Interval Obligation Rules and Related Results

Osman Palancı[1], Sırma Zeynep Alparslan Gök[1] and Gerhald Wilhelm Weber[2]

[1] *Suleyman Demirel University*
Faculty of Arts and Sciences, Department of Mathematics,
32260 Isparta, Turkey. E-mail: osmanpalanci@sdu.edu.tr
E-mail: zeynepalparslan@yahoo.com
[2] *Middle East Technical University*
Institute of Applied Mathematics,
06531 Ankara, Turkey.
E-mail: gweber@metu.edu.tr

Abstract In this study, we extend the well-known obligation rules by using interval calculus. We introduce interval obligation rules for minimum interval cost spanning tree (micst) situations. It turns out that the interval obligation rule and the interval Bird rule are equal under suitable conditions. Further, we show that such rules are interval cost monotonic and induce population monotonic interval allocation schemes (pmias). Some examples of pmias and interval obligation rules for micst situations are also given.

Keywords: Graphs and networks, minimum cost spanning tree situations, interval data, obligation rules, population monotonic allocation scheme.

1. Introduction

A connection situation arises in the presence of a group of agents, each of which needs to be connected directly or via other agents to a source. If connections among agents are costly, then each agent will evaluate the opportunity of cooperating with other agents in order to reduce costs. In fact, if a group of agents decides to cooperate, a configuration of links which minimizes the total cost of connection is provided by a *minimum cost spanning tree (mcst)*.

The problem of finding an *mcst* can be easily solved by using different algorithms proposed in literature (for example see Graham and Hell, 1985). However, finding an mcst does not guarantee that it is going to be really implemented: agents must still support the cost of the mcst and then a cost allocation problem must be addressed. This cost allocation problem was introduced by Claus and Kleitman, 1973 and has been studied with the aid of cooperative game theory since the basic paper of Bird, 1976.

The special case of a minimization problem where no network is initially presented is old problem for Operations Research (OR). In this context, algorithms to construct a tree connecting every village to the source with minimal total cost is provided in Borůvka, 1926. Later, (Dijksta, 1959; Kruskal, 1956; Prim, 1957)found similar algorithms. A historic overview of this minimization problem can be found in Graham and Hell, 1985. Further, (Claus and Kleitman, 1973) introduced the cost allocation problem for the special case of minimum cost spanning tree problems, in which no network is initially presented. In the sequel Bird, 1976 treated this problem with game-theoretic methods and proposed for each minimum cost spanning tree a cost allocation associated with it. Furthermore, we note that discrete or combinatorial optimization embodies a vast and significant area of combinatorics that interfaces many related subjects. Included among these are linear programming, OR and game theory.

Since then, many authors have noted that this kind of cost allocation problems may arise on many different physical networks such as telephone lines, highways, electric power systems, computer chips, water delivery systems, rail lines etc. On the other hand, numerous studies in the literature have shown that to retrieve the information needed to assess the exact cost of all the links of a real network is a very hard task (Janiak and Kasperski, 2008; Montemanni, 2006; Yaman et.al., 1999; Yaman et.al., 2001). So, we argue that it is more realistic to imagine connection situations where the costs of links are identifiable at a level of uncertainty, i.e., only the range of the costs is known, and no probability information on the realization of costs is given. Such connection situations with uncertain costs may be represented using graphs where the costs associated to the edges are intervals of real numbers.

In this context a practical example of a cost allocation problem is studied in Moretti et.al., 2011 which is inspired by the application suggested by Yaman et.al., 2001. In this example, a design of a telecommunication network of users that want to be connected with a service provider is considered. Here, the agents are the users, the source is the service provider and the cost of a link is proportional to its traffic load. Suppose that routing delays on links are not known with certainty. This uncertainty is caused by the time varying nature of the traffic load of the network. It is then desirable to develop a network that hedges against all possible configurations of the costs, that we will call scenarios, which may occur. On the other hand, the cost of the total traffic load must be shared among users and, consequently, incentives to cooperation should be sustainable before and after the realization of an optimal network.

As in the classical case, where edge costs are real numbers, also in the situation where edge costs are intervals of real numbers, a cost allocation problem arises. With the goal to study this kind of cost allocation problems, in this paper we extend the notion of an obligation rule by using interval calculus, and we study some cost monotonicity properties. It turns out that cost monotonicity, under interval uncertainty, provides a population monotonic interval allocation scheme.

