Pricing in Queueing Systems M/M/m with Delays*

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Abstract A non-cooperative m-person game which is related to the queueing system $M/M/m$ is considered. There are n competing transport companies which serve the stream of customers with exponential distribution with parameters μ_i, $i = 1, 2, ..., m$ respectively. The stream forms the Poisson process with intensity λ. The problem of pricing and determining the optimal intensity for each player in the competition is solved.

Keywords: Duopoly, equilibrium prices, queueing system.

1. Introduction

A non-cooperative n-person game which is related to the queueing system $M/M/m$ is considered. There are n competing transport companies, which serve the stream of customers with exponential distribution with parameters μ_i, $i = 1, 2, ..., m$ respectively. The stream forms the Poisson process with intensity λ. Suppose that $\lambda < \sum_{i=1}^{m} \mu_i$. Let companies declare the price for the service. Customers choose the service with minimal costs. This approach was used in the Hotelling's duopoly (Hotelling, 1929; D'Aspremont, Gabszewicz, Thisse, 1979; Mazalova, 2012) to determine the equilibrium price in the market. But the costs of each customer are calculated as the price for the service and expected time in queue. Thus, the incoming stream is divided into m Poisson flows with intensities λ_i, $i = 1, 2, ..., m$, where $\sum_{i=1}^{m} \lambda_i = \lambda$. So the problem is following, what price for the service and the intensity for the service is better to announce for the companies. Such articles as (Altman, Shimkin, 1998; Levhari, Luski, 1978; Hassin, Haviv, 2003), and (Mazalova, 2013; Koryagin 2008; Luski, 1976) are devoted to the similar game-theoretic problems of queuing processes.

2. The competition of two players

Consider the following game. There are two competitive transport companies which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The transport companies declare the price of the service c_1 and c_2 respectively. So the customers choose the service with minimal costs, and the incoming stream is divided into two Poisson flows with intensities λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \lambda$. In this case the costs of each customer will be

$$c_i + \frac{\lambda_i}{\mu_i(\mu_i - \lambda_i)}, \quad i = 1, 2,$$

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where $\lambda_i/\mu_i(\mu_i - \lambda_i)$ is the expected time of staying in a queue (Taha, 2011). So, the balance equations for the customers for choosing the service are

$$c_1 + \frac{\lambda_1}{\mu_1(\mu_1 - \lambda_1)} = c_2 + \frac{\lambda_2}{\mu_2(\mu_2 - \lambda_2)}.$$

So, the payoff functions for each player are

$$H_1(c_1, c_2) = \lambda_1 c_1, \quad H_2(c_1, c_2) = \lambda_2 c_2,$$

We are interested in the equilibrium in this game.

Nash equilibrium. For the fixed c_2 the Lagrange function for finding the best reply of the first player is defined by

$$L_1 = \lambda_1 c_1 + k\left(c_1 + \frac{\lambda_1}{\mu_1(\mu_1 - \lambda_1)} - c_2 - \frac{\lambda_2}{\mu_2(\mu_2 - \lambda_2)}\right) + \gamma(\lambda_1 + \lambda_2 - \lambda). \quad (1)$$

For finding the local maxima by differentiating (1) we get

$$\frac{\partial L_1}{\partial c_1} = \lambda_1 + k = 0$$

$$\frac{\partial L_1}{\partial \lambda_1} = c_1 + \frac{k}{\mu_1(\mu_1 - \lambda_1)} + \frac{k\lambda_1}{\mu_1(\mu_1 - \lambda_1)^2} + \gamma = 0$$

$$\frac{\partial L_1}{\partial \lambda_2} = -\frac{k}{\mu_2(\mu_2 - \lambda_2)} - \frac{k\lambda_2}{\mu_2(\mu_2 - \lambda_2)^2} + \gamma = 0$$

from which

$$c_1 = \lambda_1\left(\frac{1}{\mu_1(\mu_1 - \lambda_1)} + \frac{1}{\mu_2(\mu_2 - \lambda_2)} + \frac{\lambda_1}{\mu_1(\mu_1 - \lambda_1)^2} + \frac{\lambda_2}{\mu_2(\mu_2 - \lambda_2)^2}\right),$$

Symmetric model. Start from the symmetric case, when the services are the same, i. e. $\mu_1 = \mu_2 = \mu$. It is obvious from the symmetry of the problem, that in equilibrium $c_1^* = c_2^* = c^*$ and $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$. So

$$c^* = \frac{\lambda}{2}\left(\frac{2}{\mu(\mu - \frac{\lambda}{2})} + \frac{\lambda}{\mu(\mu - \frac{\lambda}{2})^2}\right). \quad (2)$$

So, if one of the players uses the strategy (2), the maximum of payoff of another player is reached at the same strategy. That means that this set of strategies is equilibrium.

