An Axiomatization of the Proportional Prenucleolus

Natalia Naumova

St.Petersburg State University, Faculty of Mathematics and Mechanics
Universitetsky pr. 28, Petrodvorets, St.Petersburg, 198504, Russia
E-mail: Nataliai.Naumova@mail.ru

Abstract The proportional prenucleolus is defined on the class of all positive TU games with finite sets of players. The set of axioms used by Sobolev (1975) for axiomatic justification of the prenucleolus is modified. It is proved that the proportional prenucleolus is a unique value that satisfies 4 axioms: efficiency, anonymity, proportionality, and proportional DM consistency. The proof is a modification of the proof of Sobolev's theorem.
For strictly increasing concave function U defined on $(0, +\infty)$ with range equal to R^1, a generalization of the proportional prenucleolus is called U–prenucleolus. The axioms proportionality and proportional DM consistency are generalized for its justification.

Keywords: cooperative games; proportional nucleolus; prenucleolus; consistency.

1. Introduction

For cooperative TU games the nucleolus was defined by Schmeidler, 1969. First it was used for constructive proof of existence of the bargaining set M_1^i. The prenucleolus was defined in Sobolev, 1975 and later in Maschler, Peleg, Shapley, 1979. A unique axiomatic justification of the prenucleolus was given by Sobolev, 1975. He proved that the prenucleolus is a unique value that satisfies 4 axioms: efficiency, anonymity, covariance, and Davis-Maschler consistency. Since that time there appeared only some weakening of his axioms (Orshan, 1993, Peleg and Sudholter, 2007).

For games with positive characteristic function, the proportional prenucleolus, where excesses in the "classical" case are replaced by ratios of coalitional claims to total shares of players in these coalitions, is natural. We propose a modiication of Sobolev's set of axioms, where covariance is replaced by proportionality and Davis-Maschler consistency is replaced by proportional DM consistency. The proof is a modification of the proof of Sobolev.

For strictly increasing concave function U, U–prenucleolus is a generalization of the proportional prenucleolus, the generalization of the proportionality axiom is called U–excess property, and the generalization of the proportional DM consistency is called U–DM consistency. If $U(t) = \ln(t)$, then U–prenucleolus is the proportional prenucleolus and the axioms for it justification coincide with axioms for justification of the proportional prenucleolus.

The paper is organized as follows. Section 2 contains the main definitions and the main statements of the paper. The results of Sobolev that will be used in the proof are described in Section 3. The proof of axiomatic justification of U–prenucleolus is given in Section 4.

2. Definitions and main theorems

Consider a class of positive TU games

$$\mathcal{G}^+ = \{(N, v) : |N| < \infty, \, v(S) > 0 \quad \text{for} \quad \emptyset \neq S \subset N\}.$$

A*value* on $\mathcal{G}^+$ is a map that assigns to every $(N, v) \in \mathcal{G}^+$ a vector $x \in \mathrm{R}^{|N|}_{++}$.

A *preimputation* of $(N, v) \in \mathcal{G}^+$ is a vector $x \in \mathrm{R}^{|N|}_{++}$ such that $\sum_{i \in N} x_i = v(N)$. For $S \subset N$, denote $x(S) = \sum_{i \in S} x_i$.

For preimputation z of $(N, v) \in \mathcal{G}^+$, let the collection of coalitions $\{S : S \subset N, S \neq \emptyset\}$ be enumerated such that $z(S_i)/v(S_i) \leq z(S_{i+1})/v(S_{i+1})$. Denote

$$\theta((N, v), z) = \{z(S_i)/v(S_i)\}_{i=1}^{2^{|N|}-1}.$$

The preimputation y of (N, v) belongs to the *proportional prenucleolus* of (N, v) iff

$$\theta((N, v), y) \geq_{lex} \theta((N, v), z) \quad \text{for all preimputations} \quad z \quad \text{of} \quad (N, v).$$

For each $(N, v) \in \mathcal{G}^+$, the proportional prenucleolus of (N, v) is a singleton. It follows from the results of Vilkov, 1974, Justman, 1977, Sobolev, 1975.

The following axiomatization of the proportional prenucleolus is a modification of Sobolev's axiomatization of the prenucleolus (Sobolev, 1975, see also Peleg and Sudholter, 2007).

Let a value f be defined on $\mathcal{G}^+$. Consider the following properties of f.
Efficiency. $\sum_{i \in N} f_i(N, v) = v(N)$.