We note that Suijs, 2003 studied *mcst* problems in which the connection costs are represented by random variables. In our paper, costs are not random variables, but instead, they are closed and bounded intervals of real numbers.

We start with some preliminaries in the next section. In Section 3, interval obligation rules are introduced; in the same section, the relation between the interval obligation rules and the interval Bird rule are given. In Section 4, it is shown that interval obligation rules are interval cost monotonic and induce pmias. A summary on our work are given in Section 5.

2. Preliminaries

In this section we give some terminology on graph theory, interval calculus and some basic definitions and useful results from the theory of cooperative interval games (Alparslan Gök, 2009; Alparslan Gök, 2010; Alparslan Gök et. al., 2009a; Alparslan Gök et.al., 2011; Alparslan Gök et.al., 2009b; Diestel, 2000; Moretti et.al., 2011; Tijs, 2003)

An (undirected) *graph* is a pair $< V, E >$, where V is a set of vertices or nodes and E is a set of edges e of the form $\{i, j\}$ with $i, j \in V$, $i \neq j$. The *complete graph* on a set V of vertices is the graph $< V, E_V >$, where $E_V = \{\{i, j\} | i, j \in V \text{ and } i \neq j\}$. A *path* between i and j in a graph $< V, E >$ is a sequence of nodes $i = i_0, i_1, \ldots, i_k = j$, $k \geq 1$, such that all the edges $\{i_s, i_{s+1}\} \in E$, for $s \in \{0, \ldots, k-1\}$, are distinct. A *cycle* in $< V, E >$ is a path from i to i for some $i \in V$. Two nodes $i, j \in V$ are *connected* in $< V, E >$ if $i = j$ or if there exists a path between i and j in $< V, E >$. A *connected component* of V in a graph $< V, E >$ is a maximal subset of V with the property that any two nodes in this subset are connected in $< V, E >$.

A *minimum interval cost spanning tree* (*micst*) *situation* is a situation where $N = \{1, 2, \ldots, n\}$ is a set of agents who are willing to be connected as cheaply as possible to a source (i.e., a supplier of a service) denoted by 0, based on an interval-valued weight (or cost) function.

For each $S \subseteq N$, we also use the notation $S_0 = S \cup \{0\}$, and the notation W for the *interval weight function*, i.e., a map which assigns to each edge $e \in E_{N_0}$ a closed interval $W(e) \in I(\mathbb{R}_+)$. The interval cost $W(e)$ of each edge $e \in E_{N_0}$ ($N_0 = N \cup \{0\}$) will be denoted by $[\underline{W}(e), \overline{W}(e)]$. No probability distribution is assumed for edge costs. We denote an *micst situation* with set of users N, source 0, and interval weight function W by $< N_0, W >$ (or simply W). Further, we denote by $\mathcal{IW}^{N_0}$ the set of all *micst situations* $< N_0, W >$ (or W) with node set N_0.

The *cost of a network* $\Gamma \subseteq E_{N_0}$ in an micst situation $W \in \mathcal{IW}^{N_0}$ is $W(\Gamma) = \sum_{e \in \Gamma} W(e)$. A network Γ is a *spanning network* on $S_0 = S \cup \{0\}$, with $S \subseteq N$, if for every $e \in \Gamma$ we have $e \in E_{S_0}$ and for every $i \in S$ there is a path in $< S_0, \Gamma >$ from i to the source. For any *micst* situation $W \in \mathcal{IW}^{N_0}$ it is possible to determine at least one *spanning tree* on N_0, i.e., a spanning network without cycles on N_0, *of minimum interval cost* (such a network is also called an *micst on* N_0 *in* W or, shorter, an *micst for* W). Note that the number of edges which form a spanning tree on N_0 is n. In the following, we will denote by $\mathcal{T}_{N_0}$ the set of all spanning trees for N_0 and by $\mathcal{M}_{N_0}^{W} \subseteq \mathcal{T}_{N_0}$ the set of all *micst* for N_0 in W, for each $W \in \mathcal{IW}^{N_0}$.

