Phenomenon of Narrow Throats of Level Sets of Value Function in Differential Games*

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Abstract A number of zero-sum differential games with fixed termination instant are given, in which a level set of the value function has one or more time sections that are almost degenerated (have no interior). Presence of such a peculiarity make very high demands on the accuracy of computational algorithms for constructing value function. Analysis and causes of these degeneration situations are important during study of applied pursuit problems.

Keywords: linear differential games, fixed termination instant, level sets of value function, geometric methods, narrow throats

1. Introduction

During investigating zero-sum differential games, the main topic is constructing and studying the value function of the game. One of the traditional approaches to value function construction is to solve the corresponding Hamilton–Jacobi–Bellman–Isaacs partial differential equation. Another approach is based on the representation of the value function as a collection of its level sets (Lebesgue sets). These sets are built by means of a geometric method.

This representation is the most intuitive when the phase vector of the game is two-dimensional or when the game can be reduced to such a situation. In this case, any level set is located in a three-dimensional space *time* $\times$ *two-dimensional phase space* and can be effectively constructed and visualized to graphic study of its structure and peculiarities. The result of constructions is often a collection of polygons that approximate its time sections (t-sections) on some time grid.

A very important thing both from theoretic and numerical points of view is loss of interior by t-sections of a level set at some instant. Further its evolution (in the backward time) can lead to complete degeneration of the set (its t-sections become empty), or can bring back the interior. The last case corresponds to the situation when we say that the level set has a *narrow throat*.

Earlier, the authors have investigated appearance of narrow throats in linear differential game with fixed termination instant and terminal convex payoff function (Kumkov et al., 2005). That game appears during study an interception problem of one weak-maneuvering object by another one. This paper contains a number of

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examples of another games with convex payoff function, in which there are narrow throats. Also, we consider games having non-convex payoff. They arise from a pursuit game with two pursuers and one evader. The study is made numerically by algorithms and programs worked out by the authors.

2. Games with Convex Payoff Function

2.1. Problem Formulation

Let us consider a zero-sum linear differential game (Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988):

$$\begin{aligned} &\dot{\mathbf{z}} = A(t)\mathbf{z} + B(t)u + C(t)v, \\ &t \in [t_0; T],\ \mathbf{z} \in R^n,\ u \in P \subset R^p,\ v \in Q \subset R^q, \\ &\varphi\big(z_i(T), z_j(T)\big) \to \min_u \max_v . \end{aligned} \tag{1}$$

The first player governs the control u and minimizes the payoff φ; the second player choosing its control v maximizes the payoff. The sets P and Q that constrain the players' controls are convex compacta in their spaces. The payoff function φ depends on values of two components of the phase vector at the termination instant and is convex.

It is necessary to construct level sets of the value function and study them from the point of view of narrow throat presence.

2.2. Equivalent Differential Game

A standard approach to study linear differential games with fixed termination instants and payoff function depending on a part of phase coordinates at the termination instant assumes a passage to a new phase vector; see, for example, (Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988). These new variables are regarded as the values of the target components forecasted to the termination instant under zero players' controls. Often they are called *zero effort miss coordinates* (Shima and Shinar, 2002; Shinar and Shima, 2002). In our case, we pass to new coordinates x_1 and x_2, where $x_1(t)$ is the value of the component z_i forecasted from the current instant t to the termination instant T, and x_2 is the forecasted value of the component z_j.

To obtain constructively the forecasted values, one uses a matrix combined of two rows of the fundamental Cauchy matrix $X(T,t)$ for the system $\dot{\mathbf{z}} = A(t)\mathbf{z}$. These rows correspond to the target components of the phase vector. In our case, we use the ith and jth rows of the Cauchy matrix. The change of variables is described by the formula $x(t) = X_{i,j}(T,t)\mathbf{z}(t)$. (The subindices i, j of the matrix X denote taking the corresponding rows of the fundamental Cauchy matrix.)

The equivalent game has the following form:

$$\begin{aligned} &\dot{x} = D(t)u + E(t)v, \\ &t \in [t_0; T],\ x \in R^2, u \in P,\ v \in Q,\ \varphi\big(x_1(T), x_2(T)\big), \\ &D(t) = X_{i,j}(T,t)B(t),\ E(t) = X_{i,j}(T,t)E(t). \end{aligned} \tag{2}$$

Further to analyze the evolution in time of the time sections of the level sets of the value function, it is useful to involve the sets $\mathcal{P}(t) = D(t)P$, $\mathcal{Q}(t) = E(t)Q$, which are called *vectograms* of the players at the instant t. The sense of the vectograms is the collection of velocities that can be given to the system by the players at the corresponding time instant. If one has that $\mathcal{Q}(t) \subset \mathcal{P}(t)$, then it can be said that at the instant t the first player *has (dynamic) advantage*. In the case of opposite inclusion, we say about advantage of the second player.

