

The Nash Equilibrium in Multy-Product Inventory Model

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Abstract In this paper game theory model of inventory control of a set of products is treated. We consider model of price competition. We assume that each retailer can use single-product and multi-product ordering . Demand for goods which are in stock is constant and uniformly distributed for the period of planning. Retailers are considered as players in a game with two-level decision making process. At the higher level optimal solutions of retailers about selling prices for the non-substituted goods forming Nash equilibrium are based on optimal inventory solution (order quantity or cycle duration) as a reaction to chosen prices of the players. We describe the price competition in context of modified model of Bertrand. Thus at the lower level of the game each player chooses internal strategy as an optimal reaction to competitive player's strategies which are called external. Optimal internal strategies are represented in analytical form. Theorems about conditions for existences of the Nash equilibrium in the game of price competition are proved.

Keywords:game theory, non-coalition game, Bertrand oligopoly, Nash equilibrium, logistics.

1. Introduction

Inventory management of physical goods and other products or elements is an integral part of logistic systems common to all sectors of the economy including industry, agriculture, and defense. Since the logistic costs account for up to 20% of the costs of Russian companies under the modern conditions, the issue of reducing costs for optimization of logistics systems is particularly relevant. The first paper on mathematical modeling in inventory management was written by Harris (Harris ,1915) in 1915. We may also note famous book by Hadley and Whitin (Hadley and Whitin, 1963), as well as books by Hax and Candea (Hax and Candea, 1963) and Tersine (Tersine, 1994). Inventory control systems with single decision maker capture many important aspects of inventory management. On the other hand they usually don't take into account decisions of other competitors on the market. Game theory is a mathematical theory of decision making by participants in conflicting or cooperating situations. Its goal is to explain, or to provide a normative guide for, rational behavior of individuals confronted with strategic decision or involved in social interaction. The theory is concerned with optimal strategic behavior, equilibrium situations, stable outcomes, bargaining, coalition formation, equitable allocations, and similar concepts related to resolving group differences. The field of game theory may be divided roughly in two parts, namely non-cooperative game theory and cooperative game theory. Models in non-cooperative game theory assume that each player

in the game optimizes its own objective and does not care about the effect of its decisions on others. The focus is on finding optimal strategies for each player. Binding agreements among the players are not allowed. Up to date, many researchers use non-cooperative game theory to analyse supply chain problems. Non-cooperative game theory uses the notion of a strategic equilibrium or simply equilibrium to determine rational outcomes of a game. Numerous equilibrium concepts have been proposed in the literature (van Damme, 1991). A lot of researches are devoted to analytical design of contracting arrangements to eliminate inefficiency of decision making in supply chain with several players like echelon inventory game and local inventory game (see Cachon review, 2003). Two-level strategic decision making in Bertrand type competitive inventory model for the first time was treated in (Mansur Gastratov, Victor Zakharov, 2011). Some widely used concepts are dominant strategy, Nash equilibrium, and sub-game perfect equilibrium. Nash Equilibrium says that strategies chosen by all players are said to be in Nash equilibrium if no player can benefit by unilaterally changing their strategy. Nash (Nash, 1951) proved that every finite game has at least one Nash equilibrium. Historically, most researchers establish the existence of an equilibrium based on the study of the concavity or quasi-concavity of profit function. Dasgupta and Maskin (Dasgupta, Maskin, 1986), Parlar (Parlar, 1988), Mahajan and van Ryzin (Mahajan, van Ryzin, 2001), Netessine et al. (Mahajan, van Ryzin, 2001), among others establish the existence of a Nash equilibrium based on the two above-mentioned properties of the profit function. A starting paper on mathematical models of inventory management was Harris (1915). Parlar (Parlar, 1988) was the first to analyse the inventory problem in game theory frameworks. There are two main classes of models depending on whether price or quantity is regarded as the decision variable. The static models of Cournot (Cournot, 1838) and Bertrand (Bertrand, 1883) were developed long before modern game theoretic methods. Hence to control the prices is faster and easier than to good's quantities, we use the Bertrand oligopoly model of price competition (Friedman, 1983).

