Competition Form of Bargaining

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Abstract We consider the noncooperative zero-sum game, related with the competitions. Players submit the competition projects, that are characterized by a finite set of parameters. The arbitrator or arbitration committee uses a stochastic procedure with the probability distribution to determine the most preferred project. This distribution is known to all participants. Payoff ot the winner depend on the parameters of his project. The three-dimensional mathematical model of this problem is constructed, which is then extended to the multi-dimensional case. The equilibria in the games with four and n persons are found, as well as the corresponding payoffs are computed.

Keywords: Model of competition, bargaining, stochastic procedure, n-person game, Nash equilibrium.

1. Introduction

Recently in the search for performer of any work or any service provider, tenders are widely used. A lot of companies participate in it, both large and just developing. Tender means the competition form of bargaining. A competition presupposes rivalry among participants for the right to carry out their projects. This means that in the market there are several potential performers with similar capabilities, and the initiator's of tender offer is interesting for them. As a result, both sides stand to benefit: the organizer receives the best performer, and performer gets a big contract and good profit.

This paper presents a multi-dimensional game-theoretic model of the tender as the competition of projects. The n persons participate in the competition. Their proposals, or projects, are characterized by a finite set of parameters. As parameters such project can include, for example, description of cost, time of implementation, number of participants, etc.

2. The model

2.1. Game with Four Players

We consider a noncooperative non-zero sum game, related with competition. Assume there are four persons, or players. They represent competition projects, characterized by a set of three parameters (x, y, z). Let the player I seeks for maximize the amount of $x + y + z$, and players II, III and IV - for minimize the parameter x, y or z, respectively.

An arbitrator or arbitration committee considers proposals received and selects one of projects, using the stochastic procedure with the probability distribution

$$f(x, y, z) = g(x) \cdot g(y) \cdot g(z), \text{ where } g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2},$$

which is known to participants. In this case the winner receives payoff, which depends on the parameters of project. In the paper (Mazalov and Tokareva, 2010) an equilibrium in competition of projects among three persons on the plane is obtained.

Because of the symmetry of the model, the optimal strategies of players will be found in this form
player I: (c, c, c),
player II: $(-a, 0, 0)$,
player III: $(0, -a, 0)$,
player IV: $(0, 0, -a)$.

Fix these proposals of players II, III and IV. Let player I submit the project (x_1, y_1, z_1), where $x_1, y_1, z_1 \geqslant 0$. Then the space of projects is split into four subspaces, limited by the planes: $\alpha_1 : y = x$, $\alpha_2 : z = x$, $\alpha_3 : z = y$,

$$\alpha_4 : z = -\frac{x_1 + a}{z_1}x - \frac{y_1}{z_1}y + \frac{x_1^2 + y_1^2 + z_1^2 - a^2}{2z_1},$$

$$\alpha_5 : z = -\frac{x_1}{z_1}x - \frac{y_1 + a}{z_1}y + \frac{x_1^2 + y_1^2 + z_1^2 - a^2}{2z_1},$$

$$\alpha_6 : z = -\frac{x_1}{z_1 + a}x - \frac{y_1}{z_1 + a}y + \frac{x_1^2 + y_1^2 + z_1^2 - a^2}{2(z_1 + a)}.$$

These planes intersect at the point with coordinates $x = y = z = x_0$, where

$$x_0 = \frac{x_1^2 + y_1^2 + z_1^2 - a^2}{2(x_1 + y_1 + z_1 + a)}.$$

Consider the subspace V_1, bounded by the planes α_4, α_5 and α_6. Depict the projection of lines, which considered planes intersect, on the plane XOY (Fig. 1) and the mutual arrangement of all obtained regions in space (Fig. 2). If the arbitrator's decision is in the region V_1, player I wins, and his payoff is

$$H_1(x_1, y_1, z_1) = (x_1 + y_1 + z_1) \cdot \mu(V_1), \tag{1}$$

where $\mu(V_1)$ is measure of the set V_1, which is equal to

$$\mu(V_1) = \int\limits_{-\infty}^{x_0} g(x)dx \int\limits_{-\infty}^{x} g(y)dy \int\limits_{q(x,y)}^{\infty} g(z)dz + \int\limits_{-\infty}^{x_0} g(x)dx \int\limits_{x}^{u(x)} g(y)dy \int\limits_{p(x,y)}^{\infty} g(z)dz +$$