2.3. Numerical Construction of Level Sets

Fix a value c and describe construction of an approximation of the level set $\mathcal{W}_c$ of the value function $\mathcal{V}$ of game (2). The set will correspond to the chosen constant c.

For a numerical construction, at first, let fix a time grid $\{t_j\}$, $t_0 < t_1 < \ldots < t_N = T$. The constructions are made in the backward time from the termination instant T. Let at some instant t_{j+1} we have an approximation $W_c(t_{j+1})$ of the t-section $\mathcal{W}_c(t_{j+1})$ of the level set $\mathcal{W}_c$. Then the approximation $W_c(t_j)$ of the t-section $\mathcal{W}_c(t_j)$ is described by the following formula (Pschenichnyi and Sagaidak, 1970):

$$W_c(t_j) = \bigl(W_c(t_{j+1}) + (-\Delta_j)D(t_j)P\bigr) \overset{*}{-} \Delta_j E(t)Q. \qquad (3)$$

Here, $\Delta_j = t_{j+1} - t_j$; $D(t_j)$ and $E(t_j)$ are the matrices from dynamics (2) computed at the instant t_j; P and Q are the sets constraining the controls of the first and second players. The sign "+" denotes the operation of algebraic sum (Minkowski sum), and "$\overset{*}{-}$" denotes the geometric difference (Minkowski difference).

The initial set $W_c(T)$ for the procedure is taken as a convex polygon M_c close in the Hausdorff metrics to the convex level set $\mathcal{M}_c = \bigl\{(x_1, x_2) : \varphi(x_1, x_2) \le c\bigr\}$ of the payoff function. Convexity of the set $\mathcal{M}_c$ is due to the convexity of the payoff function.

It is known that in linear differential games with fixed termination instant, convexity of the target set provides convexity of all t-sections $W_c(t_j)$ of the corresponding solvability set (the maximal stable bridge). Therefore, in procedure (3) we can apply algorithms for processing convex sets. In iteration procedures suggested by one of the authors (Isakova et al., 1984; Kumkov et al., 2005), convex sets in the plane are described by their support functions. (There is a one-to-one correspondence between a convex compact non-empty set S and its support function $\rho(l; S) = \max\bigl\{\langle l, s\rangle : s \in S\bigr\}$, which is positively-homogeneous; here, $\langle \cdot, \cdot \rangle$ denotes a dot product.) With that, to construct the support function of Minkowski sum of two sets we should just construct the sum of the support functions of the summands. To obtain the support function of Minkowski difference of two sets, it is necessary to build convex hull of difference of support functions of the initial sets. Also there is a very helpful fact that the support function of a convex polygon is piecewise-linear with areas of linearity in the cones between outer normals to its neighbor edges. Due to all these properties, it is possible to suggest effective procedures for addition of sets, subtraction, and convex hull construction.

During the backward constructions, the current section $W_c(t_{j+1})$ is summed with the dynamic capabilities $(-\Delta_j)D(t_j)P = (-\Delta_j)\mathcal{P}(t_j)$ of the first player and further subtracted by dynamic capabilities $\Delta_j E(t_j)Q = \Delta_j \mathcal{Q}(t_j)$ of the second player. Thus, the change of the t-section is connected to the correlation of the players' vectograms. If the first player's vectogram is "greater" than the vectogram of the second one (that is, if the first player has advantage), then the t-section grows in the backward time. In the opposite situation when the second player has advantage, vice versa, the section contracts in the backward time. If neither $\mathcal{P}(t) \subset \mathcal{Q}(t)$, nor $\mathcal{Q}(t) \subset \mathcal{P}(t)$, then the first player has advantage in some directions and disadvantage in others. Studying the situation of advantage of one or other player allows to estimate qualitatively the evolution of the level set in time without its exact construction.

2.4. Examples

One-to-One Interception Problem. In the works (Shinar et al., 1984; Shinar and Zarkh, 1996; Melikyan and Shinar, 2000), a three-dimensional problem of interception in near space or upper atmosphere is considered. The pursuer P is an intercept-missile; the evader E is a weak maneuverable target (for example, another missile or a large aircraft). The geometry of the interception is drawn in Fig. 1. The three-dimensional problem reduces naturally to a two-dimensional one. The longitudinal velocities of the objects are rather large, and the approach time is small. Thus, the control accelerations a_P and a_E that are orthogonal to the current velocity of the corresponding object cannot turn significantly the velocity vectors. Due to this, the longitudinal motion of the objects can be considered as uniform. Also, the minimal approach distance, which is the natural payoff in this game, can be changed by the lateral distance at the instant of nominal longitudinal passage of the objects. This instant is fixed as the termination one.