The Harris – Wilson formula is a traditional method for determining the order or production quantity if you know the total uniform consumption during a period of time. The formula tries to find an optimal balance between the two costs to minimize the total cost, which is known as the economic order quantity (EOQ). The classical EOQ formula is essentially a trade-off between the ordering cost, assumed to be a flat fee per order, and inventory holding cost. A larger order quantity reduces ordering frequency, and, hence ordering cost. On the other hand, a smaller order quantity reduces average inventory but requires more frequent ordering and higher ordering cost. This formula dating for 1913 is extremely well-known (Harris, 1915).

2. Preliminaries

2.1. Non-cooperative games

Let's consider a system

$$\Gamma = \langle N, \{\Omega_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n \rangle. \quad (1)$$

This system is called a non-cooperative game, where

$N = \{1, 2, \dots, n\}$ – set of players,

Ω_i – set of strategies of player i ,

Π_i – payoff function of player i .

Players make an interactive decisions simultaneously choosing their strategies x_i from strategy sets Ω_i . The agreements and coalition formations are forbidden. Vector $x = (x_1, \dots, x_n)$ is called situation in the game. As a result players are paid payoff $\Pi_i = \Pi_i(x)$. A game is a formal representation of a situation in which a number of decision makers (players) interact in a setting of strategic interdependence. By that, we mean that the welfare of each decision maker depends not only on his own actions, but also on the actions of the other players. Moreover, the actions that are best for him to take may depend on what he expects the other players to do. We say that game theory analyzes interactions between rational, decision-making individuals who may not be able to predict fully the outcomes of their actions. We call $x^* = (x_1^*, \dots, x_n^*)$ a Nash equilibrium if for all admissible strategies $x_i \in \Omega_i$, $i = 1, \dots, n$ the following inequalities hold

$$\Pi_i(x^*) \geq \Pi_i(x_1^*, x_2^* \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_n^*).$$

Theorem 1. (Kukushkin, Morozov, 1984) *In game (1) there exists Nash equilibrium in pure strategies if for each $i \in N$ strategy set Ω_i is compact and convex, and payoff function $\Pi_i(x)$ is concave with respect to x_i and continuous on $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$.*

Assume for any $i \in N$ the function $\Pi_i(x)$ is continuously differentiable with respect to x_i . From (Tirol, 2000) we can see that first-order necessary condition for Nash equilibrium is the following

$$\frac{\partial \Pi_i(x^*)}{\partial x_i} = 0, i \in N. \quad (2)$$

Suppose the payoff function $\Pi_i(x)$, $i = 1, \dots, n$ is concave for all $x_i \in \Omega_i$. In this case solution of system (2) appears to be a Nash equilibrium in pure strategies in non-cooperative game

$$\Gamma = \langle N, \{\Omega_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n \rangle.$$

2.2. Oligopoly

There are two most notable models in oligopoly theory: Cournot oligopoly, and Bertrand oligopoly. In the Cournot model, firms control their production level, which influences the market price. In the Bertrand model, firms choose the price to charge for a unit of product, which affects the market demand.

Definition 1. Non-cooperative oligopoly is a market where a small number of firms act independently but are aware of each other's actions.

In the oligopoly model we suppose that:

1. firms are rational;
2. firms reason strategically.

Firms or players meet only once in a single period model. The market then clears one and for all. There is no repetition of the interaction and hence, no opportunity for the firms to learn about each other over time. Such models are appropriate for markets that last only a brief period of time. Cournot and Bertrand oligopolies are modeled as strategic games, with continuous action sets (either production levels or prices). We study competitive markets in which firms use price as their strategic variable.

3. The Price Competition

Let's consider a market with n retailers sell m products: $i = 1, \dots, n, j = 1, \dots, m$. In this model each supplier forms for the planning period T single-product orders to supplier.