Asymmetric model. Assume now, that transport services are not equal, i. e. $\mu_1 \neq \mu_2$, suppose that $\mu_1 > \mu_2$. Find the equilibrium in the pricing problem in this case. The system of equations that determine the equilibrium prices of transport companies is

$$c_1^* + \frac{\lambda_1}{\mu_1(\mu_1 - \lambda_1)} = c_2^* + \frac{\lambda_2}{\mu_2(\mu_2 - \lambda_2)}$$

$$c_1^* = \lambda_1\left(\frac{1}{\mu_1(\mu_1 - \lambda_1)} + \frac{1}{\mu_2(\mu_2 - \lambda_2)} + \frac{\lambda_1}{\mu_1(\mu_1 - \lambda_1)^2} + \frac{\lambda_2}{\mu_2(\mu_2 - \lambda_2)^2}\right),$$

$$c_2^* = \lambda_2\left(\frac{1}{\mu_1(\mu_1-\lambda_1)} + \frac{1}{\mu_2(\mu_2-\lambda_2)} + \frac{\lambda_1}{\mu_1(\mu_1-\lambda_1)^2} + \frac{\lambda_2}{\mu_2(\mu_2-\lambda_2)^2}\right),$$

$$\lambda_1 + \lambda_2 = \lambda.$$

In Table 1 the values of the equilibrium prices with different μ_1, μ_2 at $\lambda = 10$ and are given.

Table 1: The value of (c_1^*, c_2^*), (p_1^*, p_2^*) and (λ_1, λ_2) at $\lambda = 10$

		μ_2				
μ_1		6	7	8	9	10
7	$(c_1;c_2)$	(5,41;5,1)	(2,5;2,5)			
	$(\lambda_1;\lambda_2)$	(5,15;4,85)	(5;5)			
8	$(c_1;c_2)$	(4,04;3,64)	(1,75;1,65)	(1,11;1,11)		
	$(\lambda_1;\lambda_2)$	(5,25;4,75)	(5,14;4,86)	(5;5)		
9	$(c_1;c_2)$	(3,4;2,98)	(1,4;1,26)	(0,87;0,82)	(0,625;0,625)	
	$(\lambda_1;\lambda_2)$	(5,33;4,67)	(5,27;4,73)	(5,14;4,86)	(5;5)	
10	$(c_1;c_2)$	(3,06;2,62)	(1,21;1,04)	(0,73;0,66)	(0,52;0,59)	(0,4;0,4)
	$(\lambda_1;\lambda_2)$	(5,39;4,61)	(5,36;4,64)	(5.26;4,74)	(5,13;4,87)	(5;5)

3. The competition of m players.

Let us increase the number of players. There are m competitive transport companies which serve the stream of customers with exponential distribution with parameters μ_i, $i = 1, 2, ..., m$ respectively. The transport companies declare the price of the service c_i, $i = 1, 2, ..., m$ and the customers choose the service with minimal costs. The incoming stream is divided into n Poisson flows with intensities λ_i, $i = 1, 2, ..., m$, where $\sum\limits_{i=1}^{m} \lambda_i = \lambda$. Thus, the balance equations for the customers for choosing the service are

$$c_1 + \frac{\lambda_1}{\mu_1(\mu_1-\lambda_1)} = c_i + \frac{\lambda_i}{\mu_i(\mu_i-\lambda_i)}, \quad i = 1, ..., m.$$

The payoff functions for each player are

$$H_i(c_1, ..., c_i) = \lambda_i c_i, \quad i = 1, ..., m.$$

Find the equilibrium in this game.For the fixed c_i, $i = 2, ..., m$ the Lagrange function for finding the best reply of the first player is defined by

$$L_1 = c_1\lambda_1 + \sum_{i=2}^{m} k_i\left(c_1 + \frac{\lambda_1}{\mu_1(\mu_1-\lambda_1)} - c_i - \frac{\lambda_i}{\mu_i(\mu_i-\lambda_i)}\right) + \gamma(\sum_{i=1}^{m}\lambda_i - \lambda). \quad (3)$$