Let $I(\mathbb{R})$ be the set of all closed intervals in $\mathbb{R}$. A *cooperative interval cost game* is an ordered pair $< N, \widehat{c} >$, where $N = \{1, 2, ..., n\}$ is the set of players and $\widehat{c} : 2^N \rightarrow I(\mathbb{R})$ is the characteristic function with $\widehat{c}(\emptyset) = [0, 0]$, which assigns to each coalition $S \in 2^N$ a closed and bounded interval $[\underline{\widehat{c}}(S), \overline{\widehat{c}}(S)]$. A classical cooperative game $< N, c >$ can be identified with $< N, \widehat{c} >$, where $\widehat{c}(S) = [c(S), c(S)]$ for each $S \in 2^N$. The family of all interval games with player set N is denote by IG^N. Instead of $\widehat{c}(\{i\}), \widehat{c}(\{i, j\})$, etc., we often write $\widehat{c}(i), \widehat{c}(i, j)$, etc..

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then, $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$ and $\alpha I = [\alpha \underline{I}, \alpha \overline{I}]$.

In this paper we also need a partial substraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. We recall that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $I \preccurlyeq J$, if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$. We say that I is better than J, which we denote by $I \succ J$, if and only if $I \succcurlyeq J$ and $I \neq J$.

Further, we use the notation $I(\mathbb{R}_+)$ for the set of all closed nonnegative intervals in $\mathbb{R}$.

In this paper, $n-$tuples of intervals $I = (I_1, ..., I_n)$ where $I_i \in I(\mathbb{R})$ for each $i \in N$, will play a key role. For further use we denote by $I(\mathbb{R})^N$ the set of all $n-$dimensional vectors whose components are elements in $I(\mathbb{R})$. Let $I_i = [\underline{I}_i, \overline{I}_i]$

be the interval payoff of player i, and let $I = (I_1, ..., I_n)$ be an interval payoff vector. Then, according to Moore, 1985, we have
$nolimits_{i \in S} I_i = [\sum_{i \in S} \underline{I}_i, \sum_{i \in S} \overline{I}_i] \in I(\mathbb{R})$ for each $S \in 2^N \setminus \{\emptyset\}$.

The *interval core* $\mathcal{C}(N, \widehat{c})$ of the interval cost game $\widehat{c}$ is defined by

$$\mathcal{C}(N, \widehat{c}) := \left\{ (I_1, ..., I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = \widehat{c}(N), \sum_{i \in S} I_i \preccurlyeq \widehat{c}(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

The interval core consists of those interval payoff vectors which assure the distribution of the uncertain worth of the grand coalition such that each coalition of players can expect a weakly better interval payoff than what that group can expect on its own, implying that no coalition has any incentives to split off. Here, $\sum_{i \in N} I_i = \widehat{c}(N)$ is the *efficiency condition* and $\sum_{i \in S} I_i \preccurlyeq \widehat{c}(S), S \in 2^N \setminus \{\emptyset\}$, are the *stability conditions* of the interval payoff vectors.

Given an element $\mathbf{a} = (a_1, \ldots, a_n) \in (E_{N_0})^n$, we denote by $\mathbf{a}_{|j}$ the restriction of $\mathbf{a}$ to the first j components, that is $\mathbf{a}_{|j} = (a_1, \ldots, a_j)$ for each $j \in N$. Further, for each $j \in N$, we denote by $\Pi(\mathbf{a}_{|j})$ the partition of N_0 defined as

$$\Pi(\mathbf{a}_{|j}) = \{T \subseteq N_0 | T \text{ is a connected component in } < N_0, \{a_1, \ldots, a_j\} >\}.$$

In the following, we will use the notation $\Pi(\mathbf{a}_{|0})$ to denote the singleton partition of N_0.

For each $\Gamma \in \mathcal{T}_{N_0}$ and each $W \in \mathcal{IW}^{N_0}$, we denote by $\mathbf{A}^{\Gamma,W} \subseteq (E_{N_0})^n$ the set of vectors $\mathbf{a} = (a_1, \ldots, a_n)$ of n distinct edges in Γ such that $W(a_1) \preccurlyeq ... \preccurlyeq W(a_n)$,

Note that $W(a_i)$ is monotonically increasing with respect to " $\preccurlyeq$ " :

$$\mathbf{A}^{\Gamma,W} = \{\mathbf{a} \in (\Gamma)^n | W(a_1) \preccurlyeq \ldots \preccurlyeq W(a_n), a_j \neq a_k \text{ for all } j, k \in N\}.$$

