1, 1)$. Given $a \leq 1$, this solution is:

$$W(S, a) \equiv \arg\max_{x \in S}[\frac{1}{2}x_1^a + \frac{1}{2}x_2^a]^{\frac{1}{a}}.$$[18]

On any other (not normalized) problem, the solution point is obtained by appropriate utility rescaling. In light of the resemblance to the well-known concept from Consumer Theory, I will call these solutions *normalized CES solutions*. Both N and KS are normalized CES solutions: N corresponds to $\lim_{a \to 0} W(., a)$ and KS corresponds to $\lim_{a \to -\infty} W(., a)$.[19] Thus, Sobel's family offers a smooth parametrization of a class of solutions, of which N and KS are special members. It turns out that on this class KSNR and MD are equivalent.

Theorem 4. *A normalized CES solution satisfies Kalai-Smorodinsky-Nash robustness if and only if it satisfies midpoint domination.*

6. An alternative "at least as independent as KS" relation

Recall the basic definition from Section 3: μ is at least as independent as KS given (S, d) if for every relevant "sub-problem" (Q, e) the solution point according to μ is between the solution point of KS and the solution point "of IIA." Underlying this definition is a notion of "betweenness," a notion which is not immune to criticism. Specifically, it suffers the following drawback. Consider the case where $KS(Q, e)$ is to the left (and north) of $KS(S, d)$, $\mu(Q, e)$ is to the right (and south) of $KS(S, d)$, but the distance between $\mu(Q, e)$ and $KS(S, d)$ is only a tiny $\epsilon > 0$; namely, the solution μ hardly changed its recommendation in the move from (S, d) to (Q, e),

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[17] See his paper for the definition of the axiom. de Clippel's result is related to an earlier characterization which is due to Chun (1990).

[18] The maximizer is unique for $a < 1$; $a = 1$ corresponds to the *relative utilitarian solution*, which (in general) is multi-valued.

[19] Maximizing $W(S, a)$ describes a well-defined method for solving arbitrary bargaining problems, not necessarily normalized. In this more general case, $\lim_{a \to -\infty} W(., a)$ corresponds to the *egalitarian solution* (Kalai (1977)). See Bertsimas et al (2012) for a recent detailed paper on the matter.

but this change is opposite in its direction to that of KS. According to the aforementioned definition, such a solution μ is **not** at least independent as KS given (S, d). Therefore, it is not at least as independent as KS even if it coincides with KS (or with an IIA-satisfying solution) on all other problems.

Thus, one may argue that an appropriate definition of "more independent than" should not rely on direction (and hence should not rely on betweenness) and should take distance into consideration. This definition, therefore, should be based on an appropriate metric of "IIA violations." Here is one such definition.

Say that μ is *metrically at least as independent as* KS given (S, d), if $\mu(S, d) = KS(S, d) \equiv x$ and for every relevant sub-problem (Q, e) it is true that:

$$\max_{i\in\{1,2\}}|\mu_i(Q, e) - x_i| \leq \max_{i\in\{1,2\}}|KS_i(Q, e) - x_i|.$$

Metrically at least as independent as KS means that the corresponding property holds for every (S, d) such that $\mu(S, d) = KS(S, d)$. In Theorem 1, "at least as independent as KS" can be replaced by "metrically at least as independent as KS" provided that attention is restricted solutions that (i) are efficient, and (ii) satisfy the following axiom, in the statement of which (S, d) is an arbitrary problem.

Weak Contraction Monotonicity (WCM): For every i and every number r, $\mu_i(V, d) \leq \mu_i(S, d)$, where $V \equiv \{x \in S : x_i \leq r\}$.[20]

Note that WCM is implied (separately) by IIA and by IM.

Theorem 5. *Let μ be solution that satisfies weak Pareto optimality, midpoint domination on triangles, independence of non-individually-rational alternatives, and weak contraction monotonicity. Suppose that μ is metrically at least as independent as the Kalai-Smorodinsky solution. Then μ satisfies Kalai-Smorodinsky-Nash robustness.*

Finally, we also have the following result.

