Stable Cooperation in Graph-Restricted Games*

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Abstract In the paper we study stable coalition structures in the games with restrictions on players' cooperation and communication. Restriction on cooperation among players is given by a coalition structure, whereas restriction on their communication is described by a graph. Having both a coalition structure and a graph fixed, a payoff distribution can be calculated based on worth of each coalition of players. We use the concept of stability for a coalition structure similar to Nash stability, assuming that the graph structure is fixed. The results are illustrated with examples.

Keywords: cooperation, coalition structure, graph, characteristic function, stability, Shapley value, Myerson value, ES-value.

1. Introduction

In cooperative games with coalition structure it is supposed that players belonging to one coalition can communicate with each other. However, some interactions among players in one coalition may be not possible due to several reasons. Such restrictions lead us to consider games with coalition structure and constrained communication. In the paper, we associate constrained communication with an undirected graph showing the structure of players' communication. In addition, not all coalition structures are appropriate for all players. For this purpose we try to find "proper" or "stable" in some sense coalition structures from the set of all coalition structures. When we talk about stability of the coalition structure, we mean stability in the sense of players payoffs, prescribed by a single-valued solution—a rule that assigns a single payoff distribution to each game. Specifically, we consider a rule based on the Shapley value (Shapley, 1953) as a solution in such a game. Moreover, the stable solution should satisfy the property of individual rationality.

In (Aumann and Drèze, 1974) games with a coalition structure were considered, and the solution called the Aumann–Drèze value, based on the Shapley value, was proposed. On the contrary, in graph-restricted games proposed in (Myerson, 1977), the Myerson value was introduced as a solution, and it was also based on the Shapley value. A solution, based on a combination of both types of restriction—coalition structure and a graph, was introduced in (Vázquez-Brage et al., 1996). Such a solution is a generalization of Owen value (Owen, 1977) and the Myerson value. Other solutions, e.g. (Khmelnitskaya, 2010), were developed later.

In (Hart and Kurz, 1983) the concept of stability for a coalition structure in a strategic setting was introduced. Other stability concepts were studied later in (Haeringer G., 2001), (Tutic, 2010). In the present study we use a concept similar

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to Nash stability, proposed in (Bogomolnaia and Jackson, 2002), supposing that players' payoffs are defined based on some cooperative solution.

Characteristic function estimating the worth of any coalition, plays the key role in the game. Note that in our discussion we do not claim the superadditivity of this function. If characteristic function is superadditive in the game with restrictions on players' cooperation, it is obvious that the grand coalition is always stable, but other coalition structures may be stable as well. However, in non-superadditive case, we cannot a priori determine stable coalition structures without analyzing properties of the characteristic function. It is worth mentioning that existence of the stable coalition structure for an arbitrary characteristic function was proved in (Sedakov et al., 2013) for at most three-person case, and it was shown that for more than three players we cannot guarantee the existence of such structures. Stability of the coalition structure was considered for both the Shapley value and the ES-value[1] (or the CIS-value[2]) (Driessen and Funaki, 1991). In the paper we find stable coalition structures, provided that communication among players is restricted by a fixed graph. Unlike the idea in (Caulier et al., 2013), where the stable network is determined, provided that coalition structure is given, we use the opposite idea. Specifically, we try to determine stable coalition stricture, assuming that players' communication is restricted by an a priori given graph structure.

The paper is organized as follows. In Section 2 we introduce the game with coalition structure and restricted cooperation by a undirected graph. Then, in Section 3 we define the stable coalition structure, whereas in Section 4 we find stable coalition structures in a game with the major player.

2. Cooperative game with coalition structure restricted by graph

Consider the class of cooperative games with coalition structure determined as follows:

Definition 1. Cooperative game with coalition structure is a system $\Gamma = (N, v, \pi)$, where $N = \{1, \ldots, n\}$ is the set of players, $v : 2^N \to R$ is a characteristic function with $v(\varnothing) = 0$ and π is a coalition structure $\pi = \{B_1, \ldots, B_m\}$, i.e. $B_1 \cup \ldots \cup B_m = N$, $B_i \cap B_j = \varnothing$ for all $i, j \in N, i \neq j$.