$$+ \int\limits_{-\infty}^{x_0} g(x)dx \int\limits_{u(x)}^{\infty} g(y)dy \int\limits_{r(x,y)}^{\infty} g(z)dz + \int\limits_{x_0}^{\infty} g(x)dx \int\limits_{-\infty}^{v(x)} g(y)dy \int\limits_{q(x,y)}^{\infty} g(z)dz +$$

$$+ \int\limits_{x_0}^{\infty} g(x)dx \int\limits_{v(x)}^{\infty} g(y)dy \int\limits_{r(x,y)}^{\infty} g(z)dz. \tag{2}$$

There

$$u(x) = -\frac{x_1 + z_1 + a}{y_1}x + \frac{x_1^2 + y_1^2 + z_1^2 - a^2}{2y_1},$$

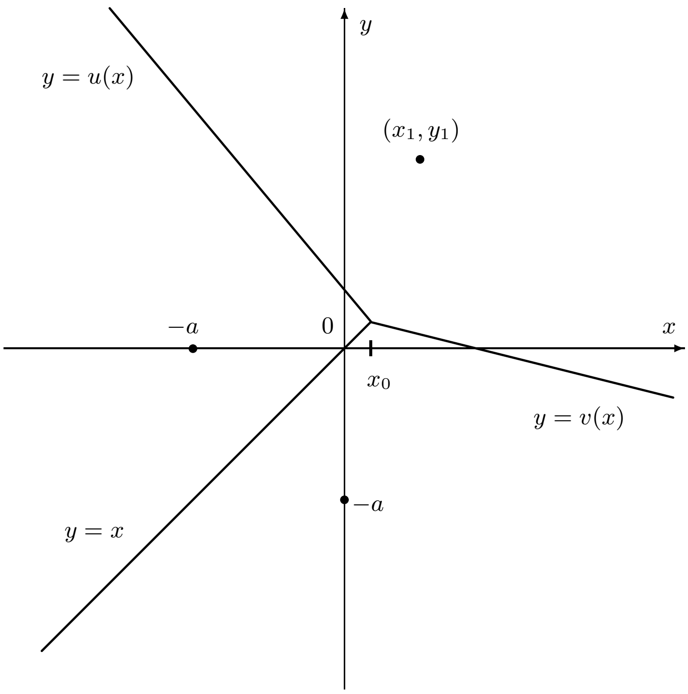

Fig. 1: Projection on the plane XOY

$$v(x) = -\frac{x_1}{y_1+z_1+a}x + \frac{x_1^2+y_1^2+z_1^2-a^2}{2(y_1+z_1+a)},$$

$$p(x,y) = -\frac{x_1+a}{z_1}x - \frac{y_1}{z_1}y + \frac{x_1^2+y_1^2+z_1^2-a^2}{2z_1},$$

$$q(x,y) = -\frac{x_1}{z_1}x - \frac{y_1+a}{z_1}y + \frac{x_1^2+y_1^2+z_1^2-a^2}{2z_1},$$

$$r(x,y) = -\frac{x_1}{z_1+a}x - \frac{y_1}{z_1+a}y + \frac{x_1^2+y_1^2+z_1^2-a^2}{2(z_1+a)}.$$

From equations (1) and (2) we obtain

$$H(x_1,y_1,z_1) = (x_1+y_1+z_1)\cdot\left[1 - \int_{-\infty}^{x_0} g(x)dx\left(\int_{-\infty}^{x} g(y)\cdot G(q(x,y))dy + \int_{x}^{u(x)} g(y)\cdot G(p(x,y))dy + \int_{u(x)}^{\infty} g(y)\cdot G(r(x,y))dy\right) - \int_{x_0}^{\infty} g(x)dx\left(\int_{-\infty}^{v(x)} g(y)\cdot G(q(x,y))dy + \int_{v(x)}^{\infty} g(y)\cdot G(r(x,y))dy\right)\right], \qquad (3)$$

