Completions for Space of Preferences

Victor V. Rozen

Saratov State University,
Astrakhanskaya St. 83, Saratov, 410012, Russia
E-mail: rozenvv@info.sgu.ru

Abstract A preferences structure is called a complete one if it axiom linearity satisfies. We consider a problem of completion for ordering preferences structures. In section 2 an algorithm for finding of all linear orderings of finite ordered set is given. It is shown that the indicated algorithm leads to construction of the lattice of ideals for ordered set. Further we find valuations for a number of linear orderings of ordered sets of special types. A problem of contraction of the set of linear completions for ordering preferences structures which based on a certain additional information concerning of preferences in section 4 is considered. In section 5, some examples for construction and evaluations of the number of all linear completions for ordering preferences structures are given.

Keywords: preferences structure, ordering preferences structure, completion of preferences structure, a valuation for the number of linear completions.

1. Introduction

A *space of preferences* (or *preferences structure*) can be defined as a triplet of the form

$$\langle A, \alpha, \beta \rangle \,, \tag{1}$$

where α and β are binary relations on a set A satisfying the following axioms:

$$\begin{array}{ll} 1. & \alpha \cap \alpha^{-1} = \emptyset \qquad (asymmetry); \\ 2. & \beta^{-1} = \beta \qquad (symmetry); \\ 3. & \Delta_A \subseteq \beta \qquad (reflexivity); \\ 4. & \alpha \cap \beta = \emptyset \qquad (disjointness). \end{array} \tag{2}$$

We mean
A as a *set of alternatives*;
α as a *strict preference relation*;
β as an *indifference relation.*
As usually we put $\rho = \alpha \cup \beta$ and use the notation:
$a \overset{\rho}{\lesssim} b \overset{df}{\Leftrightarrow} a \overset{\alpha}{<} b$ or $a \overset{\beta}{\sim} b$.

Then a space of preferences can be written as a pair $\langle A, \rho \rangle$, where the strict preference relation and the indifference relation can be presented as

$$\begin{aligned} \alpha &= \rho \backslash \rho^{-1}, \\ \beta &= \rho \cap \rho^{-1}. \end{aligned} \tag{3}$$

The main special properties for preferences structures are the following ones:

$$\text{Transitivity}: \quad a_1 \overset{\rho}{\lesssim} a_2, a_2 \overset{\rho}{\lesssim} a_3 \Rightarrow a_1 \overset{\rho}{\lesssim} a_3; \qquad (Tr)$$

$$\text{Antisymmetry}: \quad a_1 \overset{\rho}{\lesssim} a_2, a_2 \overset{\rho}{\lesssim} a_1 \Rightarrow a_1 = a_2; \qquad (Antsym)$$

$$\text{Linearity}: \quad a_1 \overset{\rho}{\lesssim} a_2 \quad \text{or} \quad a_2 \overset{\rho}{\lesssim} a_1. \qquad (Lin)$$

Definition 1. Preferences structure satisfying the conditions (*Tr*) and (*Antsym*) is called *an ordering preferences structure* and satisfying the conditions (*Tr*), (*Antsym*) and (*Lin*) is called a *linear (or complete) ordering preferences structure.*

Definition 2. A preferences structure $\langle A, \alpha_1, \beta_1 \rangle$ is called a *completion* of a preferences structure $\langle A, \alpha, \beta \rangle$ if inclusions

$$\alpha \subseteq \alpha_1, \beta \subseteq \beta_1 \qquad (4)$$

hold and at least once of these inclusions is strict.

Remark 1. A preferences structure $\langle A, \alpha, \beta \rangle$ has not completions if and only if it is a linear one.

Thus the most interesting are completions of a preferences structure to a linear preferences structure. In this paper, we study some questions concerning of completions for ordering preferences structure. The main problems of our investigation are:

(PI) *The problem of description of all completions for ordering preferences structure to a linear one* and

(PII) *The problem of contraction of the set of linear completions based on certain additional information concerning of these completions.*

2. Linear orderings of ordered sets

2.1. An algorithm for finding of all linear orderings

It is well known the following classical result (Birkhoff, 1967).

Szpilrajn Theorem. *Any partial ordering can be enlarged to a linear ordering.*

Thus in terms of our paper, any ordering preferences structure has a completion to a linear one. However, Szpilrajn theorem is not a constructive propositional since it does not indicate a method for construction of linear completions.