Suppose that communication among players can be restricted by a graph g consisting of finite set of nodes which is the set of players N and the set of links. If players i and j are linked, then $\{ij\} \in g$. Denote the complete graph by g^N.

Given characteristic function $v(S)$ and graph g which restricts communication among players, determine "new" characteristic function using the following approach (Myerson, 1977):

$$v^g(S) = \sum_{T \in S/g} v(T), \tag{1}$$

where S/g is a unique partition of S s.t. $S/g = \{\{i \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} | j \in S\}$. Construction of characteristic function v^g using this approach has a useful property: if characteristic function v is superadditive, then v^g is also superadditive. Denote a cooperative game with coalition structure π restricted by graph g as $\Gamma^g = (N, v^g, \pi)$.

[1] The equal surplus division value.
[2] Center of gravity of the imputation set.

Example 1. (Coauthor model) The model represents the work of researchers who write papers. The link in the graph means players work on a joint paper. Coalition structure can be interpreted as a partition of the players among research institutes. In Fig. 1 we can see an example of coalition structure $\pi = \{\{1,2\},\{3,4,5\}\}$ with a given graph $g = \{12, 23, 34, 25\}$ with the set of players $N = \{1,2,3,4,5\}$.

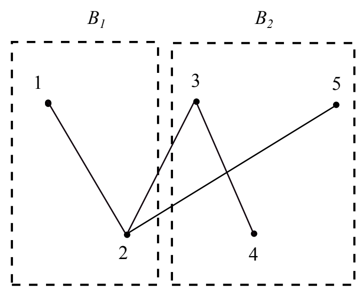

Fig. 1: Coalition structure with a given graph for Example 1.

Definition 2. An n-dimensional profile $x = (x_1, \ldots, x_n) \in R^n$ is a payoff distribution in game Γ^g if it is efficient, i. e. for all $B_j \in \pi$:

$$\sum_{i \in B_j} x_i = v^g(B_j).$$

Definition 3. Payoff distribution x is an allocation in game Γ^g with coalition structure π if for all $i \in N$ it is individually rational:

$$x_i \geq v^g(\{i\}).$$

A solution (or a cooperative solution) is the rule which prescribes a subset of the n-dimensional space for each game Γ^g. If the prescribed subset consists of one point, the solution is called the single-valued. We consider the Myerson value and the ES-value as single-valued cooperative solutions. Let $B(i) \in \pi$ be the coalition containing player $i \in N$. The Myerson value is the payoff distribution $\mu = (\mu_1, \ldots, \mu_n)$ which can be calculated by formula:

$$\mu_i = \sum_{S \subseteq B(i), i \in S} \frac{(|B(i)| - |S|)!(|S| - 1)!}{|B(i)|!} [v^g(S) - v^g(S \setminus \{i\})] \quad (2)$$

for all $i \in N$. The ES-value $\psi = (\psi_1, \ldots, \psi_n)$ is defined as:

$$\psi_i = v^g(\{i\}) + \frac{v^g(B(i)) - \sum_{j \in B(i)} v^g(\{j\})}{|B(i)|}. \quad (3)$$

3. Stable coalition structure

Consider the problem of choosing stable coalition structure π, i.e. the structure which is stable against the individual player's deviations when other players do not deviate from structure π.

Definition of stable coalition structure with respect to a solution for cooperative game Γ was proposed in (Parilina and Sedakov, 2012), and can be reformulated for the game Γ^g restricted by graph g in the following way:

Definition 4. Coalition structure $\pi = \{B_1, \ldots, B_m\}$ is stable with respect to a solution if for each $i \in N$ the following inequality holds:

$$x_i \geq x_i',$$

where x, x' are payoff distributions calculated according to the chosen solution for games (N, v^g, π), (N, v^g, π') respectively, $\pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$ for any $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-B_i} = \pi \setminus B_i \subset \pi$, i.e. $B(i) \in \pi$ is the coalition which contains i.

Contrary to the concept of network stability in the games with fixed coalition structure (Caulier et al., 2013), we propose the model in which graph structure is fixed and all possible coalition structures can be realized. The problem of stability of a coalition structure arises. The main distinction in these two approaches is in what should be chosen as an unchangeable property of a game, graph vs. coalition structure.