Differentiating (3),we find

$$\frac{\partial L_1}{\partial c_1} = \lambda_1 + \sum_{i=2}^{m} k_i = 0,$$

$$\frac{\partial L_1}{\partial \lambda_1} = c_1 + \frac{\sum\limits_{i=2}^{m} k_i}{\mu_1(\mu_1 - \lambda_1)} + \frac{\sum\limits_{i=2}^{m} k_i \lambda_1}{\mu_1(\mu_1 - \lambda_1)^2} + \gamma = 0,$$

$$\frac{\partial L_1}{\partial \lambda_i} = -\frac{k_i}{\mu_i(\mu_i - \lambda_i)} - \frac{k_i \lambda_i}{\mu_i(\mu_i - \lambda_i)^2} + \gamma = 0, \quad i = 2, ..., m.$$

from which

$$c_i^* = \lambda_i \left(\frac{1}{\sum_{j=0, j \neq i}^{m} (\mu_j - \lambda_j)^2} + \frac{1}{(\mu_i - \lambda_i)^2} \right),$$

$$c_i^* + \frac{\lambda_i}{\mu_i(\mu_i - \lambda_i)} = c_{i+1}^* + \frac{\lambda_{i+1}}{\mu_{i+1}(\mu_{i+1} - \lambda_{i+1})}, \quad i = 0, ..., m-1 \tag{4}$$

$$\sum_{i=1}^{m} \lambda_i = \lambda.$$

4. The competition of 2 players on graph.

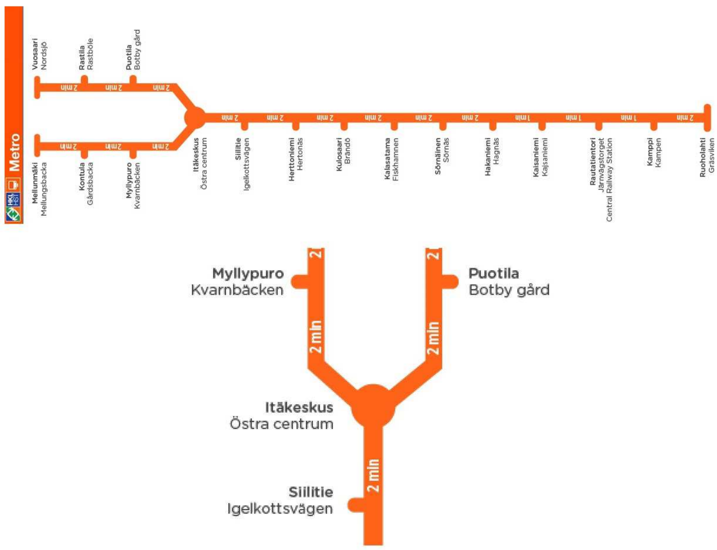

Fig. 1: Competition of 2 players on graph G_1

Consider competition on the graph G_1, which is equivalent to a part of the Helsinki Metro. Let's define the game as $\Gamma =< I, II, G_1, Z_1, Z_2, H_1, H_2 >$, where I, II are 2 competitive transport companies which serve the stream of customers with exponential distribution with parameters μ_i, $i = 1, 2$ on graph $G_1 =< V, E >$. $V = \{1, 2, 3, 4\}$ is the set of vertices, $E = \{e_{12}, e_{23}, e_{24}\}$ - the set of edges. $Z_i = \{R_1^i, R_2^i\}$

is the set of routes of player i. Each rout is a sequence of vertices. So there are two routs $R_1^i = \{1,2,3\}$ and $R_2^i = \{1,2,4\}$, $i = 1,2$. The stream of passengers forms the Poisson process with intensity Λ, where

$$\Lambda = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & 0 & \lambda_{23} & \lambda_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

The transport companies declare the price of the service c_{kj}^i, $i = 1,2$, $k = 1,2$, $j = 2,3,4$, $j \neq k$ and the customers choose the service with minimal costs. The incoming stream Λ is divided into two Poisson flows with intensities $\lambda_{kj} = \lambda_{kj}^1 + \lambda_{kj}^2$, $k = 1,2$, $j = 2,3,4$, $j \neq k$. We are interested in equilibrium in this game.