Theorem 6. *A normalized CES solution that satisfies weak contraction monotonicity also satisfies Kalai-Smorodinsky-Nash robustness.*[21]

7. Discussion

Motivated by the goal to reconcile independence and monotonicity in bargaining within the class of standard solutions, I have introduced the requirement that the bargaining solution be "at least as independent as KS." That is, I took *the* monotonic standard solution as the measuring stick for how far one can depart from independence.

Weaker versions of IIA have previously been considered in the literature. The best known axiom in this regard is Roth's (1977) *independence of irrelevant alternatives other than the disagreement point and the ideal point*, which applies IIA only to pairs of problems that, in addition to a common disagreement point, share the

[20] Implicit here is the assumption that $(V, d) \in \mathcal{B}$. Obviously, $V = \emptyset$ for a sufficiently small r.

[21] It is an open question whether the converse is true; namely, whether WCM and KSNR are equivalent on the class of normalized CES solutions.

same ideal point. Thomson (1981) generalized Roth's axiom to a family of axioms, parametrized by a *reference function.* Such an axiom applies IIA only to pairs of problems that share the same reference-function-value. Though these type of weakening of IIA are more common in the literature, they seems not be a good fit for the task of reconciling independence and monotonicity: Roth's axiom seems too weak and Thomson's seems too abstract. The property "at least as independent as KS," or $\succeq_{IIA} KS$, though having a somewhat special structure, seems more well-suited for the goal of the current paper.

This property, in turn, when considered in combination with MDT and INIR (a combination which is almost without loss of generality within the class of standard solutions), implies KSNR. KSNR, in turn, is of interest in its own right. On its basis I have derived axiomatizations of the two solutions: N is characterized by KSNR and IIA and KS is characterized by KSNR and IM. These new characterizations are not a consequence of the known theorems of Nash (1950) and Kalai-Smorodinsky (1975), since they hold on the entire class of bargaining solutions, not only on the class of the standard ones. Note also that the fact that a solution satisfies KSNR does not imply that it is standard. For example, the following is a non-standard solution that satisfies KSNR: $E^{KSNR}(S,d) \equiv m(S,d) + (e,e)$, where e is the maximal number such that the aforementioned expression is in S.

Appendix

Proof of Theorem 4: It is known that MD is equivalent to $a \leq 0$ (see Sobel (1981)). I will prove that KSNR is also equivalent to $a \leq 0$. It is enough to prove this equivalence for $a < 0$ (because $a = 0$ corresponds to the Nash solution). For simplicity (and without loss), I will consider only (normalized) strictly comprehensive problems—those for which $S = \{(x, f(x)) : x \in [0,1]\}$, where f is a differentiable strictly concave decreasing function; the case of an arbitrary problem follows from standard limit arguments.

The parameter a corresponds to the solution

$$W(S,a) \equiv \text{arg max}_{x \in S_{\mathbf{0}}} [\frac{1}{2}x_1^a + \frac{1}{2}x_2^a]^{\frac{1}{a}}.$$

In the case of a normalized strictly comprehensive problem, the object of interest is

$$\text{arg max}_{0 \leq x \leq 1} [\frac{1}{2}x^a + \frac{1}{2}f(x)^a]^{\frac{1}{a}} \equiv W(x,a).$$

Therefore $\frac{d}{dx}W(x,a) = e[\frac{a}{2}x^{a-1} + \frac{a}{2}f(x)^{a-1}f'(x)]$, where e is a shorthand for a strictly positive expression.

At the optimum, the derivative of this expression is zero (i.e., an FOC holds).

$$[\frac{f(x(a))}{x(a)}]^{1-a} = -f'(x(a)),$$

where $x(a)$ is player 1's solution payoff given the parameter a. The derivative of the RHS with respect to a is $-f''(x(a))x'(a)$, hence the sign of $x'(a)$ is the same as the sign of the derivative of the LHS. To compute the latter, recall the formula

$$\Psi'(a) = \Psi(a) \times \{h'(a)\log[g(a)] + \frac{h(a)g'(a)}{g(a)}\},$$

where $\Psi(a) \equiv [g(a)]^{h(a)}$.