Anonymity. Let for games (N, v) and (N', w) there exists a bijection $\pi : N \to N'$ such that $v(S) = w(\pi S)$ for all $S \subset N$. Then $f_i(N, v) = f_{\pi i}(N', w)$.

Proportionality. For any games $(N, v), (N, w) \in \mathcal{G}^+$, any $x, y \in \mathrm{R}^{|N|}_{++}$,

$$\frac{x(S)}{v(S)} = \frac{y(S)}{w(S)} \quad \text{for all} \quad S \subset N$$

implies

$$x = f(N, v) \quad \text{iff} \quad y = f(N, w).$$

The proportionality property means that the value f depends only on the values of proportional excesses. It was used in Yanovskaya, 2002 for axiomatization of some proportional solutions instead of covariance property.

Proportional DM consistency. Let $x = f(N, v)$, then for each $S \subset N$, $S \neq \emptyset$, $x_S = f(S, v^{x,S})$, where

$$v^{x,S}(P) = \begin{cases} v(N) - x(N \setminus S) & \text{for} \quad P = S, \\ \max_{T \subset N \setminus S} \frac{v(P \cup T) x(P)}{x(P \cup T)} & \text{for} \quad P \subset S, \, P \neq S. \end{cases}$$

The proportional DM consistency is a modification of Davis-Maschler consistency.

Theorem 1. *The proportional prenucleolus is a unique value defined on $\mathcal{G}^+$ that satisfies efficiency, proportionality, anonymity, and proportional DM consistency properties.*

The generalization of this theorem will be proved in Section 4.

Now consider a generalization of the proportional prenucleolus. Let U be a strictly increasing concave function defined on $(0,+\infty)$ with $U((0,+\infty)) = \mathrm{R}^1$.

For preimputation z of $(N,v) \in \mathcal{G}^+$, let the collection of coalitions $\{S : S \subset N, S \neq \emptyset\}$ be enumerated such that

$$U(z(S_i)) - U(v(S_i)) \leq U(z(S_{i+1})) - U(v(S_{i+1})).$$

Denote

$$\theta((N,v),z) = \{U(z(S_i)) - U(v(S_i))\}_{i=1}^{2^{|N|}-1}.$$

The preimputation y of (N,v) belongs to the *U–prenucleolus* of (N,v) iff

$$\theta((N,v),y) \geq_{lex} \theta((N,v),z) \quad \text{for all preimputations} \quad z \quad \text{of} \quad (N,v).$$

As U is a concave function on $(0,+\infty)$, the functions $U(x(S))$ are continuous for all S, hence for each $(N,v) \in \mathcal{G}^+$, the U–prenucleolus of (N,v) is a nonempty set. Moreover, it is a singleton. The proof is the same as in the case of "classical" prenucleolus.

The proportionality and the proportional DM consistency properties are generalized as follows.

U–**excess property.** For any games $(N,v),(N,w) \in \mathcal{G}^+$, any $x,y \in \mathrm{R}^{|N|}_{++}$,

$$U(x(S)) - U(v(S)) = U(y(S)) - U(w(S)) \quad \text{for all} \quad S \subset N, S \neq \emptyset$$

implies

$$x = f(N,v) \quad \text{iff} \quad y = f(N,w).$$

The proportionality axiom is equvalent to ln–excess property.

U–**DM consistency.** Let $x = f(N,v)$, then for each $S \subset N$, $S \neq \emptyset$, $x_S = f(S, v^{x,S})$, where

$$v^{x,S}(P) = \begin{cases} 0 \quad \text{for} \quad P = \emptyset, \\ v(N) - x(N \setminus S) \quad \text{for} \quad P = S, \\ U^{-1}\left(U(x(P)) + \max_{T \subset N \setminus S}[U(v(P \cup T)) - U(x(P \cup T))]\right) \\ \text{for} \quad P \subset S,\ P \notin \{S, \emptyset\}. \end{cases}$$

It means that

$$U(v^{x,S}(P)) - U(x(P)) = \max_{T \subset N \setminus S}[U(v(P \cup T)) - U(x(P \cup T))].$$

Note that $v^{x,S}$ is well defined. Indeed, since U is a strictly increasing and continuous function with range equal to R^1, $U^{-1}(t)$ is well defined for all $t \in \mathrm{R}^1$.

U–DM consistency is a modification of Davis-Maschler consistency. The proportional DM consistency is equivalent to ln–DM consistency.