If graph g is complete, i.e. $g = g^N$, the following proposition directly follows from (Sedakov et al., 2013).

Proposition 1. *If graph g is complete and $|N| \leqslant 3$, there always exists a stable coalition structure with respect to the Myerson value and ES-value.*

Unfortunately, stable coalition structure with respect to the ES-value may not exist in a case of more than 3 players (Sedakov et al., 2013).

Now consider a general case of graph g, and superadditive characteristic function v.

Proposition 2. *If characteristic function v^g is superadditive, then coalition structure $\{N\}$ is stable relative to the Myerson value and the ES-value.*

Proof. Following Definition 4 coalition structure $\{N\}$ turns into structure $\pi' = \{\{N \setminus i\}, \{i\}\}$ if player $i \in N$ deviates. The ith component of the Myerson value and the ES-value in coalition structure π' is equal to $v^g(\{i\})$ which is not larger than the ith component of the corresponding payoff distributions in coalition structure π because of superadditivity of characteristic function v^g. □

Proposition 3. *If characteristic function v^g is superadditive and graph g is split up into a finite number of connected components, i.e. $g = g_1 \cup g_2 \cup \ldots \cup g_k$ where $g_i \cap g_j = \varnothing$ for any $i \neq j$, then coalition structure $\{B_1, \ldots, B_k\}$, where coalition B_l contains all the players connected by component g_l and no players else, is stable with respect to the Myerson value and the ES-value.*

Proof. Consider coalition structure $\pi = \{B_1, \ldots, B_k\}$ described in Proposition. Prove that there are no players who can benefit deviating from structure π. If player $i \in B(i) \in \pi$ deviates, his deviation may lead to the coalition structure in which he plays as an individual player, and his component of the Myerson value or the ES-value will be equal to $v^g(\{i\})$ which is not larger than his component calculated for coalition structure π because of superadditivity of characteristic function.

Player i may also deviate joining some coalition $B_j \in \pi$ which is different from coalition $B(i)$. But his component of the Myerson value or the ES-value will be $v^g(\{i\})$ because he does not have any connections with players from B_j which follows from graph g definition. Therefore, coalition structure π is stable with respect the Myerson or the ES-values. □

4. Game with the major player

Let the set of players be $N = \{1, \ldots, n\}$, $|N| = n \geqslant 2$. Suppose there is the major player referred as player 1, and all other players, i.e. players from set $N \setminus \{1\}$, are supposed to be symmetric. It means that for any coalition $S \subseteq N$ and any $i, j \in N \setminus \{1\}, i \neq j$ the following condition is satisfied: $v(S) - v(S \setminus \{i\}) = v(S) - v(S \setminus \{j\})$. The worth of coalition S, $v(S)$, is interpreted as its work efficiency and defined as

$$v(S) = \begin{cases} 0, & \text{if } S = \{i\}, i \in N \setminus \{1\}, \\ \gamma s, & \text{if } s > 1, 1 \notin S, \\ \alpha(s-1) + \dfrac{\beta}{s}, & \text{if } s \geqslant 1, 1 \in S, \end{cases} \quad (4)$$

where $s = |S|$ is a number of players in coalition S, and α, β, γ are positive parameters satisfying the conditions $\gamma \leqslant \alpha \leqslant \beta$. The value of characteristic function (4) for singleton $S = \{i\}$, $i \in N \setminus \{1\}$ represents that this particular player gains nothing working alone. Parameter β is the work efficiency of the major player. If s players from the set $N \setminus \{1\}$ cooperate, they could gain proportionally to the size of coalition with coefficient γ, i.e. γs. Here parameter γ is an efficiency of one player if he cooperates with players from $N \setminus \{1\}$. If the major player belongs to coalition S, the efficiency of coalition S is directly proportional to the number of players from $N \setminus \{1\}$ in coalition S, and inversely proportional to the number of players in S (since the major player works himself as well as controls the others).

Suppose that communication among players is restricted by graph g which is a "star": the major player can communicate with all other players, whereas any player from $N \setminus \{1\}$ communicates only with the major player (see Fig. 2).