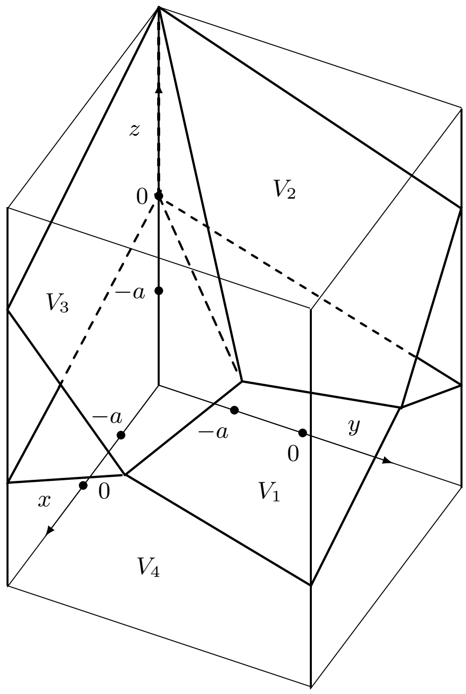

Fig. 2: Schematic representation of the regions

where $G(x)$ is the normal distribution function. The function (3) has a maximum, depending on a, at the point $x_1 = y_1 = z_1 = c$.

Suppose now that player I selects the strategy (c, c, c), while the offers of players II and III remain the same: $(-a, 0, 0)$ and $(0, -a, 0)$, respectively, and player IV offers $(0, 0, -b)$. We find the best response of player II to the strategies of players I, III and IV. The space of projects splits into subspaces. Consider the boundary subspaces of V_2:

$$\alpha_1 : y = x,$$

$$\alpha_2 : z = \frac{a}{b}x - \frac{b^2 - a^2}{2b},$$

$$\alpha_4 : z = -\frac{c + a}{c}x - y + \frac{3c^2 - a^2}{2c}.$$

The abscissa of the intersection point of these three planes is

$$m = \left(\frac{3c^2 - a^2}{2c} + \frac{b^2 - a^2}{2b} \right) \cdot \frac{1}{2 + a/b + a/c}.$$

In the region under consideration player II wins, and his payoff is equal

$$H_2(a) = a \cdot \mu(V_2) = a \cdot \left[\int\limits_{-\infty}^{m} g(x)dx \int\limits_{x}^{u(x)} g(y)dy \int\limits_{w(x)}^{s(x,y)} g(z)dz \right] =$$

$$= a \cdot \left[\int_{-\infty}^{m} g(x)dx \int_{x}^{u(x)} (G(w(x)) - G(s(x,y))) \cdot g(y)dy \right], \qquad (4)$$

where

$$w(x) = \frac{a}{b}x - \frac{b^2 - a^2}{2b},$$

$$s(x,y) = -\frac{c+a}{c}x - y + \frac{3c^2 - a^2}{2c}.$$

Using symmetry, we can conclude that minimum of the function (4) is reached when $a = b$. The optimal parameters a and c can be found approximately by the methods of numerical simulation

$$a = b \approx 1.5834, \quad c \approx 1.3207.$$

In equilibrium the players get payoffs

$$H_1 \approx 0.9949, \quad H_2 = H_3 = H_4 \approx 0.3952$$

with probabilities, respectively

$$\mu(V_1) \approx 0.2511, \quad \mu(V_2) = \mu(V_3) = \mu(V_4) \approx 0.2496.$$

2.2. Game with n+1 Players

Suppose now that $n+1$ players submit their projects for the competition, which are characterized by a set of n parameters $(u_1, u_2, ..., u_n)$. Let player I still interested in maximizing the amount of $u_1 + u_2 + ... + u_n$, and the other players, starting from the second, seek for minimize the parameter u_{i-1}, where $i = \overline{1,n}$ is the number of player.