Theorem 2. *Let U be a strictly increasing concave continuous function defined on $(0,+\infty)$ with $U((0,+\infty)) = \mathrm{R}^1$.*

The U–prenucleolus is a unique value defined on $\mathcal{G}^+$ that satisfies efficiency, U–excess property, anonymity, and U–DM consistency conditions.

This theorem will be proved in Section 4.

3. Sobolev's construction.

Let N be a finite set of players.

A collection $\mathcal{D}$ of coalitions is a *balanced collection on* N if there exist positive numbers $\{\delta_S\}_{S\in\mathcal{D}}$ satisfying

$$\sum_{S\in\mathcal{D}:i\in S}\delta_S = 1 \quad \text{for all} \quad i\in N.$$

The vector $\{\delta_S\}_{S\in\mathcal{D}}$ is called a *vector of balancing weights of* $\mathcal{D}$.

A *coalitional family* is a pair $(N, \{\mathcal{B}_l\}_{l\in L})$, where
1) N and L are finite nonempty sets,
2) $\mathcal{B}_l \subset 2^N$ for all $l\in L$,
3) $\mathcal{B}_l \cap \mathcal{B}_t = \emptyset$ for $l,t\in L$, $l\neq t$.

Let $\mathcal{H} = (N, \{\mathcal{B}_l\}_{l\in L})$ be a coalitional family. A permutation π of N is a *symmetry* of $\mathcal{H}$ if for every $l\in L$ and every $S\in\mathcal{B}_l$, $\pi(S)\in\mathcal{B}_l$. $\mathcal{H}$ is *transitive* if for every pair $(i,j)\in N\times N$ there exists a symmetry π of $\mathcal{H}$ such that $\pi(i)=j$.

If N is a finite set, $i\in N$, and $\mathcal{B}\subset 2^N$, then denote $\mathcal{B}^i = \{S\in\mathcal{B} : i\in S\}$.

Let $\mathcal{H}_i = (N_i, \{\mathcal{B}_{i,l}\}_{l\in L_i})$, $i=1,2$ be coalitional families. The *product* of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the coalitional family defined by
$N^\star = N_1\times N_2$,
$L^\star = \{(1,l):\ l\in L_1\}\cup\{(2,l):\ l\in L_2\}$,
$\mathcal{B}_{(1,l)} = \{S\subset N^\star :\ S = T\times N_2,\ T\in\mathcal{B}_{(1,l)}\}$ for all $l\in L_1$,
$\mathcal{B}_{(2,l)} = \{S\subset N^\star :\ S = N_1\times T,\ T\in\mathcal{B}_{(2,l)}\}$ for all $l\in L_2$.

Let N be a set of players. Let $\mathcal{B}_k$, $k=1,\ldots,p$ be balanced on N collections of coalitions such that

$$\mathcal{B}_i\subset\mathcal{B}_{i+1} \quad \text{for all} \quad i<p \quad \text{and} \quad \mathcal{B}_p = 2^N\setminus\{\emptyset\}.$$

Fix $k\in\{1,\ldots,p\}$. The vector of balancing weights $\{\delta_S\}_{S\in\mathcal{B}_k}$ of $\mathcal{B}_k$ can be chosen such that $\delta_S = m_S/m$ for $S\in\mathcal{B}_k$, where the numbers m and m_S are natural numbers. Hence

$$\sum_{S\in\mathcal{B}_k^i} m_S = m \quad \text{for all} \quad i\in N.$$

Denote $t=\sum_{S\in\mathcal{B}_k} m_S$.

The players $i,j\in N$ are *equivalent with respect to* $\mathcal{B}_k$ if $\mathcal{B}_k^i = \mathcal{B}_k^j$.

Let H_i be the equivalence class of player i.

Denote $r=\max\{|H_i| :\ i\in N\}$.

Sobolev, 1975 proved that there exists a coalitional family $(N_k^\star, \mathcal{B}_k^\star)$ associated with $(N,\mathcal{B}_k)$ with the following properties.
$N\subset N_k^\star$ and $|N_k^\star| = rC_t^m$.

$$\mathcal{B}_k^\star = \{T_{S,q}^\star :\ S\in\mathcal{B}_k,\ 1\le q\le m_S\}.$$

The sets $T_{S,q}^\star$, $S\in\mathcal{B}_k$, $1\le q\le m_S$ are distinct.

$$T_{S,q}^\star\cap N = S \quad \text{for all} \quad S\in\mathcal{B}_k,\ 1\le q\le m_S;$$

$$|\mathcal{B}_k^{\star i}| = m \quad \text{for each} \quad i \in N_k^\star;$$

$$|H_i^\star| = r \quad \text{for each} \quad i \in N_k^\star,$$

where $H_i^\star = \{j \in N_k^\star : \ \mathcal{B}_k^{\star i} = \mathcal{B}_k^{\star j}\}$.