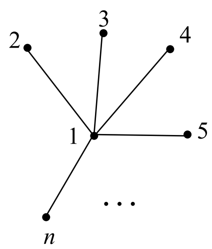

Fig. 2: Graph g restricting communication in a game with the major player.

The value of characteristic function v^g restricted by graph g becomes simpler than v, and using (1) and (4) can be written as follows:

$$v^g(S) = \begin{cases} 0, & \text{if } 1 \notin S \\ \alpha(s-1) + \dfrac{\beta}{s}, & \text{if } 1 \in S. \end{cases} \quad (5)$$

Proposition 4. *If $\alpha \in [\beta/2, \beta]$, characteristic function* (5) *is superadditive.*

Proof. Prove that for any disjoint coalitions $S, T \subset N$: $v^g(S \cup T) \geqslant v^g(S) + v^g(T)$. If both coalitions S and T do not contain the major player, this inequality holds. Let coalition S contain the major player, i.e. $1 \in S$ and $|S| = s$, $|T| = t$. The superadditivity condition can be rewritten into the system:

$$\begin{cases} \alpha(s-1) + \dfrac{\beta}{s} \leqslant \alpha(s+t-1) + \dfrac{\beta}{s+t}, & \text{for any } S,T : s \geqslant 2, t \geqslant 1, s+t \leqslant n, \\ \beta \leqslant \alpha t + \dfrac{\beta}{t+1}, & \text{for any } T : 1 \leqslant t \leqslant n-1, \end{cases}$$

The system of inequalities above has a solution if

$$\alpha \geqslant \beta \cdot \max_{\substack{s \geqslant 2, t \geqslant 1, \\ s+t \leqslant n}} \left\{ \frac{1}{1+t}, \frac{s}{s+t} \right\} = \beta \cdot \max_{\substack{s \geqslant 2, t \geqslant 1, \\ s+t \leqslant n}} \left\{ \frac{1}{t+1}, \frac{1}{1+\frac{t}{s}} \right\} = \frac{\beta}{2}.$$

Taking into account restriction $\alpha \leqslant \beta$, we obtain $\beta/2 \leqslant \alpha \leqslant \beta$. □

Next proposition gives the explicit formulas for the Myerson value and the ES-value.

Proposition 5. *In the game with the major player determined by characteristic function* (5)*, the components of the Myerson value $\mu = (\mu_1, \ldots, \mu_n)$ have the form:*

$$\mu_i = \begin{cases} \dfrac{(n-1)\alpha}{2} + \dfrac{H_n \beta}{n}, & \text{if } i = 1 \\ \dfrac{\alpha}{2} - \dfrac{(H_n - 1)\beta}{n(n-1)}, & \text{if } i \neq 1, \end{cases} \tag{6}$$

where $H_n = \sum_{k=1}^{n} 1/k$.

The ES-value is

$$\psi_i = \begin{cases} \dfrac{(n-1)\alpha}{n} + \dfrac{(n^2 - n + 1)\beta}{n^2}, & \text{if } i = 1 \\ \dfrac{(n-1)\alpha}{n} - \dfrac{(n-1)\beta}{n^2}, & \text{if } i \neq 1. \end{cases} \tag{7}$$

Moreover, μ_i, $i \in N \setminus \{1\}$ and ψ_i, $i \in N$ are increasing functions of n.

Proof. First, calculate the Myerson value for the game with characteristic function (5). The component μ_i of player $i \in N \setminus \{1\}$ is:

$$\mu_i = \sum_{s=2}^{n} \frac{(n-s)!(s-1)!}{n!} C_{n-1}^{s-2} \left[\alpha - \frac{\beta}{s(s-1)} \right] = \frac{\alpha}{2} - \frac{(H_n - 1)\beta}{n(n-1)}.$$

The component of the major player is

$$\mu_1 = \sum_{s=2}^{n} \frac{(n-s)!(s-1)!}{n!} C_{n-1}^{s-1} \left[\alpha(s-1) - \frac{\beta}{s} \right] = \frac{(n-1)\alpha}{2} + \frac{H_n \beta}{n}.$$