Present the stochastic procedure with the probability distribution for the arbitrator as follows

$$f(u_1, u_2, ..., u_n) = \prod_{i=1}^{n} g(u_i), \text{ where } g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2},$$

We assume that this distribution is known to all participants of the competition. According to the symmetry of the model, the optimal strategies of players have the form of n-dimensional vector $(u_1, u_2, ..., u_n)$. Suppose that for player I the components of this vector are the same $u_1 = u_2 = ... = u_n = c$, and for all other players such vector has only one non-zero component $u_{i-1} = -a$, where $i = \overline{1,n}$ is the number of player.

Fix these proposals of players, from the second player. Let player I submit the project $(x_1, x_2, ..., x_n)$, where $x_1, x_2, ..., x_n \geqslant 0$. Then the n-dimensional space of projects is split into $n+1$ subspaces, limited by hyperplanes:

$$\pi_1 : (x_1 + a)u_1 + x_2u_2 + ... + x_nu_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

$$\pi_2 : x_1u_1 + (x_2 + a)u_2 + ... + x_nu_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

...

$$\pi_n : x_1u_1 + x_2u_2 + ... + (x_n + a)u_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right).$$

$$\pi_{n+1} : u_1 = u_2,$$

$$\pi_{n+2} : u_1 = u_3,$$

...

$$\pi_{n+k} : u_{n-1} = u_n, \text{ where } k = \frac{n(n-1)}{2}.$$

All these hyperplanes intersect at the point with coordinates $u_1 = u_2 = ... = u_n = u_0$, where

$$u_0 = \frac{\sum\limits_{i=1}^{n} x_i^2 - a^2}{2\left(\sum\limits_{i=1}^{n} x_i + a\right)}.$$

Consider the subspace V_1, bounded by the n-dimensional planes π_1, π_2, ..., π_n. If the arbitrator's decision is in the region V_1, player I wins, and his payoff is

$$H_1(x_1, x_2, ..., x_n) = (x_1 + x_2 + ... + x_n) \cdot \mu(V_1), \tag{5}$$

where $mu(V_1)$ is the measure of V_1, which is equal to

$$\mu(V_1) = \int\limits_{-\infty}^{u_0}\int\limits_{-\infty}^{u_1}\int\limits_{\alpha_3(u_1,u_2)}^{\infty} ... \int\limits_{\pi_1(u_1,...,u_n)}^{\infty} \prod_{i=1}^{n} g(u_i)\, du_1...du_n + ...+$$

$$+\int\limits_{-\infty}^{u_0}\int\limits_{u_1}^{l_1(u_1)}\int\limits_{u_1}^{\alpha_2(u_1,u_2)} ... \int\limits_{\pi_2(u_1,...,u_n)}^{\infty} \prod_{i=1}^{n} g(u_i)\, du_1...du_n + ...+$$

$$+\int\limits_{-\infty}^{u_0}\int\limits_{-\infty}^{u_1}\int\limits_{u_2}^{\alpha_3(u_1,u_2)} ... \int\limits_{\pi_3(u_1,...,u_n)}^{\infty} \prod_{i=1}^{n} g(u_i)\, du_1...du_n + ...+$$

$$+\int\limits_{-\infty}^{u_0}\int\limits_{-\infty}^{u_1}\int\limits_{-\infty}^{u_2} ... \int\limits_{\pi_4(u_1,...,u_n)}^{\infty} \prod_{i=1}^{n} g(u_i)\, du_1...du_n + ...+$$

$$+\int\limits_{u_0}^{\infty}\int\limits_{l_2(u_1)}^{\infty}\int\limits_{\alpha_3(u_1,u_2)}^{\infty} ... \int\limits_{-\infty}^{\omega_{n-1}(u_1,...,u_{n-1})} \int\limits_{\pi_n(u_1,...,u_n)}^{\infty} \prod_{i=1}^{n} g(u_i)\, du_1...du_n. \tag{6}$$

There

$$\pi_1 \cap \pi_2 = \omega_1 : \quad (x_1 + x_2 + a)u_1 + x_3u_3 + ... + x_nu_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

...