Denote the product of coalitional families $(N_k^\star, \mathcal{B}_k^\star)$, $k = 1, \ldots, p$ by

$$\mathcal{H} = \left(\widehat{N}, \{\widehat{\mathcal{B}}_l\}_{l \in \{1,\ldots,p\}}\right).$$

Thus $\widehat{N} = \prod_{k=1}^p N_k^\star$ and for every $k = 1, \ldots, p$,

$$\widehat{\mathcal{B}}_k = \{\widehat{S} \subset \widehat{N} : \ \widehat{S} = N_1^\star \times \ldots \times N_{k-1}^\star \times S \times N_{k+1}^\star \times \ldots \times N_p^\star \quad \text{for some} \quad S \in \mathcal{B}_k^\star\}.$$

Define $\widehat{\mathcal{B}}_{p+1} = 2^{\widehat{N}} \setminus \bigcup_{k=1}^p \widehat{\mathcal{B}}_k$,

$$\widehat{\mathcal{H}} = \left(\widehat{N}, \{\widehat{\mathcal{B}}_l\}_{l \in \{1,\ldots,p+1\}}\right).$$

Sobolev proved that $\widehat{\mathcal{H}}$ is a transitive coalitional family. (see Sobolev, 1975, pp.137–145 or Peleg and Sudholter, 2007, pp.109-114.)

4. Proof of the main theorem

4.1. Auxiliary results

Let (N, v) be a cooperative TU game, x be a preimputation of (N, v), U be a function defined on R^1. For every $\alpha \in \mathrm{R}^1$, denote

$$\mathcal{D}(U, N, v, x, \alpha) = \{S \subset N : \ U(x(S)) - U(v(S)) \leq \alpha, \ S \neq \emptyset\}.$$

We use the following modification of the theorem proved by Kohlberg, 1971.

Theorem 3. *Let (N, v) be a cooperative TU game, U be a strictly increasing concave function defined on R^1 or on $(0, +\infty)$. A preimputation x of (N, v) is the U–prenucleolus of (N, v) if and only if each nonempty $\mathcal{D}(U, N, v, x, \alpha)$ is a balanced collection of coalitions on N.*

Proof. Since U is a strictly increasing continuous concave function, the U–prenucleolus exists and it is a singleton.

Formally, Kohlberg proved this theorem for the case $U(t) = t$, but the proof in Maschler, Solan, Zamir, 2013 (pp.816-821) is valid for our case since U is a strictly increasing function. □

Lemma 1. *Let U be a strictly increasing concave function defined on $(0, +\infty)$ with $U((0, +\infty)) = \mathrm{R}^1$.*

Then the U–prenucleolus satisfies efficiency, U–excess property, anonymity, and U–DM consistency conditions.

Proof. Efficiency and anonymity properties of the U–prenucleolus are evident. U–excess property follows from Kohlberg theorem. Let us check U–DM consistency.

Let x be U–prenucleolus of (N, v), $S \subset N$. For $\alpha \in \mathrm{R}^1$, denote

$$\mathcal{D}(U, S, v^{x,S}, x_S, \alpha) = \{P \subset S : \; U(x(P)) - U(v^{x,S}(P)) \leq \alpha, \; P \neq \emptyset\}.$$

Let $\mathcal{D}(U, S, v^{x,S}, x_S, \alpha) \neq \emptyset$. By Kohlberg theorem, we need to prove that $\mathcal{D}(U, S, v^{x,S}, x_S, \alpha)$ is a balanced collection of coalitions on S. By the definition of $v^{x,S}$, $\mathcal{D}(\alpha) = \mathcal{D}(U, N, v, x, \alpha) \neq \emptyset$ and for each $P \in \mathcal{D}(U, S, v^{x,S}, x_S, \alpha)$, there exists $Q \in \mathcal{D}(\alpha)$ such that $Q \supset P$. By Kohlberg theorem, $\mathcal{D}(\alpha)$ is a balanced collection of coalitions on N. Let $\{\delta_Q\}_{Q \in \mathcal{D}(\alpha)}$ be a vector of balancing weights of $\mathcal{D}(\alpha)$ on N. For $P \in \mathcal{D}(U, S, v^{x,S}, x_S, \alpha)$, take

$$\lambda_P = \sum_{Q \in \mathcal{D}(\alpha): \; Q \supset P} \delta_Q,$$

then $\{\lambda_P\}_{P \in \mathcal{D}(U,S,v^{x,S},x_S,\alpha)}$ is a vector of balancing weights of $\mathcal{D}(U, S, v^{x,S}, x_S, \alpha)$. □

4.2. Proof of Theorem 2.

Proof. By Lemma 1, U–prenucleolus satisfies 4 axioms.