Consider an ordering preferences structure which on a set of alternatives A is given. In algebra terminology, such a structure can be presented as an ordered set $(A, \leq)$ (i.e. $\leq$ is a binary relation on A satisfying conditions reflexivity, antisymmetry and transitivity). In this notations, the strict preference relation α coincides with strict order $<$ and the indifference relation β is identity relation.

We now state an algorithm for finding of all linear orderings of a finite ordered set that is an algorithm for finding of all linear completions of an ordering preferences structures. Remark that formally a linear ordering of k-element subset $B \subseteq A$ can be represented as one-one isotonic function φ from B into $\{1, \ldots, k\}$, where $\varphi(a)$ is a number of element $a \in B$ under this linear ordering. The required algorithm is based on the following lemma.

Lemma 1. *Suppose an ordered set A contains n elements and a^* is a maximal element. Assume that we have a linear ordering φ of a subset $A \backslash a^*$ (by numbers $1, 2, \ldots, n-1$). Preserve the function φ for elements of $A \backslash a^*$ and put $\varphi(a^*) = n$ then φ becomes a linear ordering of all set A.*

Thus we can obtain all completions of the set A, having completions of subsets which are a result of extraction of maximal elements. Further we use the same method for these subsets until the empty set $\emptyset$ appears. To realize this algorithm we need in the following steps.

Step 1. Define an auxiliary graph γ by the following rule. Vertexes of graph γ are some subsets of A and for two subsets $A_1, A_2 \subseteq A$ put $A_1 \overset{\gamma}{\prec} A_2$ if and only if A_2 is a result of extraction of some maximal element belonging to A_1. Then starting of the set A, we construct some sequence of conjugate subsets with respect to graph γ. It is evident that in graph γ the length of any path is equal to n.

Step 2. For each one element subset which is a vertex of graph γ write its single linear ordering.

Step 3. Let B be a k-element subset ($k = 2, \ldots, n$) which is a vertex of graph γ. Assume we have a linear ordering for each subset of the form $B \backslash a$, where a is a maximal element of B. Then we preserve these linear orderings for elements belonging $B \backslash a$ and set $\varphi(a) = k$.

Step 4. As the final step of this algorithm we obtain all linear orderings for set A which is a vertex of graph γ.

An example for finding of all completions of ordered set in section 5 is given.

2.2. Ideals of ordered set

Definition 3. Let $\langle A, \leq \rangle$ be an arbitrary ordered set. A subset $B \subseteq A$ is called an *ideal* in $\langle A, \leq \rangle$ if the following condition

$$a \in B, a' \leq a \Rightarrow a' \in B$$

holds. For any subset $X \subseteq A$ we define a set of its minorants $X^{\downarrow}$ by setting

$$X^{\downarrow} = \{a \in A\colon\ (\exists x \in X)\, a \leq x\}\,. \tag{5}$$

For any $X \subseteq A$, subset $X^{\downarrow}$ is the smallest (under inclusion) ideal which contains X; if X is an ideal then $X^{\downarrow} = X$. It is said that $X^{\downarrow}$ is the ideal generated by subset X. Particularly an ideal generated by one element subset $\{a\}$ is called a *main ideal* and denoted by $a^{\downarrow}$. A mapping $X \to X^{\downarrow}$ which every subset $X \subseteq A$ put in correspondence the ideal generated by this subset is a closure operation, hence the set $Id\,(A)$ of all ideals of ordered set $\langle A, \leq \rangle$ forms (under inclusion) a complete lattice in the sense (Birkhoff, 1967). Since the intersection and the union of any family of ideals is an ideal also then the lattice of ideals $Id\,(A)$ is distributive. We now indicate some method for construction of the lattice $Id\,(A)$.

Theorem 1. *Let $\langle A, \leq \rangle$ be a finite ordered set. Then*

1. A subset which is a result of extraction from ideal its maximal element is an ideal also;

2. Any ideal can be realized from ideal A with help of procedure of extraction of maximal elements by a finite number steps.

Proof (of theorem 1). **1.** Let $B \subseteq A$ be an ideal of the ordered set $\langle A, \leq \rangle$ and $b^* \in B$ a maximal element of B. Show that $B \backslash b^*$ also is an ideal. Indeed suppose $a \in B \backslash b^*$ and $a' \leq a$; since subset B is an ideal and $a \in B$ then $a' \in B$. Assumption $a' = b^*$ implies $b^* \leq a$. The equality $b^* = a$ is false since we obtain $b^* \in B \backslash b^*$. Then $b^* < a$ that is impossible for maximal element $b^* \in B$. Thus $a' \neq b^*$ hence $a' \in B \backslash b^*$.