The three-dimensional geometric coordinates can be introduced as it is shown in Fig. 1. The origin O is put at the position of the pursuer P. The axis OX coincides with the nominal line-of-sight. The axis OY is orthogonal to OX and is located in the plane defined by the vectors of the nominal velocities of the objects. The axis OZ is orthogonal to OX and OY.

After excluding the longitudinal motion along the axis OX from consideration, we pass to a two-dimensional problem of lateral motion in the plane OYZ. The control of the evader defines its acceleration directly; the pursuer has a more complicated dynamics. Its control affects the acceleration through a link of the first order:

$$\begin{array}{ll} \ddot{r}_P = F, & \\ \dot{F} = (u - F)/l_P, & t \in [0;T],\ r_P,\, r_E \in R^2, \quad u \in P, \quad v \in Q, \\ \ddot{r}_E = v, & \varphi\big(x(T), y(T)\big) = \big\|r_P(T) - r_E(T)\big\|. \end{array} \qquad (4)$$

Here, r_P and r_E are the radius-vectors of the positions of the pursuer and evader in the plane OYZ; l_P is the *time constant* that describes the inertiality of servomechanisms transferring the *control command signal* u to the acceleration F; v is the

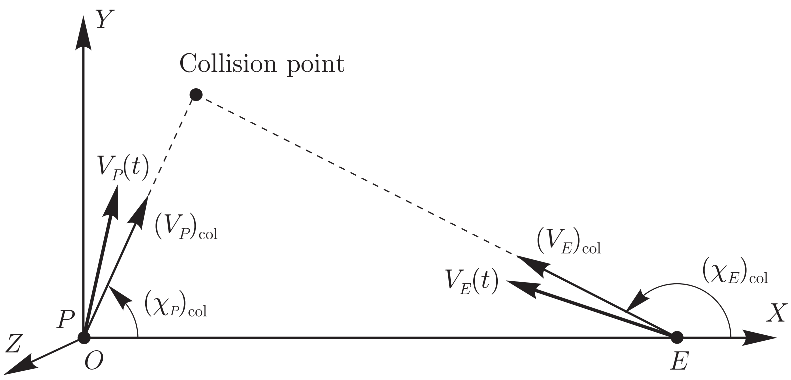

Fig. 1: The geometry of the three-dimensional interception. The actual realizations of the velocity vectors $V_P(t)$ and $V_E(t)$ are close to the nominal values $(V_P)_{\rm col}$ and $(V_E)_{\rm col}$

evader's control; T is the termination instant coinciding with the instant of longitudinal passage of objects along the nominal motions. The sets P and Q constraining the controls of the players are ellipses. These ellipses are obtained after projection of the original round vectograms on accelerations (that are orthogonal to the nominal velocities $(V_P)_{\rm col}$ and $(V_E)_{\rm col}$) into the plane OYZ. The parameters of the ellipse (the semiaxes) are defined by the maximal acceleration of the corresponding object (a_P or a_E) and by the angle between the vector of its velocity and the line-of-sight ($(\chi_P)_{\rm col}$ or $(\chi_E)_{\rm col}$).

To pass to a standard game with the payoff depending on two components of the phase vector, we use the following change of variables:

$$\begin{array}{ll} z_1 = (r_P)_Y - (r_E)_Y, & z_2 = (r_E)_Z - (r_E)_Z, \\ z_3 = (\dot r_P)_Y, & z_4 = (\dot r_E)_Y, \\ z_5 = (\dot r_P)_Z, & z_6 = (\dot r_E)_Z, \\ z_7 = (\ddot r_P)_Y, & z_8 = (\ddot r_P)_Z. \end{array} \tag{5}$$