Let f be a value defined on $\mathcal{G}^+$ that satisfies efficiency, U–excess property, anonymity, and U–DM consistency conditions. Let $(N, v) \in \mathcal{G}^+$ and x be the U–prenucleolus of (N, v). We have to prove that $f(N, v) = x$. Define (N, w) by

$$w(S) = \begin{cases} 0 \quad \text{for} \quad S = \emptyset, \\ U^{-1}\left[U(|S|) + U(v(S)) - U(x(S))\right] \quad \text{for} \quad S \neq \emptyset. \end{cases}$$

Then

$$U(x(S)) - U(v(S)) = U(|S|) - U(w(S)) \quad \text{for all} \quad S \subset N, S \neq \emptyset.$$

In view of Kohlberg theorem, U–excess property of the U–prenucleolus implies that the vector $1^{|N|} = (1, 1, \ldots, 1) \in \mathrm{R}^{|N|}$ is the U–prenucleolus of (N, w). By U–excess property of f, it is sufficient to prove that $f(N, w) = 1^{|N|}$.

Let $\{U(|S|) - U(w(S)) : \; S \subset N, S \neq \emptyset\} = \{\mu_1, \ldots, \mu_p\}$. Denote

$$\mathcal{B}_k = \{S \subset N : \; U(|S|) - U(w(S)) \leq \mu_k, S \neq \emptyset\} \quad \text{for all} \quad k = 1, \ldots, p.$$

By Kohlberg's theorem, $\mathcal{B}_k$ is a balanced collection on N. Take Sobolev's construction and notations for $\mathcal{B}_k$, $k = 1, \ldots, p$.

Define $(\widehat{N}, \widehat{w})$ as follows.

$$\widehat{w}(S) = \begin{cases} 0 \quad \text{if} \quad S = \emptyset, \\ |\widehat{N}| \quad \text{if} \quad S = \widehat{N}, \\ U^{-1}\left[U(|S|) - \mu_k\right] \quad \text{if} \quad S \in \widehat{\mathcal{B}}_k \setminus \{\emptyset, \widehat{N}\} \quad \text{for} \quad k = 1, \ldots, p, \\ U^{-1}\left[U(|S|) - \mu_p\right] \quad \text{if} \quad S \in \widehat{\mathcal{B}}_{p+1} \setminus \{\emptyset, \widehat{N}\}. \end{cases}$$

As $\widehat{\mathcal{H}}$ is transitive, for each $i, j \in \widehat{N}$ there exists a permutation π of $\widehat{N}$ such that $\pi(i) = j$ and $S \in \widehat{\mathcal{B}}_k$ implies $\pi(S) \in \widehat{\mathcal{B}}_k$ for all $k = 1, \ldots, p + 1$. As $|S| = |\pi(S)|$,

$\widehat{w}(S) = \widehat{w}(\pi(S))$ for all $S \subset \widehat{N}$, therefore by efficiency and anonymity properties of f, we get

$$f(\widehat{N}, \widehat{w})_i = 1 \quad \text{for all} \quad i \in \widehat{N}.$$

Let

$$\widehat{N}^0 = \{\widehat{i} = (i, \ldots, i) \in \widehat{N} : \ i \in N\}.$$

Consider $(\widehat{N}^0, \widehat{w}^0)$, where

$$\widehat{w}^0(P) = \begin{cases} 0 \quad \text{for} \quad P = \emptyset, \\ |\widehat{N}^0| \quad \text{for} \quad P = \widehat{N}^0, \\ U^{-1}\left(U(|P|) + \max_{T \subset \widehat{N} \setminus \widehat{N}^0}[U(\widehat{w}(P \cup T)) - U(|P \cup T|)]\right) \\ \text{for} \quad P \subset \widehat{N}^0,\ P \neq \widehat{N}^0, \emptyset. \end{cases}$$

Then by U–DM consistency, $f(\widehat{N}^0, \widehat{w}^0) = (1, \ldots, 1) \in \mathrm{R}^{|N|}$.