Taking $g(a) \equiv \frac{f(x(a))}{x(a)}$ and $h(a) \equiv 1-a$, we see that the signs of the derivative of the LHS is the same as the sign of

$$-\log[\frac{f(x(a))}{x(a)}]+\frac{(1-a)}{f(x(a))x(a)}x'(a)(f'(x(a))x(a)-f(x(a))) \equiv -\log[\frac{f(x(a))}{x(a)}]+Zx'(a),$$

where Z is a shorthand for a negative expression. Therefore, the sign of $x'(a)$ is the same as that of $-\log[\frac{f(x(a))}{x(a)}]+Zx'(a)$.

Let $k \equiv KS_1(S,\mathbf{0})$ and $n \equiv N_1(S,\mathbf{0})$.

Case 1: $x(a) < k$. In this case, $\frac{f(x(a))}{x(a)} > 1$, hence $-\log[\frac{f(x(a))}{x(a)}] < 0$. This means that $x'(a) < 0$. To see this, assume by contradiction that $x'(a) \geq 0$. This means that $\text{sign}[-\log[\frac{f(x(a))}{x(a)}]+Zx'(a)] = -1 = \text{sign}[x'(a)]$, a contradiction. Therefore $n < x(a) < k$, so KSNR holds.

Case 2: $x(a) > k$. In this case, $\frac{f(x(a))}{x(a)} < 1$, hence $-\log[\frac{f(x(a))}{x(a)}] > 0$. This means that $x'(a) > 0$. To see this, assume by contradiction that $x'(a) \leq 0$. This means that $\text{sign}[-\log[\frac{f(x(a))}{x(a)}]+Zx'(a)] = 1 = \text{sign}[x'(a)]$, a contradiction. Therefore $k < x(a) < n$, so KSNR holds. □

Proof of Theorem 5: Let μ be a solution that satisfies the axioms listed in the theorem and let $(S,d) \in \mathcal{B}$. Assume by contradiction that KSNR is violated for (S,d). Wlog, we can assume that S_d is not a rectangle and that $d = \mathbf{0}$. Let $T \equiv$ conv hull$\{(2n_1,0)(0,2n_2),\mathbf{0}\}$, where $n \equiv N(S,\mathbf{0})$. By MDT, $\mu(T,\mathbf{0}) = N(T,\mathbf{0}) = KS(T,\mathbf{0}) = n$. Let $k \equiv KS(S,\mathbf{0})$. Wlog, suppose that $x_1 < \min\{n_1,k_1\}$. Let simply denote the "metrically at least as independent as KS" relation by $\succeq$. Note that because of $\succeq$, $n_1 = k_1$ is impossible. Therefore $n_1 \neq k_1$.

Case 1: $k_1 < n_1$. Here we have $|x_1 - n_1| > |k_1 - n_1|$ and hence, because of $\succeq$, $|k_2 - n_2| > |k_1 - n_1|$.

Case 1.1: $|x_1 - n_1| \geq |x_2 - n_2|$. In this case $|k_2 - n_2| \leq |x_2 - n_2| \leq |x_1 - n_1| \leq |k_2 - n_2|$. The first inequality follows from the fact that x (which, by WPO, is on the boundary) is to the north-west of k, the second inequality is because we are in Case 1.1, and the third is by $\succeq$.

Case 1.2: $|x_1 - n_1| < |x_2 - n_2|$. By $\succeq$, $|x_2 - n_2| \leq |k_2 - n_2|$.
indent Whatever the case—1.1 or 1.2—it follows that $|x_2 - n_2| = |k_2 - n_2|$, which is impossible, since k is in the relative interior of $P(S)$.