In this case, the payoff function (which is the lateral miss) depends on the values of z_1 and z_2 at the instant T:

$$\varphi\big(z_1(T), z_2(T)\big) = \sqrt{z_1^2(T) + z_2^2(T)}.$$

Proceeding to a two-dimensional equivalent game, we obtain the dynamics

$$\begin{array}{l} \dot x = D(t)u + E(t)v, \\ t \in [0;T],\ x \in R^2,\ u \in P,\ v \in Q, \\ \varphi\big(x(T)\big) = \big\|x(T)\big\| = \sqrt{x_1^2(T) + x_2^2(T)}, \end{array}$$

where

$$\begin{array}{ll} D(t) = \zeta(t) \cdot I_2, & \zeta(t) = (T - t) + l_P e^{-(T-t)/l_P} - l_P, \\ E(t) = \eta(t) \cdot I_2, & \eta(t) = -(T - t), \end{array} \tag{6}$$

and I_2 is a unit 2×2 matrix. The sets P and Q are

$$u \in P = \left\{ u \,:\, u' \begin{bmatrix} 1/\cos^2(\chi_P)_{\rm col} & 0 \\ 0 & 1 \end{bmatrix} u \le a_P^2 \right\},$$

$$v \in Q = \left\{ v \,:\, v' \begin{bmatrix} 1/\cos^2(\chi_E)_{\rm col} & 0 \\ 0 & 1 \end{bmatrix} v \le a_E^2 \right\}.$$

Example 1. Below, we give the results (Kumkov et al., 2005) of numerical study of problem (4). The following parameters have been used: $l_P = 1.0$,

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \le 1.30^2 \right\}, \quad Q = \left\{ v \in R^2 : \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \le 1 \right\}.$$

Using notations of the original formulation, we have

$$\frac{|V_E|}{|V_P|} = 1.054, \quad \frac{a_P}{a_E} = 1.3, \quad \cos\chi_P = 0.67, \quad \cos\chi_E = 0.71, \quad l_P = 1.$$

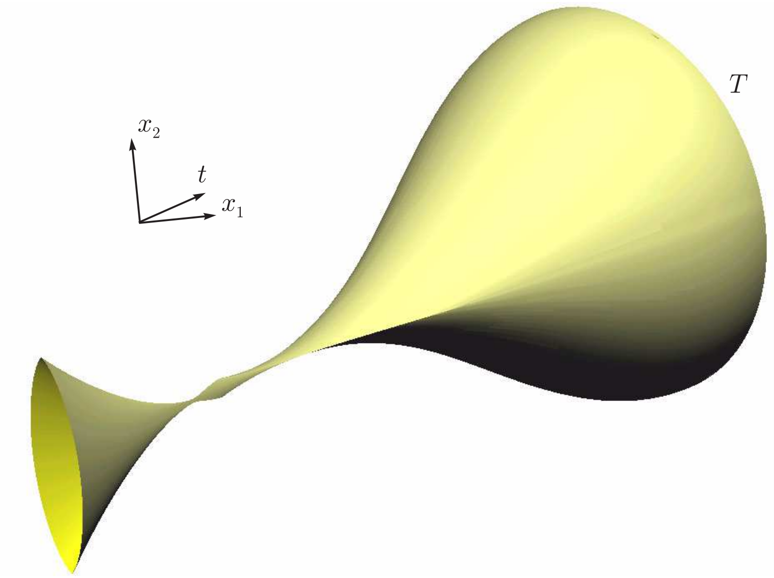

Fig. 2: Example 1. A general view of the level set of the value function with a narrow throat

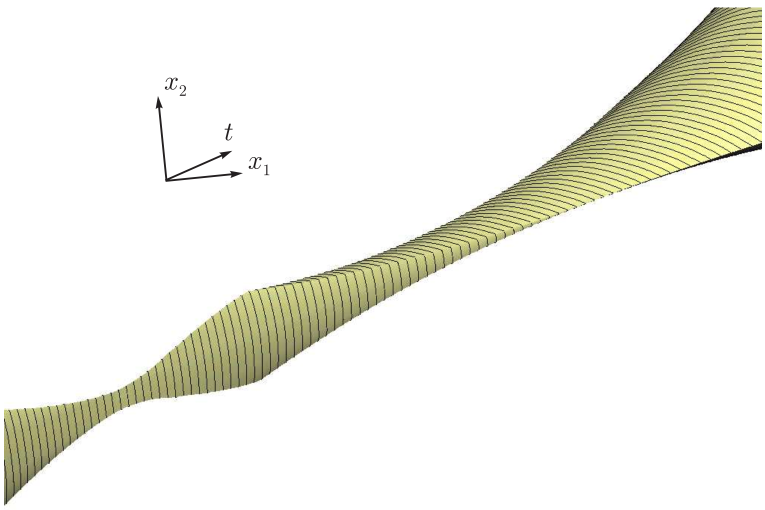

Fig. 3: A large view of the narrow throat

In Fig. 2, one can see a general view of the level set W_c computed for $c = 2.391$, which is a bit greater than the critical one (that is, to the one corresponding to the level set, which t-section has no interior at some instant). The main interesting properties of this tube is that it has the narrow throat and that the direction of elongation of t-sections changes near the throat. A large view of the narrow throat is given in Fig. 3. Such a complicated shape of the throat is conditioned by the process of passage of the advantage from the second player to the first one in this time interval.