Let q_{ij} be the quantity of product j in the order; $q_i = (q_{i1}, \dots, q_{im})$ – the order vector of supplier i . After the order was received retailer assigns the prices for every good for selling: $p_i = (p_{i1}, \dots, p_{im}), i = 1, \dots, n$ – price vector of player i , where p_{ij} – the price assigned by retailer i for product j .

Assume the demand for items of the product is known and uniform during a period of planning. $D_{ij}(p_{1j}, \dots, p_{nj})$ – demand function for good j with price appointed by player i and other players. This is the inverse function of p_i . In the price competition the demand on the good depends on prices appointed by other players. Due to the single-product orders the total inventory cost function of retailer i could be expressed as

$$\begin{aligned} TC_i(p_1, \dots, p_n, q_{i1}, \dots, q_{im}) &= \\ &= \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^O \frac{D_{ij}(p_{1j}, \dots, p_{nj})}{q_{ij}} + c_{ij}^H \frac{q_{ij}}{2} \right], \end{aligned}$$

where

$p_i = (p_{i1}, \dots, p_{im}),$

c_{ij}^O – order cost per unit of good j for player i ,

c_{ij}^H – holding cost per unit of product j for retailer i during period T ,

c_j – procurement price of product j fixed by supplier.

We assume that prices satisfy the conditions:

$$p_{ij} > c_j, \quad i = 1, \dots, n, \quad j = 1, \dots, m.$$

The payoff function is expressed as

$$\begin{aligned} \Pi_i(p_1, \dots, p_n, q_i) &= \\ &= \sum_{j=1}^m p_{ij} D_{ij}(p_{1j}, \dots, p_{nj}, q_i) - TC_i(p_1, \dots, p_n, q_i). \end{aligned} \tag{3}$$

Following single-product inventory game theory model (Mansur Gastratov, Victor Zakharov, 2011) we consider expanding this model for multi-product one under the condition of single-product ordering strategies of the players (retailers). As in single-product model retailer has to calculate inventory decision on order quantity of a product as an optimal reaction for the products prices assigned by all players. That is we also can introduce internal and external strategies of the players as follows.

Definition 2. We define $q_i = (q_{i1}, \dots, q_{im})$ as internal strategy, and $p_i = (p_{i1}, \dots, p_{im})$ as external strategy of player i .

To take into account influence of external strategy of player to internal one we could find optimal reaction of the retailer to prices assigned by all players. We would realize two-stage procedure.

To find optimal internal strategy retailer has to solve the following problem

$$\begin{aligned} & \min_{(q_{i1}, \dots, q_{im})} TC_i(p_{i1}, \dots, p_{im}, q_i) = \\ & = \min_{(q_{i1}, \dots, q_{im})} \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^O \frac{D_{ij}(p_{1j}, \dots, p_{nj})}{q_{ij}} + c_{ij}^H \frac{q_{ij}}{2} \right]. \end{aligned}$$

The value of the economic order quantity is defined as internal player strategy $q_i = (q_{i1}, \dots, q_{im})$. In this case it is possible to use the Harris-Wilson formula because of demand function has additive form. Now we have

$$q_{ij}^* = \sqrt{\frac{2c_{ij}^O D_{ij}(p_{1j}, \dots, p_{nj})}{c_{ij}^H}}. \tag{4}$$

Now it is possible to substitute the optimal q_{ij}^* in (3). We get the new payoff function:

$$\tilde{\Pi}_i(p_{i1}, \dots, p_{im}) = \sum_{j=1}^m p_{ij} D_{ij}(p_{1j}, \dots, p_{nj}, q_i) - TC_i^*(p_{i1}, \dots, p_{im}),$$

where

$$\begin{aligned} & TC_i^*(p_{i1}, \dots, p_{im}) = \\ & = \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^O \frac{D_{ij}(p_{1j}, \dots, p_{nj})}{\sqrt{\frac{2c_{ij}^O D_{ij}(p_{1j}, \dots, p_{nj})}{c_{ij}^H}}} + c_{ij}^H \frac{\sqrt{\frac{2c_{ij}^O D_{ij}(p_{1j}, \dots, p_{nj})}{c_{ij}^H}}}{2} \right]. \end{aligned}$$