An micst *game* $< N, \widehat{c}_W >$ (or simply $\widehat{c}_W$) corresponding to an micst situation $W \in \mathcal{IW}^{N_0}$ is defined by

$$\widehat{c}_W(T) := \min\{W(\Gamma) | \Gamma \text{ is a spanning network on } T_0\}$$

for every $T \in 2^N \setminus \{\emptyset\}$, with the convention that $\widehat{c}_W(\emptyset) = [0, 0]$. Also, an *interval solution* is a map $\mathcal{F} : \mathcal{IW}^{N_0} \to I(\mathbb{R})^N$ assigning to every *micst* situation $W \in \mathcal{IW}^{N_0}$ a unique allocation in $I(\mathbb{R})^N$.

Finally, we give the notion of population monotonic interval allocation scheme (pmias) for the game $< N, \widehat{c} >$. We say that for a cost game $\widehat{c}$, a *scheme* $A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is a pmias of $\widehat{c}$ if

i) $\sum_{i \in S} A_{iS} = \widehat{c}(S)$ for all $S \in 2^N \setminus \{\emptyset\}$, and

ii) $A_{iS} \succcurlyeq A_{iT}$ for all $S, T \in 2^N$ and $i \in N$ with $i \in S \subset T$.

3. Interval obligation rules

Consider $\Delta(N)$ to be an usual simplex on N, defined by $\Delta(N) = \{x \in \mathbb{R}_+^N | \sum_{i \in N} x_i = 1\}$. The sub-simplex $\Delta(S)$ of $\Delta(N)$ given by $\Delta(S) = \{x \in \Delta(N) | \sum_{i \in S} x_i = 1\}$ is called the set of *obligation vectors* of S. An *obligation function* is a map $O : 2^N \setminus \{\emptyset\} \to \Delta(N)$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ an obligation vector

$$O(S) \in \Delta(S)$$

in such a way that for each $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$ it holds

$$O_i(S) \geq O_i(T).$$

Such an obligation function O on $2^N \setminus \{\emptyset\}$ induces an *obligation map* $\hat{O} : \Theta(N_0) \rightarrow \mathbb{R}^N$ such that

$$\hat{O}_i(\theta) := \sum_{S \in \theta, 0 \notin S} O_i(S),$$

for each $i \in N$ and each $\theta \in \Theta(N_0)$; here, $\Theta(N_0)$ is the family of partitions of N_0.

Note that if $\theta = \{N_0\}$, then the resulting empty sum is assumed, by definition, to be the n-vector of zeroes: $\hat{O}(\theta) = 0 \in \mathbb{R}^N$ (for details see Tijs et.al., 2006). Obligation maps are basic ingredients for interval obligation rules. Now, we introduce the notion of the interval obligation rule.

Definition 1. Let $\hat{O}$ be an obligation map on $\Theta(N_0)$. The *interval obligation rule* $\phi^{\hat{O}} : \mathcal{IW}^{N_0} \rightarrow I(\mathbb{R})^N$ is defined by

$$\phi^{\hat{O}}(W) := \sum_{j=1}^{n} W(a_j)(\hat{O}(\Pi(\mathbf{a}_{|j-1})) - \hat{O}(\Pi(\mathbf{a}_{|j})))$$

for each *micst* situation $W \in \mathcal{IW}^{N_0}$, each $\Gamma \in \mathcal{M}_{N_0}^W$ and $\mathbf{a} \in \mathbf{A}^{\Gamma,W}$, and where $\Pi(\mathbf{a}_{|j-1}))$ and $\Pi(\mathbf{a}_{|j})$, for each $j = 1, \ldots, n$, are partitions of the set N_0.