The balance equations are

$$c_{12}^1 + a_1^1 = c_{12}^2 + a_1^2,$$

$$c_{23}^1 + a_2^1 = c_{23}^2 + a_2^2,$$

$$c_{24}^1 + a_3^1 = c_{24}^2 + a_3^2,$$

$$c_{13}^1 + a_1^1 + a_2^1 = c_{13}^2 + a_1^2 + a_2^2,$$

$$c_{14}^1 + a_1^1 + a_3^1 = c_{14}^2 + a_1^2 + a_3^2,$$

$$\lambda_{kj} = \lambda_{kj}^1 + \lambda_{kj}^2, \quad k = 1,2, \quad j = 2,3,4, \quad j \neq k,$$

where

$$a_1^i = \frac{\lambda_{12}^i + \lambda_{13}^i + \lambda_{14}^i}{\mu^i(\mu^i - \lambda_{12}^i - \lambda_{13}^i - \lambda_{14}^i)},$$

$$a_2^i = \frac{\lambda_{13}^i + \lambda_{23}^i}{\frac{\mu^i}{2}(\frac{\mu^i}{2} - \lambda_{13}^i - \lambda_{23}^i)},$$

$$a_3^i = \frac{\lambda_{14}^i + \lambda_{24}^i}{\frac{\mu^i}{2}(\frac{\mu^i}{2} - \lambda_{14}^i - \lambda_{24}^i)}.$$

The payoff functions for each player are

$$H_i((c_{R_1}^1, c_{R_2}^1, c_{R_1}^2, c_{R_2}^2) = \sum_{k=1}^{2} \sum_{j=2, j\neq k}^{4} \lambda_{kj}^i c_{kj}^i, \quad i = 1,2.$$

The Lagrange function for finding the best reply of the first player is defined by

$$L_1 = \sum_{k=1}^{2} \sum_{j=2, j\neq k}^{4} \lambda_{kj}^i c_{kj}^i + k_1\left(c_{12}^1 + a_1^1 - c_{12}^2 - a_1^2\right) + k_2\left(c_{23}^1 + a_2^1 - c_{23}^2 - a_2^2\right) +$$

$$+k_3\left(c^1_{24}+a^1_3-c^2_{24}-a^2_3\right)+k_4\left(c^1_{13}+a^1_1+a^1_2-c^2_{13}-a^2_1-a^2_2\right)+$$
$$+k_5\left(c^1_{14}+a^1_1+a^1_3-c^2_{14}-a^2_1-a^2_3\right).$$

Differentiating this equation we find

$$\frac{\partial L_1}{\partial c^1_{12}}=\lambda^1_{12}+k_1=0,\quad \frac{\partial L_1}{\partial c^1_{23}}=\lambda^1_{23}+k_2=0,$$

$$\frac{\partial L_1}{\partial c^1_{24}}=\lambda^1_{24}+k_3=0,\quad \frac{\partial L_1}{\partial c^1_{13}}=\lambda^1_{13}+k_4=0,$$

$$\frac{\partial L_1}{\partial c^1_{14}}=\lambda^1_{14}+k_5=0.$$

Since $\lambda^1_{kj}=\lambda_{kj}-\lambda^2_{kj}$, $k=1,2$, $j=2,3,4$, $j\neq k$, we get

$$\frac{\partial L_1}{\partial \lambda^1_{12}}=c^1_{12}+(k_1+k_4+k_5)\left(\frac{\partial a^1_1}{\partial\lambda^1_{12}}+\frac{\partial a^2_1}{\partial\lambda^2_{12}}\right),$$

$$\frac{\partial L_1}{\partial \lambda^1_{23}}=c^1_{23}+(k_2+k_4)\left(\frac{\partial a^1_2}{\partial\lambda^1_{23}}+\frac{\partial a^2_2}{\partial\lambda^2_{23}}\right),$$

$$\frac{\partial L_1}{\partial \lambda^1_{24}}=c^1_{24}+(k_3+k_5)\left(\frac{\partial a^1_3}{\partial\lambda^1_{24}}+\frac{\partial a^2_3}{\partial\lambda^2_{24}}\right),$$

$$\frac{\partial L_1}{\partial \lambda^1_{13}}=c^1_{13}+(k_1+k_4+k_5)\left(\frac{\partial a^1_1}{\partial\lambda^1_{13}}+\frac{\partial a^2_1}{\partial\lambda^2_{13}}\right)+(k_2+k_4)\left(\frac{\partial a^1_2}{\partial\lambda^1_{13}}+\frac{\partial a^2_2}{\partial\lambda^2_{13}}\right),$$

$$\frac{\partial L_1}{\partial \lambda^1_{14}}=c^1_{14}+(k_1+k_4+k_5)\left(\frac{\partial a^1_1}{\partial\lambda^1_{14}}+\frac{\partial a^2_1}{\partial\lambda^2_{14}}\right)+(k_3+k_5)\left(\frac{\partial a^1_2}{\partial\lambda^1_{14}}+\frac{\partial a^2_2}{\partial\lambda^2_{14}}\right),$$