On the next stage the modified price competition of Bertrand oligopoly is considered. Retailers choose the goods' prices according to the price competition with other players in non-cooperative game:

$$\Gamma = \left\langle N, \left\{ \tilde{\Pi}_i \right\}_{i=1}^n, \left\{ \Omega_i \right\}_{i=1}^n \right\rangle, \tag{5}$$

where $N = 1, \dots, n$ – set of players,

Ω_i – strategy set of player i ,

where $\Omega_i = \Omega_{i1} \times \Omega_{i2} \times \dots \times \Omega_{im}$,

$\Omega_{ij} = \{p_{ij} \mid p_{ij} > c_j\}$, $i = 1, \dots, n$, $j = 1, \dots, m$.

$\tilde{\Pi}_i(p_{i1}, \dots, p_{im})$ – payoff function of player i .

This function depends on external player strategies $(p_{i1}, \dots, p_{im}) \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$.

Every player i chooses external strategy $p_i \in \Omega_i$, which gives the decision of problem

$$\begin{aligned} & \max_{(p_1, \dots, p_n)} \Pi_i(p_{i1}, \dots, p_{im}) = \max_{(p_1, \dots, p_n)} \left[\sum_{j=1}^m p_{ij} D_{ij}(p_{1j}, \dots, p_{nj}) - \right. \\ & \left. - \sum_{j=1}^m \left(c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^O \frac{D_{ij}(p_{1j}, \dots, p_{nj})}{\sqrt{\frac{2c_{ij}^O D_{ij}(p_{1j}, \dots, p_{nj})}{c_{ij}^H}}} + c_{ij}^H \frac{\sqrt{\frac{2c_{ij}^O D_{ij}(p_{1j}, \dots, p_{nj})}{c_{ij}^H}}}{2} \right) \right]. \end{aligned}$$

The optimal strategy is achieved by finding Nash equilibrium (Nash strategies), which is the most commonly used solution concept in game theory. The convexity, continuity and differentiability for payoff function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is necessary and sufficient for existence of Nash equilibrium. In terms of convenience we denote

$$\widetilde{C}_{ij} = \frac{\sqrt{c_{ij}^H}}{\sqrt{2}} + \left(1 + \sqrt{c_{ij}^O}\right).$$

Now we can rewrite the cost function as

$$TC_i^*(p_{i1}, \dots, p_{im}) = \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + \widetilde{C}_{ij} D_{ij}(p_{1j}, \dots, p_{nj})^{1/2} \right]$$

and payoff function is expressed as

$$\begin{aligned} & \widetilde{\Pi}_i(p_{i1}, \dots, p_{im}) = \\ & = \sum_{j=1}^m p_{ij} D_{ij}(p_{1j}, \dots, p_{nj}) - \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + \widetilde{C}_{ij} D_{ij}(p_{1j}, \dots, p_{nj})^{1/2} \right]. \end{aligned}$$

The existence of Nash equilibrium depends on demand function. There exist two cases for demand function form.

First case: the case when demand function is in linear form.

Proposition 1. *If for every $i \in N$ strategy set Ω_i is compact and convex and demand function has a linear form then there exists the Nash equilibrium in the game (5).*

Proof. The demand function is linear, continuous and differentiable. The square root function $(D_{ij}(p_{1j}, \dots, p_{nj}))^{1/2}$ is concave, continuous and differentiable with respect to p_{ij} on Ω_i . The payoff function $\widetilde{\Pi}_i(p_1, \dots, p_n)$ is represented as difference of linear function and square root function. From the properties of this function we get that the payoff function is concave, continua, and differentiable with respect to p_{ij} on Ω_i . Which leads to existence of unique Nash equilibrium. \square

Second case: the case of non-linear demand function. In general case for existence of Nash equilibrium we need the special conditions.