Second, calculate the ES-value for this game. Following formula (3), we obtain:

$$\psi_i = \frac{\alpha(n-1) + \beta/n - \beta}{n} = \frac{(n-1)\alpha}{n} - \frac{(n-1)\beta}{n^2}, \quad i \in N \setminus \{1\}$$

and for the major player:

$$\psi_1 = \beta + \frac{\alpha(n-1) + \beta/n - \beta}{n} = \frac{(n-1)\alpha}{n} + \frac{(n^2 - n + 1)\beta}{n^2}.$$

Now prove that μ_i, $i \in N$, $i \neq 1$ is an increasing function of n. For this purpose, find the difference between components $\mu_i(n+1)$ and $\mu_i(n)$ calculated for games with $n+1$ and n players respectively. We can easily show that for all $n \geqslant 2$ this difference is positive:

$$\mu_i(n+1) - \mu_i(n) = -\frac{(H_{n+1} - 1)\beta}{n(n+1)} + \frac{(H_n - 1)\beta}{n(n-1)} = \frac{(2(H_n - 1) - \frac{n-1}{n+1})\beta}{n(n+1)(n-1)} > 0$$

because $H_n - 1 \geqslant 1/2$ and $(n-1)/(n+1) < 1$. It proves that μ_i, $i \in N \setminus \{1\}$ is an increasing function on n.

Similarly, consider the difference between major player's components of the ES-value in games with $n+1$ and n players respectively:

$$\psi_1(n+1) - \psi_1(n) = \frac{\alpha}{n(n+1)} + \beta\left(\frac{1}{n(n+1)} + \frac{2n+1}{n^2(n+1)^2}\right) > 0,$$

and the difference between components of the ES-value of any other player:

$$\psi_i(n+1) - \psi_i(n) = \frac{\alpha}{n(n+1)} + \beta\frac{n^2 - n - 1}{n^2(n+1)^2} > 0.$$

□

Examine the problem of stability of coalition structures with respect to the Myerson value and ES-value. First, consider the case of the Myerson value, second, the case of the ES-value.

Proposition 6. *If* $\frac{2\beta}{n-1}(1 - H_n/n) \leqslant \alpha \leqslant \beta$, *coalition structure* $\{N\}$ *is stable with respect to the Myerson value. If* $\alpha \leqslant \beta/2$, *coalition structure* $\{\{1\}, B_2, \ldots, B_m\}$ *is stable with respect to the Myerson value. No other stable coalition structures are possible.*

Proof. Coalition structure $\{N\}$ is stable when the following system has a solution:

$$\begin{cases} \dfrac{(n-1)\alpha}{2} + \dfrac{H_n\beta}{n} \geqslant \beta, \\ \dfrac{\alpha}{2} - \dfrac{(H_n - 1)\beta}{n(n-1)} \geqslant 0. \end{cases} \qquad (8)$$

The first (second) inequality corresponds to the condition that the major (any other) player does not benefit deviating from coalition $\{N\}$ and playing individually. The system (8) is equivalent to the condition:

$$\alpha \geqslant \frac{2\beta}{n-1} \max\left\{\frac{H_n - 1}{n}; 1 - \frac{H_n}{n}\right\} = \frac{2\beta}{n-1}\left(1 - \frac{H_n}{n}\right),$$

and given restriction $\alpha \leqslant \beta$, we obtain the result of the proposition. It is easy to show that the lower bound $\frac{2\beta}{n-1}(1 - H_n/n)$ tends to zero as n tends to infinity.

Consider coalition structure $\{\{1\}, B_2, \ldots, B_m\}$, where $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m|$. This coalition structure is stable when the following system has a solution:

$$\begin{cases} \beta \geqslant \dfrac{b_j\alpha}{2} + \dfrac{H_{b_j+1}\beta}{b_j+1}, \quad j = 2, \ldots, m \\ 0 \geqslant \dfrac{\alpha}{2} - \dfrac{(H_2-1)\beta}{2}, \end{cases} \tag{9}$$

where $b_j = |B_j|$. The first inequality can be rewritten into the following:

$$\alpha \leqslant \min_{j=2,\ldots,m} \frac{2\beta}{b_j}\left(1 - \frac{H_{b_j+1}}{b_j+1}\right). \tag{10}$$