We now state the following lemma.

Lemma 2. *Let $B \subseteq A$ be an ideal in ordered set $\langle A, \leq \rangle$ and $B \neq A$. Then there exists such element $a_1 \in A \backslash B$ that*

α) subset $B \cup \{a_1\}$ is an ideal and

β) the element a_1 is a maximal one in $B \cup \{a_1\}$.

Proof (of lemma 2). Consider any ideal B in ordered set $\langle A, \leq \rangle$ where $B \neq A$. Fix some minimal element a_1 of non-empty set $A \backslash B$. Check that $B \cup \{a_1\}$ is an ideal. Suppose $a \in B \cup \{a_1\}$ and $a' < a$. If $a \in B$ then $a' \in B$ by definition of ideal hence $a' \in B \cup \{a_1\}$. In the case $a = a_1$ assume $a' \notin B$. Then $a' \in A \backslash B$ and we have $a_1 > a'$ that is false since element a_1 is minimal in $A \backslash B$. Thus $a' \in B \subseteq B \cup \{a_1\}$ and α) is proved. Show β). The assumption $b > a_1$ for some $b \in B$ implies by definition of ideal the inclusion $a_1 \in B$ in contradiction with $a_1 \in A \backslash B$ and lemma 2 is proved. □

We now prove the proposition **2** of theorem 1. Let $B \subseteq A$ be an ideal in ordered set $\langle A, \leq \rangle$ and $B \neq A$. By lemma 2 there exists such element $a_1 \in A \backslash B$ that the subset $B \cup \{a_1\}$ is an ideal and a_1 is a maximal element in $B \cup \{a_1\}$. If $B \cup \{a_1\} = A$ then the ideal B is a result of extraction of maximal element a_1 from A and our proposition is proved. If $B \cup \{a_1\} \neq A$ then using lemma 2 once more we obtain that there exists such $a_2 \in A \backslash (B \cup \{a_1\})$ that the subset $B \cup \{a_1, a_2\}$ is an ideal and a_2 is a maximal element in $B \cup \{a_1, a_2\}$. Consider two cases: $B \cup \{a_1, a_2\} = A$ and $B \cup \{a_1, a_2\} \neq A$ etc. Since the set A is finite, we have a sequence of the kind $\{a_1, a_2, \ldots, a_t\}$ where $a_s \in A \backslash (B \cup \{a_1, \ldots, a_{s-1}\})$ and the following conditions hold ($s = 1, \ldots, t$):

α^*) $B \cup \{a_1, \ldots, a_s\}$ is an ideal;

β^*) a_s is a maximal element in $B \cup \{a_1, \ldots, a_s\}$;

γ^*) $B \cup \{a_1, \ldots, a_t\} = A$.

Thus the ideal B is a result of extraction of maximal elements $\{a_t, a_{t-1}, \ldots, a_1\}$ from a chain of ideals starting of A which was to be proved. Finally for $B = A$ the proposition **2** of Theorem 1 is evident since in this case the required number of extractions of maximal elements is equal to zero. □

According to theorem 1, we remark that vertexes of an auxiliary graph γ are precisely ideals of ordered set $\langle A, \leq \rangle$. Hence we obtain

Corollary 1. *We can identify auxiliary graph γ of ordered set $\langle A, \leq \rangle$ with lattice $Id(A)$ of its ideals $Id(A)$. Namely, the set of vertexes of graph γ coincides with the set of ideals and the canonical order relation of lattice $Id(A)$ can be presented as following: $B_1 \supseteq B_2$ if and only if there exists a path from B_1 to B_2 in graph γ.*

We now remark that for finite ordered set $\langle A, \leq \rangle$ the procedure of finding its linear orderings can be reduced to finding of maximal chains in the lattice of ideals $Id(A)$. Further using the indicated algorithm for construction of linear orderings, we have

Corollary 2. *For finite ordered set $\langle A, \leq \rangle$, there exists one-one correspondence between its linear orderings and maximal chains in the lattice $Id\,(A)$ of its ideals. Hence the number of linear completions of an ordering preferences structure $\langle A, \leq \rangle$ coincides with the number of maximal chains in the lattice $Id\,(A)$.*

Remark 2. Indicated correspondence can be realized in the following manner. For linear ordering $\{a_{i_1} < a_{i_2} < \ldots < a_{i_n}\}$ of A, the corresponding maximal chain in the lattice of ideals is

$$\{\emptyset \subset \{a_{i_1}\} \subset \{a_{i_1}, a_{i_2}\} \subset \ldots \subset \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}\}.$$