Let $S \subset N$, denote $\widehat{S} = \{\widehat{i} = (i, \ldots, i) \in \widehat{N} : \ i \in S\}$. We prove that

$$w(S) = \widehat{w}^0(\widehat{S}). \tag{1}$$

If $S \in \{N, \emptyset\}$ then (1) is valid. Let $S \neq N, \emptyset$.

Step 1. Let us prove that $\widehat{w}^0(\widehat{S}) \geq w(S)$. There exists $k \leq p$ such that $\mu_k = U(|S|) - U(w(S))$, i.e., $S \in \mathcal{B}_k$. By the definition of $\mathcal{B}^\star_k$, there exists $S^\star \in \mathcal{B}^\star_k$ such that $S^\star \cap N = S$. Take

$$\widehat{Q} = N^\star_1 \times \ldots \times N^\star_{k-1} \times S^\star \times N^\star_{k+1} \times \ldots \times N^\star_p,$$

then $\widehat{Q} \in \widehat{\mathcal{B}}_k$ and $\widehat{w}(\widehat{Q}) = U^{-1}\left[U(|\widehat{Q}|) - \mu_k\right]$. As $\widehat{S} \subset \widehat{Q}$ and U is a strictly increasing function,

$$U(\widehat{w}^0(\widehat{S})) \geq U(|\widehat{S}|) + U(\widehat{w}(Q)) - U(|\widehat{Q}|) = U(|S|) - \mu_k = U(w(S)),$$

hence $\widehat{w}^0(\widehat{S}) \geq w(S)$.

Step 2. We prove that $\widehat{w}^0(\widehat{S}) \leq w(S)$. Let $\widehat{T} \subset \widehat{N} \setminus \widehat{N}^0$ and

$$U(\widehat{w}^0(\widehat{S})) = U(|\widehat{S}|) + U(\widehat{w}(\widehat{S} \cup \widehat{T})) - U(|\widehat{S} \cup \widehat{T}|). \tag{2}$$

Let $k_0 = \min\{k : \ S \in \mathcal{B}_k\}$.

If $\widehat{S} \cup \widehat{T} \in \mathcal{B}^\star_{p+1}$ then

$$U(|\widehat{S} \cup \widehat{T}|) - U(\widehat{w}(\widehat{S} \cup \widehat{T})) = \mu_p \geq \mu_{k_0} = U(|S|) - U(w(S)),$$

and, by (5), this implies $\widehat{w}^0(\widehat{S}) \leq w(S)$.

Now suppose that $\widehat{S} \cup \widehat{T} \in \mathcal{B}^\star_k$, where $k \leq p$. Then there exists $S^\star \in \mathcal{B}^\star_k$ such that

$$\widehat{S} \cup \widehat{T} = N^\star_1 \times \ldots \times N^\star_{k-1} \times S^\star \times N^\star_{k+1} \times \ldots \times N^\star_p.$$

Then $S = S^\star \cap N$.

Indeed, if $i \in S$ then

$$\widehat{i} = (i, \dots, i) \in \widehat{S} \subset \widehat{S} \cup \widehat{T},$$

hence $i \in S^\star$, so $S \subset S^\star \cap N$.

If $j \in S^\star \cap N$ then $\widehat{j} = (j, \dots, j) \in \widehat{S} \cup \widehat{T}$, but $\widehat{T} \subset \widehat{N} \setminus \widehat{N}^0$ implies $\widehat{j} \notin \widehat{T}$, hence $\widehat{j} \in S$ and $S^\star \cap N \subset S$.

As $S = S^\star \cap N$ and $S^\star \in \mathcal{B}^\star_k$, we have $S \in \mathcal{B}_k$, hence $k_0 \leq k$. Then

$$U(|\widehat{S} \cup \widehat{T}|) - U(\widehat{w}(\widehat{S} \cup \widehat{T})) = \mu_k \geq \mu_{k_0} = U(|S|) - U(w(S)),$$

and by (2) this implies $\widehat{w}^0(\widehat{S}) \leq w(S)$.

Thus, $w(S) = \widehat{w}^0(\widehat{S})$ for all $S \subset N$.

Take the following bijection $\pi: \ N \to \widehat{N}^0$. $\pi(i) = (i, \dots, i) \in \mathrm{R}^{|N|}$. As was proved above, $w(S) = \widehat{w}^0(\pi S)$ for all $S \subset N$ Anonymity of f and $f(\widehat{N}^0, \widehat{w}^0) = (1, \dots, 1) \in \mathrm{R}^{|N|}$ implies $f(N, w)_i = 1$ for all $i \in N$. This completes the proof of Theorem 2. □

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