Since the right part of (10) is the increasing function of b_j and it exceeds $\beta/2$, the solution of (9) is

$$\alpha \leqslant \min\left\{\frac{\beta}{2}, \min_{j=2,\ldots,m} \frac{2\beta}{b_j}\left(1 - \frac{H_{b_j+1}}{b_j+1}\right)\right\} = \min\left\{\frac{\beta}{2}, \frac{2\beta}{b_2}\left(1 - \frac{H_{b_2+1}}{b_2+1}\right)\right\} = \frac{\beta}{2}.$$

Consider coalition structure $\{B_1, B_2, \ldots, B_m\}$, where $|B_1| \geqslant 2$ and $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m|$. Following Definition 4, write the system of constrains guaranteeing stability of this coalition structure:

$$\begin{cases} \dfrac{\alpha(b_1-1)}{2} + \dfrac{\beta H_{b_1}}{b_1} \geqslant \beta, \\ \dfrac{\alpha(b_1-1)}{2} + \dfrac{\beta H_{b_1}}{b_1} \geqslant \max\limits_{j=2,\ldots,m}\left\{\dfrac{\alpha b_j}{2} + \dfrac{\beta H_{b_j+1}}{b_j+1}\right\}, \\ 0 \geqslant \dfrac{\alpha}{2} - \dfrac{\beta(H_{b_1+1}-1)}{b_1(b_1+1)}, \\ \dfrac{\alpha}{2} - \dfrac{\beta(H_{b_1}-1)}{b_1(b_1-1)} \geqslant 0. \end{cases} \tag{11}$$

From the last two inequalities we obtain condition:

$$0 \leqslant \frac{\alpha}{2} - \frac{\beta(H_{b_1}-1)}{b_1(b_1-1)} \leqslant \frac{\alpha}{2} - \frac{\beta(H_{b_1+1}-1)}{b_1(b_1+1)} \leqslant 0,$$

which never holds. Therefore, the system (11) does not have solution. It proves that the only possible stable coalition structures are $\{N\}$ when $\frac{2\beta}{n-1}(1 - H_n/n) \leqslant \alpha \leqslant \beta$ and $\{\{1\}, B_2, \ldots, B_m\}$, where $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m|$ if $\alpha \leqslant \beta/2$. □

Proposition 7. *If $\beta/n \leqslant \alpha \leqslant \beta$, coalition structure $\{N\}$ is stable with respect to the ES-value. If $\alpha \leqslant \beta/(b_m+1)$, coalition structure $\{\{1\}, B_2, \ldots, B_m\}$, $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m| = b_m$, is stable with respect to the ES-value. No other stable coalition structures are possible.*

Proof. Coalition structure $\{N\}$ is stable when neither the major player nor any other player benefits deviating and, therefore, becoming a singleton, i.e. the following system has a solution

$$\begin{cases} \dfrac{\alpha(n-1)}{n} + \dfrac{\beta(n^2-n+1)}{n^2} \geqslant \beta, \\ \dfrac{\alpha(n-1)}{n} - \dfrac{\beta(n-1)}{n^2} \geqslant 0, \end{cases} \tag{12}$$

if $\alpha \geqslant \beta/n$. Taking into account the restriction $\alpha \leqslant \beta$, we obtain $\beta/n \leqslant \alpha \leqslant \beta$.

Consider coalition stricture $\{\{1\}, B_2, \ldots, B_m\}$, $|B_2| \leqslant \ldots \leqslant |B_m|$, $|B_j| = b_j$, $j = 2, \ldots, m$. It is stable if neither the major player benefits joining any coalition from the set $\{B_2, \ldots, B_m\}$, nor any other player i, $i \neq 1$ benefits joining player 1:

$$\begin{cases} \beta \geqslant \max\limits_{j=2,\ldots,m} \left\{ \dfrac{\alpha b_j}{b_j+1} + \dfrac{\beta((b_j+1)^2 - (b_j+1) + 1)}{(b_j+1)^2} \right\}, \\ 0 \geqslant \dfrac{\alpha}{2} - \dfrac{\beta}{2^2}. \end{cases} \tag{13}$$