3. A valuation for a number of linear orderings

3.1. A finding of the number of linear ordering with help of auxiliary graph γ

Using inductive algorithm for construction of auxiliary graph γ (see 2.1.), we can find the number $N\,(A)$ of all linear orderings of a finite ordered set without of finding of these orderings. As a first step, we need to construct the auxiliary graph γ. Denote by $N\,(B)$ the number of all linear orderings for arbitrary set B which is a vertex of graph γ. Since linear orderings of B are extensions of linear orderings of conjugate with B vertexes then we obtain the following recurrent formula:

$$N\,(B) = \sum N\,(B \backslash a)\,, \tag{6}$$

where a is an arbitrary maximal element of subset B. Since every subset which is a final vertex in graph γ is one element hence it has a single linear ordering. Using formula (6) we can find the number of all linear orderings for any vertex of graph γ. In particular we can find the required number $N\,(A)$. An example for count of the number of all linear orderings of ordered set in section 5 will be given.

3.2. A valuation of the number of linear orderings for some special cases

Remark that formula (6) for finding of $N\,(A)$ can be used only in the case the graph γ (i.e. the lattice of ideals of ordered set) is given. However a practical construction of the graph γ for ordered set which contains some tens of elements is very hard. Further we consider certain methods for finding $N\,(A)$ in some special cases. Let $\langle A_k, \omega_k \rangle$ $(k = 1, \ldots, r)$ be a family of ordered sets and $\langle A, \omega \rangle$ is the discrete sum of this family. Denote by N_{dis} the number of all linear orderings for $\langle A, \omega \rangle$. Then we have the following formula (see Rozen, 2013):

$$N_{dis} = \frac{n!}{n_1! n_2! \ldots n_r!} N_1 \cdot N_2 \cdot \ldots \cdot N_r, \tag{7}$$

where $n_k = |A_k|\,(k = 1, \ldots, r)\,, n = \sum\limits_{k=1}^{r} n_k$.

This formula is proved by induction on r. For $r = 1$ the right part of (7) is equal to $N_1 = N_{dis}$. Let us show that (7) is truth for $r = 2$. Indeed, consider two ordered sets A and B, where the first set contains n_1 elements and the second set n_2 elements. Let $(a_1, a_2, \ldots, a_{n_1})$ and $(b_1, b_2, \ldots, b_{n_2})$ be their linear orderings, respectively. Then we can obtain a linear ordering for discrete sum $A \cup B$ in the following manner. Fix a subset $\{i_1, i_2, \ldots, i_{n_1}\}$ in the set $\{1, 2, \ldots, n_1 + n_2\}$ and let $\{i_{n_1+1}, \ldots, i_{n_1+n_2}\}$ be its complement (suppose these sequences are increasing). Then by setting $\varphi(a_s) = i_s$ $(s = 1, \ldots, n_1)$ and $\varphi(b_t) = i_{n_1+t}$ $(t = 1, \ldots, n_2)$ we obtain a linear ordering of discrete sum $A \cup B$. Hence every pair of linear orderings of A and B generates $C^{n_1}_{n_1+n_2} = \frac{(n_1+n_2)!}{n_1!n_2!}$ of linear orderings for their discrete sum $A \cup B$. Denote by N_1 the number of linear orderings of A and by N_2 the number of linear orderings of B, then the number of pairs of linear orderings is equal to $N_1 \cdot N_2$; thus we obtain $\frac{(n_1+n_2)!}{n_1!n_2!} N_1 \cdot N_2$ of linear orderings for $A \cup B$. For $r = 2$ formula (7) is shown. Remark now that discrete sum of r ordered sets can be represented as a discrete sum of two ordered sets: $A_1 \cup \ldots \cup A_{r-1} \cup A_r = (A_1 \cup \ldots \cup A_{r-1}) \cup A_r$. Using our assumption for $r = 2$, we obtain the required proposition in general case, that is (7).

As a corollary of formula (7) we now obtain a valuation for $N(A)$ in the case A is a tree ordered set. Consider a tree T with a root a_0. Then we can define on the set A of tree vertexes the tree order by the rule: $a_1 \le a_2$ if and only if there exists a path from a_1 to a_2. Remark that a_0 is the greatest element under order $\le$. For each $a_k \in A$ the set T_{a_k} consisting of vertexes $a \le a_k$ forms a tree with root a_k; it is called *subtree with root* a_k. Particularly, $T = T_{a_0}$.