Case 2: $k_1 > n_1$. Let $r \in (n_1,k_1)$ be such that $KS_1(V,\mathbf{0}) < n_1$, where $V \equiv \{x \in S : x_1 \leq r\}$. Invoking WCM and applying the arguments from Case 1 to V completes the proof. □

Proof of Theorem 6: I will prove that if a normalized CES solution violates KSNR—that is, if its defining parameter is $a > 0$—then it also violates weak contraction monotonicity. Fix then $0 < a < 1$[22] and let μ denote the corresponding solution.

[22] The case $a = 1$ can be easily treated separately (i.e., it is easy to show that the relative utilitarian solution violates WCM; for brevity, I omit an example); focusing on $a < 1$ allows for "smooth MRS considerations" as will momentarily be clear.

Claim: There is a unique $k = k(a) \in (0, 1)$ such that:

$$1 - k = k^{1-a}. \tag{3}$$

Proof of the Claim: The RHS is decreasing in k, the LHS is increasing, and opposite strict inequalities obtain at $k = 0$ and $k = 1$. QED

From here on, fix $k = k(a)$.

Consider $S = \text{conv hull}\{\mathbf{0}, (1, 0), (0, 1), (1, k)\}$. Due to (3), a and k are such that the slope of the line connecting $(0, 1)$ and $(1, k)$ (i.e., the slope of the strict Pareto frontier of S) is $|k - 1|$, which equals the "MRS" of the objective that μ maximizes, when this MRS is evaluated at the corner $(1, k)$. Therefore, $\mu(S, \mathbf{0}) = (1, k)$.

Now let $t \in (0, 1)$. Note that $(t, (k-1)t+1) \in P(S)$. Let us "chop" S at the height $(k - 1)t + 1$; namely, consider $V = V(t) \equiv \{(a, b) \in S : b \leq (k - 1)t + 1\}$. To show a violation of WCM, I will find a $t \in (0, 1)$ for which $\mu_2(V(t), \mathbf{0}) > k = \mu_2(S, \mathbf{0})$.

To this end, let us consider the normalized feasible set derived from $V = V(t)$, call it $Q = Q(t)$. Given $t \in (0, 1)$, $Q = Q(t) = \{(a, \frac{b}{(k-1)t+1}) : (a, b) \in V(t)\}$. By SINV, $\mu_2(Q(t), \mathbf{0}) = \frac{\mu_2(V(t),\mathbf{0})}{(k-1)t+1}$, or $\mu_2(V(t), \mathbf{0}) = [(k-1)t+1]\mu_2(Q(t), \mathbf{0})$. Therefore, I will show that $[(k - 1)t + 1]\mu_2(Q(t), \mathbf{0}) > k$, or $\mu_2(Q(t), \mathbf{0}) > \frac{k}{(k-1)t+1}$. Note that the south-east corner of $P(Q)$ is $(1, \frac{k}{(k-1)t+1})$. Therefore, it is enough to show that the MRS at this corner is smaller than the slope of $P(Q)$. Namely, that

$$[\frac{(k-1)t+1}{k}]^{a-1} < \frac{1-k}{(k-1)t+1},$$

for a suitably chosen t. Note that for $t \sim 1$ the LHS is approximately one, so we are done if $1 < \frac{1-k}{k}$, or $k < \frac{1}{2}$. Hence, it is enough to prove that $k(a) < \frac{1}{2}$ for $a > 0$. To this end, let us re-write (3) as

$$G \equiv k^{1-a} + k - 1 = 0.$$

Since $k(0) = \frac{1}{2}$, it is enough to prove that $k' < 0$. By the Implicit Function Theorem, the sign of k' is opposite to the sign of $[\frac{\partial G}{\partial k}]/[\frac{\partial G}{\partial a}]$. Note that $\frac{\partial G}{\partial k} = (1-a)k^{-a}+1 > 0$ and that the sign of $\frac{\partial G}{\partial a}$ is the same is that of $\frac{\partial \log k^{1-a}}{\partial a} = -\log k > 0$. $\square$

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