$$\pi_1 \cap \pi_{i+1} = \omega_i: \quad x_1u_1 + x_2u_2 + ... + x_{i-1}u_{i-1} + (x_i + x_{i+1} + a)u_i +$$

$$+ x_{i+2}u_{i+2} + ... + x_nu_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

$$...$$

$$\pi_1 \cap \pi_n = \omega_{n-1}: \quad x_1u_1 + x_2u_2 + ... + (x_{n-1} + x_n + a)u_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

$$...$$

$$\alpha_1 \cap \alpha_2 = l_1: \quad \left(\sum_{i=1}^{n-1} x_i + a\right)u_1 + x_nu_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right),$$

$$\alpha_1 \cap \alpha_3 = l_2: \quad x_1u_1 + \left(\sum_{i=2}^{n} x_i + a\right)u_n = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - a^2\right)$$

The function (6) has a maximum, depending on a, at the point $x_1 = x_2 = ... = x_n = c$. Suppose now that player I chooses a strategy $(c, c, ..., c)$, while the offers of the players with numbers $i = \overline{2, n}$ remain the same: component u_i of the vector for player i is $-a$, the remaining components are equal to zero, and the player $n+1$ offers $(0, ..., 0, -b)$. We find the best response of player II to the strategies of players with numbers $i = 1$ and $i = \overline{3, n+1}$. The space of projects is splits into subspaces. Consider the boundary subspaces of V_2:

$$\pi_1: (c+a)u_1 + \sum_{i=2}^{n} cu_i = \frac{nc^2 - a^2}{2},$$

$$\pi_{n+1}: u_1 = u_2,$$

$$...$$

$$\pi_{2n-2}: u_1 = u_{n-1},$$

$$\pi_{2n-1}: u_n = \frac{a}{b}u_1 - \frac{b^2 - a^2}{2b}.$$

The abscissa of the intersection point of these n planes is

$$m = \left(\frac{nc^2 - a^2}{2c} + \frac{b^2 - a^2}{2b}\right) \cdot \frac{1}{a/b + a/c + n - 1}.$$

In the region under consideration player II wins, and his payoff is equal

$$H_2(a) = a \cdot \mu(V_2) =$$

$$= a \cdot \int_{-\infty}^{m} \int_{u_1}^{l_1(u_1)} \int_{u_2}^{\alpha_1(u_1,u_2)} ... \int_{u_2}^{\omega_1(u_1,...,u_{n-2})} \int_{\phi(u_1)}^{\rho(u_1,...,u_n)} \prod_{i=1}^{n} g(u_i)\, du_1...du_n. \tag{7}$$

where

$$\phi(u_1) = \frac{a}{b}u_1 - \frac{b^2 - a^2}{2b},$$

$$\rho(u_1, ..., u_{n-1}) = -\frac{c+a}{c}u_1 - \sum_{i=2}^{n-1} u_i + \frac{nc^2 - a^2}{2c}.$$

Using symmetry, we can conclude that the minimum of the function (7) is reached when $a = b$. Thus, for large n the optimal parameters a and c can be approximately estimated as follows:

$$a = b \approx \frac{n+1}{n} + \varepsilon,\ \text{ãäå}\ \varepsilon > 0, \qquad c \approx \frac{n+1}{n}.$$

In equilibrium the players get payoffs

$$H_1 \approx 1, \qquad H_2 = H_3 = H_4 \approx \frac{1+\delta}{n}, \qquad \delta > 0$$

with probabilities, respectively

$$\mu(V_1) \approx \mu(V_2) = \mu(V_3) = \mu(V_4) \approx \frac{1}{n+1}.$$

3. Conclusion

We present an extension of the model, proposed in (Mazalov and Tokareva, 2010), on the three-dimensional and multi-dimensional cases. The optimal solutions is found by the methods of numerical modeling. A similar approach has been widely used for solving zero-sum game problems on the line. In the papers (Mazalov et al., 2012; Kilgour, 1994) equilibriums in games involving one arbitrator are obtained, and in the paper (Mazalov, 2010) involving arbitration committee. Under real conditions of market the experts of competition committee act as the arbitrators. They assess the expected project for each of the parameters, and on the basis of this assessment the probability distribution, corresponding to the opinion of experts, is formed. Then the players submit their proposals for the competition, and the committee can immediately reject the projects, that are dominated by the other projects.

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