Corollary 3. *Let T_{a_0} be a tree and $\{T_{a_0}, T_{a_1}, \ldots, T_{a_r}\}$ all its subtrees having not less than two vertexes. Then a number $N(T_{a_0})$ of all linear orderings of tree T_{a_0} is defined by formula:*

$$N_{T_{a_0}} = \frac{|T_{a_0}|!}{|T_{a_0}| \cdot \ldots \cdot |T_{a_r}|}, \tag{8}$$

where $|T_{a_k}|$ denotes a number of elements of subtree T_{a_k}.

Proof of corollary 3 is given by induction on numbers of levels of tree. To prove induction step one can use that if to eliminate the greatest element of tree then we obtain a discrete sum of tree orders for which formula (7) is true, and the number of linear orderings for these tree orders can be founded by assumption (8).

4. A contraction of the set of linear completions

We now consider a problem of contraction of the set of linear completions for ordering preferences structures which based on some additional information concerning of preferences. Suppose an ordering preferences structure in the form $\langle A, \omega \rangle$ is given where ω is an order relation on the set of alternatives A.

We consider here additional information of the following types.

Type 1: Information under strict preferences

This information with binary relation $\delta \subseteq A^2$ can be given where the assertion $(a_1, a_2) \in \delta$ means that alternative a_2 is strict better than alternative a_1. Such information does not contradict with an ordering ω if and only if the relation $\omega \cup \delta$ is acyclic; in this case $\omega_1 = tr(\omega \cup \delta)$ is an ordering of A which contains previous ordering ω and the relation δ also. Further finding linear completions of ordering ω_1 we obtain some part of all linear completions for ordering ω. Completions of ordering ω_1 are completions for ordering ω which conform with additional information in the form of binary relation δ.

Type 2: Information under indifference relation

In this case, additional information in the form of an equivalence relation $\varepsilon \subseteq A^2$ is given.

Definition 4. Let φ be an isotonic function from ordered set A in some chain C. Put $\varepsilon_\varphi = \{(a_1, a_2) : \varphi(a_1) = \varphi(a_2)\}$. An equivalence $\varepsilon \subseteq A^2$ is said to be *ranged equivalence* if $\varepsilon = \varepsilon_\varphi$ for some isotonic function φ.

We assume here that ε is a ranged equivalence. Then we factorize relation ω under equivalence ε and obtain factor-ordering $Tr(\omega/\varepsilon)$ on the factor-set A/ε. Let $C_1, \dots, C_r$ be all classes of equivalence ε and ω_k is a restriction of ω on subset C_k $(k = 1, \dots, r)$; put N_k the number of linear completions for ω_k and $N(\omega)$ the number of linear completions for $\langle A, \omega \rangle$. Then we have the following evaluation for the number $N(\omega, \varepsilon)$ of linear completions for the factor-structure:

$$N(\omega, \varepsilon) \leq \frac{N(\omega)}{N_1 \cdot N_2 \cdot \ldots \cdot N_r}. \tag{9}$$

The inequality (9) shows that additional information of type 2 implies a strong contraction for the number of linear completions.

Remark 3. Conditions concerning of equivalence $\varepsilon \subseteq A^2$ under which there exists the unique linear completions for factor-structure A/ε (i.e. $N(\omega, \varepsilon) = 1$) is given in (Rozen, 2011).

5. Examples

Example 1. **Finding of all linear completions for ordering preferences structures**

Consider an ordering preferences structure consisting of 6 alternatives $A = \{a, b, c, d, e, f\}$ presented by a diagram (fig. 1).

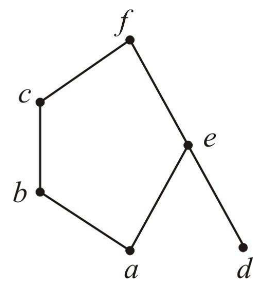

Fig. 1

To construct all its linear orderings, we need in the following steps.

Step 1. Using a procedure of extraction of maximal elements (see 2.1.), we obtain an auxiliary graph γ (fig. 2).