Proposition 2. *Let the following conditions be satisfied:*

1. *for every $i \in N$ strategy set Ω_i is compact and convex;*
2. *the demand function $D_{ij}(p_{1j}, \dots, p_{nj})$ is continuous and differentiable with respect to p_{ij} on Ω_i ;*
3. *the function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is concave with respect to p_{ij} on Ω_i , $i = 1, \dots, n$, $j = 1, \dots, m$.*

Under these conditions the Nash equilibrium in game $\Gamma = \langle N, \{\widetilde{\Pi}_i\}_{i=1}^n, \{\Omega_i\}_{i=1}^n \rangle$ exists.

Proof. If function $D_{ij}(p_{1j}, \dots, p_{nj})$ is continuous and differentiable function on Ω_i with respect to p_{ij} , then function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is continuous and differentiable with respect to p_{ij} on $\Omega_i, i = 1, \dots, n, j = 1, \dots, m$. And if function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is concave, then all conditions of Theorem 1 are satisfied and there exists the Nash equilibrium. \square

According to the Teorem 1, there exist the Nash equilibrium in this game. And, from (Tirol, 2000), the equilibrium point (p_1^*, \dots, p_n^*) is found by solving set of equations:

$$\begin{aligned} & \frac{\partial \widetilde{\Pi}_i(p_1, \dots, p_n)}{\partial p_{ij}} = \\ & = \sum_{j=1}^m \left[D_{ij}(p_{i1}, \dots, p_{im}) + (p_{ij} - c_j) \frac{\partial D_{ij}(p_{1j}, \dots, p_{nj})}{\partial p_{ij}} - \right. \\ & \left. - \frac{\widetilde{C}_{ij}}{2\sqrt{D_{ij}(p_{1j}, \dots, p_{nj})}} \frac{\partial D_{ij}(p_{1j}, \dots, p_{nj})}{\partial p_{ij}} \right], \quad i = 1, \dots, n, \quad j = 1, \dots, m. \end{aligned}$$

The issue appears when $D_{ij}(p_{1j}, \dots, p_{nj}) = 0$. In this case prices set by player are too high and demand is zero. The player does not participate in price competition. Thus, we suppose that $D_{ij}(p_{1j}, \dots, p_{nj}) > 0$. Substituting the solutions of this equation set (p_1^*, \dots, p_n^*) to (4) we will finally get the q_{ij}^* – the optimal goods quantity for player’s order.

4. The Multi-Product Orders.

As in the first case, we have the market with m products and n retailers: $i = 1, \dots, n, j = 1, \dots, m$.

Let $\tau_i T$ be duration of planning period.

Suppose that all retailers set multi-product orders assigning equal duration of cycles between deliveries as a part of period T . Let $\tau_i T$ be duration of planning period. Due to uniform demand the following equality takes place for quantity q_{ij} in multi-product order:

$$q_{ij} = \tau_i D_{ij}(p_{1j}, \dots, p_{nj}).$$

Assume that ordering cost for multi-product order for player i is equal to c_i^{MO} . Then total inventory cost function for player i can be expressed as follows:

$$TC_i(p_{i1}, \dots, p_{im}, \tau_i) = \frac{c_i^{MO}}{\tau_i} + \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^H \frac{\tau_i D_{ij}(p_{1j}, \dots, p_{nj})}{2} \right].$$

Payoff function is described as

$$\begin{aligned} \Pi_i(p_{i1}, \dots, p_{im}, \tau_i) &= \sum_{j=1}^m p_{ij} D_{ij}(p_{1j}, \dots, p_{nj}) - \\ & - \frac{c_i^{MO}}{\tau_i} - \sum_{j=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^H \frac{\tau_i D_{ij}(p_{1j}, \dots, p_{nj})}{2} \right]. \end{aligned}$$

As in the case of single-product ordering we use the two stage decision making procedure

To find optimal internal strategy of player i we have to solve the following problem:

$$\begin{aligned} & \min_{\tau_i} TC_i(p_{i1}, \dots, p_{im}, \tau_i) = \\ & = \min_{\tau_i} \left(\frac{c_i^{MO}}{\tau_i} + \sum_{i=1}^m \left[c_j D_{ij}(p_{1j}, \dots, p_{nj}) + c_{ij}^H \frac{\tau_i D_{ij}(p_{1j}, \dots, p_{nj})}{2} \right] \right). \end{aligned}$$