Symmetric model. Consider symmetric case, when the services are the same, i. e. $\mu_1=\mu_2=\mu$. It is obvious from the symmetry of the problem, that in equilibrium $c^{1*}_{kj}=c^{2*}_{kj}=c^*_{kj}$ and $\lambda^1_{kj}=\lambda^2_{kj}=\frac{\lambda_{kj}}{2}$, $k=1,2$, $j=2,3,4$, $j\neq k$. So

$$c^*_{12}=\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)}+\frac{(\lambda_{12}+\lambda_{13}+\lambda_{14})^2}{2\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)^2}$$

$$c^*_{23}=\frac{\lambda_{23}+\lambda_{13}}{\frac{\mu}{2}\left(\frac{\mu}{2}-\frac{\lambda_{23}+\lambda_{13}}{2}\right)}+\frac{(\lambda_{23}+\lambda_{13})^2}{\mu\left(\frac{\mu}{2}-\frac{\lambda_{23}+\lambda_{13}}{2}\right)^2}$$

$$c^*_{24}=\frac{\lambda_{24}+\lambda_{14}}{\frac{\mu}{2}\left(\frac{\mu}{2}-\frac{\lambda_{24}+\lambda_{14}}{2}\right)}+\frac{(\lambda_{24}+\lambda_{14})^2}{\mu\left(\frac{\mu}{2}-\frac{\lambda_{24}+\lambda_{14}}{2}\right)^2}$$

$$c^*_{13}=\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)}+\frac{(\lambda_{12}+\lambda_{13}+\lambda_{14})^2}{2\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)^2}+$$

$$+\frac{\lambda_{23}+\lambda_{13}}{\frac{\mu}{2}\left(\frac{\mu}{2}-\frac{\lambda_{23}+\lambda_{13}}{2}\right)}+\frac{(\lambda_{23}+\lambda_{13})^2}{\mu\left(\frac{\mu}{2}-\frac{\lambda_{23}+\lambda_{13}}{2}\right)^2}$$

$$c_{14}^* = \frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)}+\frac{(\lambda_{12}+\lambda_{13}+\lambda_{14})^2}{2\mu\left(\mu-\frac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{2}\right)^2}+$$

$$\frac{\lambda_{24}+\lambda_{14}}{\frac{\mu}{2}\left(\frac{\mu}{2}-\frac{\lambda_{24}+\lambda_{14}}{2}\right)}+\frac{(\lambda_{24}+\lambda_{14})^2}{\mu\left(\frac{\mu}{2}-\frac{\lambda_{24}+\lambda_{14}}{2}\right)^2}$$

In Table 2 the values of the equilibrium prices with different μ, at $\lambda_{12} = 1$, $\lambda_{23} = 1$, $\lambda_{24} = 2$, $\lambda_{13} = 3$, $\lambda_{14} = 1$ are given.

Table 2: The value of equilibrium prices at $\lambda_{12} = 1$, $\lambda_{23} = 1$, $\lambda_{24} = 2$, $\lambda_{13} = 3$, $\lambda_{14} = 1$

	μ					
prices	10	11	12	13	14	15
c_{12}^*	0,089	0,069	0,055	0,045	0,038	0,032
c_{23}^*	0,44	0,327	0,25	0,198	0,16	0,13
c_{24}^*	0,24	0,188	0,15	0,12	0,089	0,083
c_{13}^*	0,53	0,396	0,305	0,243	0,199	0,16
c_{14}^*	0,33	0,258	0,204	0,165	0,137	0,115

5. Conclusion

It is seen from the table, that the higher the intensity of service is, the lower price this transport company declare. But the prices c_{23} and c_{24}, that correspond to the edges, where the pass is divided on two roads, are greater, that c_{12}, because after this division the intensity of service is divided too.

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