Using Proposition 5, i.e. that function ψ_1 is an increasing function of the number of coalition members, we rewrite system (13) as:

$$\begin{cases} \beta \geqslant \dfrac{\alpha b_m}{b_m+1} + \dfrac{\beta((b_m+1)^2 - (b_m+1) + 1)}{(b_m+1)^2}, \\ \alpha \leqslant \dfrac{\beta}{2}. \end{cases} \tag{14}$$

and it is equivalent to inequality

$$\alpha \leqslant \min\left\{ \frac{\beta}{2}; \frac{\beta}{b_m+1} \right\} = \frac{\beta}{b_m+1}.$$

It proves the first part of the proposition.

Consider coalition structure $\{B_1, B_2, \ldots, B_m\}$, where $1 \in B_1$, $|B_1| \geqslant 2$ and $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m|$. Following Definition 4, this structure is stable if

$$\begin{cases} \dfrac{\alpha(b_1-1)}{b_1} + \dfrac{\beta(b_1^2 - b_1 + 1)}{b_1^2} \geqslant \beta, \\ \dfrac{\alpha(b_1-1)}{b_1} + \dfrac{\beta(b_1^2 - b_1 + 1)}{b_1^2} \geqslant \max\limits_{j=2,\ldots,m} \left\{ \dfrac{\alpha b_j}{b_j+1} + \dfrac{\beta((b_j+1)^2 - (b_j+1) + 1)}{(b_j+1)^2} \right\}, \\ 0 \geqslant \dfrac{\alpha b_1}{b_1+1} - \dfrac{\beta b_1}{(b_1+1)^2}, \\ \dfrac{\alpha(b_1-1)}{b_1} - \dfrac{\beta(b_1-1)}{b_1^2} \geqslant 0. \end{cases} \tag{15}$$

From the last two inequalities we obtain condition:

$$0 \leqslant \frac{\alpha(b_1-1)}{b_1} - \frac{\beta(b_1-1)}{b_1^2} \leqslant \frac{\alpha b_1}{b_1+1} - \frac{\beta b_1}{(b_1+1)^2} \leqslant 0,$$

which never holds. Therefore, the system (15) does not have solution, and any coalition structure $\{B_1, B_2, \ldots, B_m\}$ s.t. $1 \in B_1$, $|B_1| \geqslant 2$ and $|B_2| \leqslant |B_3| \leqslant \ldots \leqslant |B_m|$, is always unstable. □

In Figures 3 and 4 diagrams of all stable coalition structures with respect to both the Myerson value and the ES-value in the game with the major are shown. Here we see that in the game only two types of stable coalition structures are possible: the grand coalition and any coalition structure in which the major player is the singleton.

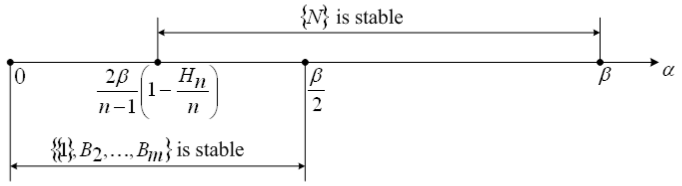

Fig. 3: Diagram of stable coalition structures with respect to the Myerson value in the game with the major player.

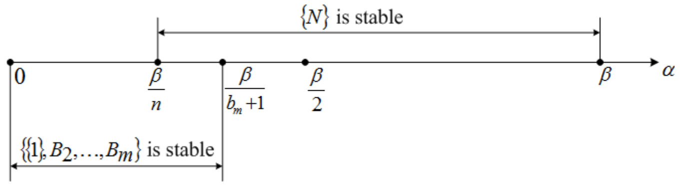

Fig. 4: Diagram of stable coalition structures with respect to the ES-value in the game with the major player.

5. Conclusion

We have considered cooperative games with coalition structures in which communication among players is restricted by graph. Assuming that any possible coalition structure can be realised in the game we suggested a method of verification of a coalition structure on stability. The idea of definition of a stable coalition structure is close to Nash equilibrium. We have proposed a model of the game with the major player in which communication among players is realised via a given "star" graph. For the game we found all possible stable coalition structures. The type of stable coalition structure depends on the parameters of the game.

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