This is the problem EQO (optimal economic order). Now it is possible to use Harris – Wilson formula because of cost function additive form. Using the dependence form

$$q_{ij} = \tau_i D_{ij}(p_{1j}, \dots, p_{nj}),$$

it's possible to find the optimal τ_i^* :

$$\tau_i^* = \sqrt{\frac{2c_i^{MO}}{\sum_{j=1}^m c_{ij}^H D_{ij}(p_{1j}, \dots, p_{nj})}}. \quad (6)$$

Substituting this optimal τ_i^* into the formula of payoff function we obtain

$$\begin{aligned} & \widetilde{\Pi}_i(p_{i1}, \dots, p_{im}) = \\ & = \sum_{j=1}^m (p_{ij} - c_j) D_{ij}(p_{1j}, \dots, p_{nj}) - \frac{2}{\sqrt{2}} (c_i^{MO})^{1/2} \left(\sum_{j=1}^m c_{ij}^H D_{ij}(p_{1j}, \dots, p_{nj}) \right)^{1/2} \end{aligned}$$

As a result we get that the payoff function depends on external strategies only.

On the next stage Bertrand oligopoly with price competition is considered. On the second stage player finds optimal prices according to the competition with other players. Let's consider non-cooperative game

$$\Gamma = \left\langle N, \left\{ \widetilde{\Pi}_i \right\}_{i=1}^n, \left\{ \Omega_i \right\}_{i=1}^n \right\rangle,$$

Ω_i – strategy set of player i ,

where $\Omega_i = \Omega_{i1} \times \Omega_{i2} \times \dots \times \Omega_{im}$,

$\Omega_{ij} = \{p_{ij} \mid p_{ij} > c_j\}$, $i = 1, \dots, n$, $j = 1, \dots, m$.

$\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ – payoff function of player i . This function depends on external player strategies $(p_{i1}, \dots, p_{im}) \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$.

Every player i chooses external strategy $p_i \in \Omega_i$, which gives the decision of problem. The aim of each player is to maximize their payoff in the price competition:

$$\widetilde{\Pi}_i(p_{i1}, \dots, p_{im}) \xrightarrow{p_{i1}, \dots, p_{im}} \max.$$

According to the Teorem 1, there exist the Nash equilibrium in this game.

The existence of Nash equilibrium depends on the form of payoff function.

Proposition 3. *Suppose the following conditions for $i = 1, \dots, n$ are satisfied:*

1. for every $i \in N$ strategy set Ω_i is compact and convex;

2. demand function has a linear form.

Under these conditions Nash equilibrium in pure strategies exists.

Proof. If function $D_{ij}(p_{1j}, \dots, p_{nj})$ has a linear form, then it is continuous and differentiable function on Ω_i with respect to p_{ij} , and function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is continuous and differentiable with respect to p_{ij} on Ω_i , $i = 1, \dots, n$, $j = 1, \dots, m$. Function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is represented as difference of linear function and square root function. From the properties of this functions follows that the payoff function is concave, continua and differentiable with respect to p_{ij} on Ω_i , $i = 1, \dots, n$, $j = 1, \dots, m$. All conditions of Theorem 1 are satisfied and there exists the Nash equilibrium. \square

Proposition 4. Suppose the following conditions for $i = 1, \dots, n$ are satisfied:

1. for every $i \in N$ strategy set Ω_i is compact and convex;
2. demand function $D_{ij}(p_{1j}, \dots, p_{nj})$ is continuous and differentiable with respect to p_{ij} on Ω_i ;
3. payoff function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is concave with respect to p_{ij} on Ω_i .

Under these conditions Nash equilibrium in pure strategies exists.