Example 1. We consider an micst situation $< N_0, W >$ with three agents denoted by $1, 2$, and 3 and the source 0. As depicted in Figure 1, to each edge $e \in E_{\{0,1,2,3\}}$ is assigned a closed interval $W(e) \in I(\mathbb{R}_+)$ representing the uncertain cost of edge e. For instance, $W(0,1) = [20, 24], W(2,3) = [10, 13]$, etc.. Now we compute the interval obligation rule In this micst situation, W, $\Gamma = \{(0,1), (1,2), (2,3)\} \in \mathcal{M}_{N_0}^W$ and

$$\mathbf{a} = (a_1, a_2, a_3) = ((2,3), (1,2), (0,1)) \in \mathbf{A}^{\Gamma,W}.$$

Then,

$$\phi_1^{\hat{O}}(W) = \frac{2}{3}[15, 20] + \frac{1}{3}[20, 24] = \left[\frac{50}{3}, \frac{64}{3}\right],$$
$$\phi_2^{\hat{O}}(W) = \frac{1}{2}[10, 13] + \frac{1}{6}[15, 20] + \frac{1}{3}[20, 24] = \left[\frac{85}{6}, \frac{107}{6}\right], \text{ and}$$
$$\phi_3^{\hat{O}}(W) = \frac{1}{2}[10, 13] + \frac{1}{6}[15, 20] + \frac{1}{3}[20, 24] = \left[\frac{85}{6}, \frac{107}{6}\right].$$

Hence,

$$\phi^{\hat{O}}(W) = \left(\left[\frac{50}{3}, \frac{64}{3}\right], \left[\frac{85}{6}, \frac{107}{6}\right], \left[\frac{85}{6}, \frac{107}{6}\right]\right).$$

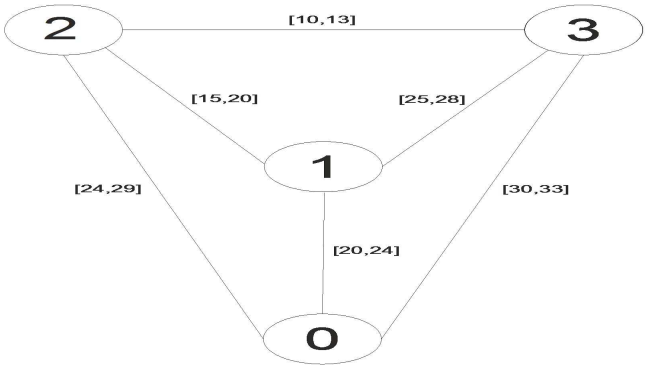

Fig. 1: An micst situation $< N_0, W >$.

Remark 1. It is obvious that if the cost of the edge connecting to source is the cheapest cost, then the interval obligation rule equals the interval Bird rule which is defined by Alparslan Gök et.al., 2014.

Example 2. Figure 2 corresponding to micst situation $< N_0, W' >$, the interval Bird allocation is

$IB(N, \{0\}, A, W') = ([10, 13], [15, 20], [20, 24])$ (see Alparslan Gök et.al., 2014).

In this micst situation W', $\Gamma = \{(0,1), (1,2), (2,3)\} \in \mathcal{M}_{N_0}^{W'}$ and

$$\mathbf{a} = (a_1, a_2, a_3) = ((0,1), (1,2), (2,3)) \in \mathbf{A}^{\Gamma, W'}.$$

Then,

$$\phi_1^{\hat{O}}(W') = (1-0)[10,13] + (0-0)[15,20] + (0-0)[20,24] = [10,13],$$
$$\phi_2^{\hat{O}}(W') = (0-0)[10,13] + (1-0)[15,20] + (0-0)[20,24] = [15,20],$$
$$\phi_3^{\hat{O}}(W') = (0-0)[10,13] + (0-0)[15,20] + (1-0)[20,24] = [20,24].$$

It is clear that

$$\phi^{\hat{O}}(W') = IB(N, \{0\}, A, W') = ([10,13], [15,20], [20,24]).$$

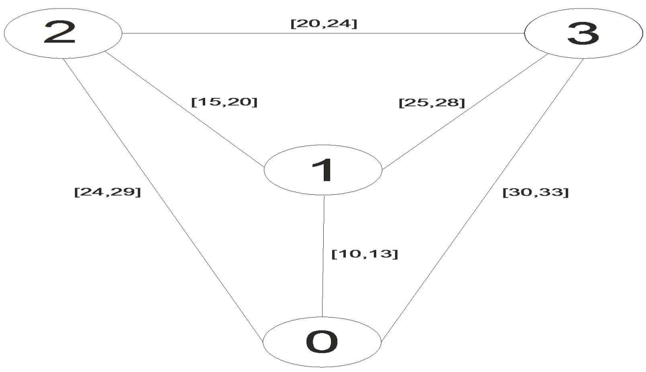

Fig. 2: An micst situation $< N_0, W' >$.