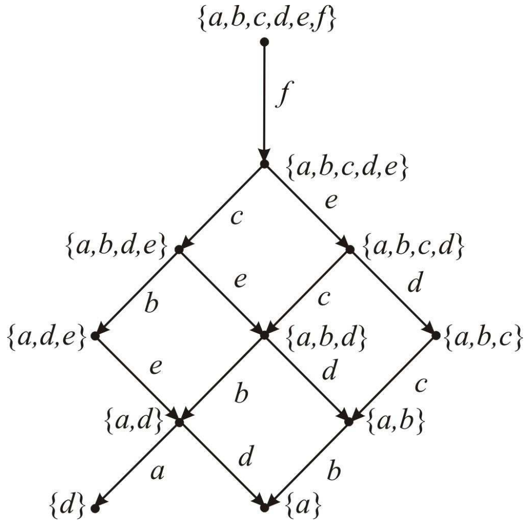

Fig. 2

Step 2. We now construct Table 1 whose rows are vertexes of graph γ (i.e. ideals) and for each ideal all its linear completions are given. Starting of one element ideals, we receive at last linear orderings of ideal $A = \{a, b, c, d, e, f\}$ in lower block of Table 1.

Table 1.

	a	b	c	d	e	f		a	b	c	d	e	f
$\{a\}$	1						$\{a,b,c,d,e\}$	1	4	5	2	3	
$\{d\}$				1				2	4	5	1	3	
$\{a,d\}$	1			2				1	3	5	2	4	
	2			1				2	3	5	1	4	
$\{a,b\}$	1	2						1	2	5	3	4	
$\{a,d,e\}$	1			2	3			1	3	4	2	5	
	2			1	3			2	3	4	1	5	
$\{a,b,d\}$	1	3		2				1	2	4	3	5	
	2	3		1				1	2	3	4	5	
	1	2		3									
$\{a,b,c\}$	1	2	3				$A=\{a,b,c,d,e,f\}$	1	4	5	2	3	6
$\{a,b,d,e\}$	1	4		2	3			2	4	5	1	3	6
	2	4		1	3			1	3	5	2	4	6
	1	3		2	4			2	3	5	1	4	6
	2	3		1	4			1	2	5	3	4	6
	1	2		3	4			1	3	4	2	5	6
$\{a,b,c,d\}$	1	3	4	2				2	3	4	1	5	6
	2	3	4	1				1	2	4	3	5	6
	1	2	4	3				1	2	3	4	5	6
	1	2	3	4									

Step 3. All linear orderings which are completions of the ordering preferences structure in fig. 1 on the following diagram are given (fig. 3)

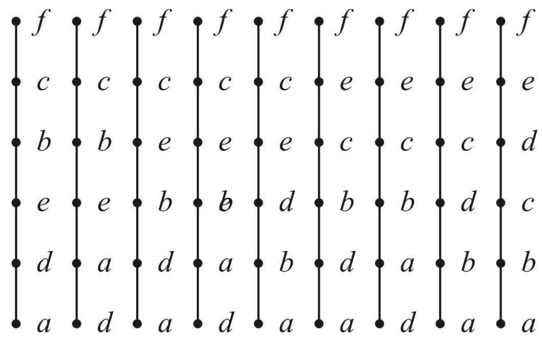

Fig. 3

Example 2. **A count of the number of all linear completions for ordering preferences structure**

Consider an ordering preferences structure in the fig. 1. To count all its linear completions we need in construction of auxiliary graph γ only. Since each subset which is a final vertex of graph γ consists of one element, it has single linear completion, hence we write 1 near every final vertex of graph γ (fig. 4). Further we write a number $N(B)$ near others vertexes B of graph γ in accordance with formula (6). The number $N(B)$ indicates a number of all linear completions for subset B. In particularly, $N(A)$ is a number of all completions for the set of all alternatives $A = \{a, b, c, d, e, f\}$.

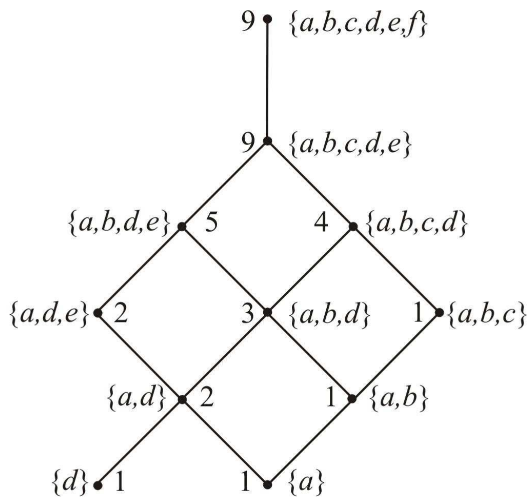

Fig. 4

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