Proof. If function $D_{ij}(p_{1j}, \dots, p_{nj})$ is continuous and differentiable function on Ω_i with respect to p_{ij} , then function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is continuous and differentiable with respect to p_{ij} on Ω_i , $i = 1, \dots, n$, $j = 1, \dots, m$. And if function $\widetilde{\Pi}_i(p_{i1}, \dots, p_{im})$ is concave, then all conditions of Theorem 2 are satisfied and there exists the Nash equilibrium. \square

From (Tirol, 2000), the equilibrium point (p_1^*, \dots, p_n^*) is found by solving set of equations:

$$\frac{\partial \widetilde{\Pi}_i(p_1, \dots, p_n)}{\partial p_{ij}} = 0,$$

$$i = 1, \dots, n.$$

After finding optimal strategy value of $p^* = (p_1^*, \dots, p_n^*)$ it is possible to substitute its to (6) and to calculate the optimal value of period τ_i^* .

5. Conclusion

In this paper game theory models for multi-product inventory control are treated in case of competition among retailers. The model of price competition in context of modied model of Bertran is considered. Each retailer can use two order types: single-product and multi-product ordering. Demand for each product is supposed to be uniform for the period of planning. In game theory model retailers are considered as players using two-level strategies. At the lower level of the game each player chooses internal strategy as an optimal reaction to competitive players strategies which are called external. Optimal internal strategies are represented in analytical form. Necessary and sufficient conditions for existence of Nash equilibrium in pure strategies for the cases of linear and non-linear demand functions are proposed.

References

- Nash, J. F. (1951). *Non-Cooperative games*. Annals of Mathematics, **54**, 286–295.
- Harris, F. (1915). *Operation and Cost*. Factory Management Series. A.W. Shaw, Chicago, 48–52.
- van Damme, E. (1991). *Stability and Perfection of Nash Equilibria*. Springer-Verlag, Berlin.
- Nash, J. F. (1950). *Equilibrium Points in n-Person Games*. Proceedings of the National Academy of Sciences, **36**, 48–59.
- Dasgupta, P., Maskin, E. (1986). *The Existence of Equilibrium in Discontinuous Economic Games*. Review of Economics Studies **53**, 1–26.
- Mahajan, S., van Ryzin, G. J. (2001). *Inventory Competition Under Dynamic Consumer Choice*. Operations Research, **49(5)**, 646–657.
- Netessine, S., Rudi, N., Wang, Y (2003). *Dynamic Inventory Competition and Customer Retention*. Working paper.
- Parlae, M. (1988). *Game Theoretic Analysis of the Substitutable Product Inventory Problem with Random Demands*. Naval Research Logistics, **35**, 397–409.
- Tirol, Jean (2000). *The markets and the market power: The organization and industry theory*. Translation from English by J. M. Donc, M. D. Facsirova, under edition A. S. Galperina and N. A. Zenkevich. SPb: Institute Economic school, in 2 Volumes. V.1. 328 p. V.2. 240 p.
- Kukushkin, N. S., Morozov, V. V. (1984). *The nonantagonistic Game Theory*. M.: Moscow State University, 103 p.
- Friedman, James (1983). *Oligopoly Theory*. Cambridge University Press, 260 p.
- Bertrand, J. (1883). *Theorie mathematique de la richesse sociale*. Journal des Savants., 67, 499–508.
- Cournot, A. A. (1883). *Recherches sur les principes mathematiques de la thorie des richesses*. Paris, L. Hachette, 198 p.
- Hax, A. C. and D. Candea (1984). *Production and inventory management*. Prentice-Hall. Englewood Cliffs, N.J., 135 p.
- Haldey, G. and T. M. Whitin (1963). *Analysis of inventory*. Prentice-Hall, Englewood Cliffs, N.J.
- Tersine, R. J. (1994). *Principles of inventory and materials management*. Elsevier North Holland, Amsterdam.
- Cachon, G. P. (2003). *Supply chain coordination with contracts*. Handbook in Operations Research and Management Science: Supply Chain Management. Vol. 11, Elsevier B. V., Amsterdam, pp. 229–340.
- Mansur Gasratov, Victor Zakharov (2011). *Games and Inventory Management*. In: Dynamic and Sustainability in International Logistics and Supply Chain Management. Cuvillier Verlag, Gottingen.