4. Interval cost monotonicity and PMIAS

In this section, we discuss some interesting interval monotonicity properties for the interval obligation rules. First, we provide the definition of interval cost monotonic solutions for micst situations.

Definition 2. An interval solution $\mathcal{F}$ is *an interval cost monotonic solution* if for all *micst* situations $W, W' \in \mathcal{IW}^{N_0}$ such that $W(e) \preccurlyeq W'(e)$ for each $e \in E_{N_0}$ it holds that $\mathcal{F}_i(W) \preccurlyeq \mathcal{F}_i(W')$ for each $i \in N$.

We prove in Theorem 7 that interval obligation rules are interval cost monotonic; the main step is the following lemma whose proof is straightforward.

Lemma 1. *Let $\hat{O}$ be an obligation map on $\Theta(N_0)$ and let $W \in \mathcal{IW}^{N_0}$. Let $\bar{e} \in E_{N_0}$ and let $h \succ W(\bar{e})$ be such that there is no $e \in E_{N_0}$ with $W(\bar{e}) \prec W(e) \prec h$. Define $\tilde{W} \in \mathcal{IW}^{N_0}$ by $\tilde{W}(e) := \tilde{W}(e)$ if $e \in E_{N_0} \setminus \{\bar{e}\}$, and $\tilde{W}(\bar{e}) = h$. Then, $\phi^{\hat{O}}(\tilde{W}) \succcurlyeq \phi^{\hat{O}}(W)$.*

The proofs of the following theorems are straightforward (see Tijs et.al., 2006).

Theorem 1. *Interval obligation rules are interval cost monotonic.*

Theorem 2. *Let $\hat{O}$ be an obligation map on $\Theta(N_0)$ and let $\phi^{\hat{O}}$ the interval obligation rule with respect to $\hat{O}$, and $W \in \mathcal{IW}^{N_0}$. Then the table $[\phi^{\hat{O}_S}(W_{|S_0})]_{S \in 2^N \setminus \{\emptyset\}}$ is a pmias for the micst game $< N, \widehat{c}_W >$.*

From Theorem 8 and the definition of a pmias, it follows that interval obligation rules provide interval cost allocations which are interval core elements of the game $< N, \widehat{c}_W >$.

Now, we give an example of interval cost monotonicity and pmias.

Example 3. Consider again the micst situation $< N_0, W >$ as depicted in Figure 1. Then, as the interval obligation rule $\phi^{\hat{O}}(W)$ previously introduced, applied to each micst situation $< S_0, W_{|S_0} >$, provides the following population monotonic interval allocation scheme:

$$[\phi^{\hat{O}_S}(W_{|S_0})]_{S \in 2^N \setminus \{\emptyset\}} = \left\{ \begin{array}{l|ccc|} S & 1 & 2 & 3 \\ \hline (123) & [16\frac{4}{6}, 21\frac{2}{6}] & [14\frac{1}{6}, 17\frac{5}{6}] & [14\frac{1}{6}, 17\frac{5}{6}] \\ (12) & [17\frac{1}{2}, 22] & [17\frac{1}{2}, 22] & * \\ (13) & [20, 24] & * & [25, 28] \\ (23) & * & [17, 21] & [17, 21] \\ (1) & [20, 24] & * & * \\ (2) & * & [24, 29] & * \\ (3) & * & * & [30, 33] \\ \hline \end{array} \right. .$$

5. Conclusions and outlook

This paper considers the class of interval obligation rules and studies their interval cost monotonicity properties. The interval obligation rules are interval cost monotonic and induce a pmias. The most important result of this study is, as already stated in Remark 3, if the cost of the edges connecting to source is the cheapest cost, then the interval obligation rule equals the interval Bird rule which is defined by Alparslan Gök et.al., 2014.

Before closing we note that the obtained results can be extended to Network Steiner problem by using Kirzhner et.al., 2012. The Steiner tree problem is superficially similar to the minimum spanning tree problem. The difference between the Steiner tree problem and the minimum spanning tree problem is that, in the Steiner tree problem, extra intermediate vertices and edges may be added to the graph in order to reduce the length of the spanning tree.

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