This example was computed in the time interval $\tau \in [0; 7]$. Here and below, $\tau = T - t$ denotes the backward time. The time step Δ equals 0.01. The level sets of the payoff function (that are rounds) and the ellipses of the constraints for the players' controls are approximated by 100-gons.

Generalized L.S.Pontryagin's Test Example. In work (Pontryagin, 1964), the following differential game

$$\ddot{\mathbf{x}} + \alpha\dot{\mathbf{x}} = u, \qquad \ddot{\mathbf{y}} + \beta\dot{\mathbf{y}} = v. \tag{7}$$

was taken as an illustration to the theoretic results. Here, α and β are some positive constants; $\mathbf{x}$, $\mathbf{y} \in R^n$; $\|u\| \le \mu$, $\|v\| \le \nu$. The termination of the game happens when the coordinates x, y of the objects coincide. The first player tries to minimize the duration of the game, the second one hinders this. Later, differential games with dynamics (7) and termination conditions depending only on the geometric coordinates of the objects were called in Russian mathematical literature as "L.S.Pontryagin's test example".

Another well-known example with the dynamics

$$\ddot{\mathbf{x}} = u, \qquad \dot{\mathbf{y}} = v \tag{8}$$

and constraints for the player's controls $\|u\| \le \mu$, $\|v\| \le \nu$ was called by L.S.Pontryagin (Pontryagin and Mischenko, 1969) as game "boy and crocodile". The first player (the "crocodile") controls its acceleration and tries to catch the second one (the "boy") to some neighborhood. The second player is more maneuverable because it controls its velocity.

Game (8) is a particular case of the game "isotropic rockets" (Isaacs, 1965), which dynamics is

$$\ddot{\mathbf{x}} + k\dot{\mathbf{x}} = u, \qquad \dot{\mathbf{y}} = v. \tag{9}$$

In works (Pontryagin, 1972; Mezentsev, 1972; **?**; Nikol'skii, 1984; Grigorenko, 1990; Chikrii, 1997), problems with dynamics more complicated than (7), (8), (9) were studied:

$$\mathbf{x}^{(k)} + a_{k-1}\mathbf{x}^{(k-1)} + \dots + a_1\dot{\mathbf{x}} + a_0\mathbf{x} = u, \qquad u \in P, \tag{10}$$

$$\mathbf{y}^{(s)} + b_{s-1}\mathbf{y}^{(s-1)} + \dots + b_1\dot{\mathbf{y}} + b_0\mathbf{y} = v, \qquad v \in Q. \tag{11}$$

Games having dynamics (10), (11) and termination conditions depending only on the geometric coordinates x, y, are often called "generalized L.S.Pontryagin's test example". In this paper, let us assume that the payoff function is defined by the formula $\varphi\big(\mathbf{x}(T), \mathbf{y}(T)\big) = \big\|\mathbf{x}(T) - \mathbf{y}(T)\big\|$. Also, let us count that $\mathbf{x}$, $\mathbf{y} \in R^2$.

A variable change similar to (5)

$$\begin{array}{ll}
z_1 = \mathbf{x}_1 - \mathbf{y}_1, & z_2 = \mathbf{x}_2 - \mathbf{y}_2, \\
z_3 = \dot{\mathbf{x}}_1, & z_4 = \dot{\mathbf{x}}_2, \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
z_{2k-1} = \mathbf{x}_1^{(k-1)}, & z_{2k} = \mathbf{x}_2^{(k-1)}, \\
z_{2k+1} = \dot{\mathbf{y}}_1, & z_{2k+2} = \dot{\mathbf{y}}_2, \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
z_{2(k+s)-3} = \mathbf{y}_1^{(s-1)}, & z_{2(k+s)-2} = \mathbf{y}_2^{(s-1)},
\end{array}$$

transforms system (10), (11) to standard form (1):

$$\dot{\mathbf{z}} = A\mathbf{z} + Bu + Cv, \quad \mathbf{z} \in R^{2(k+s)-2},\ u \in P,\ v \in Q,$$

with the matrices A, B, and C that do not depend on the time. The payoff function is terminal and convex: $\varphi\big(z_1(T), z_2(T)\big) = \sqrt{z_1^2(T) + z_2^2(T)}$.

There can be other variants of the change, which are more convenient in particular situations, but all of them assume introduction of relative geometric coordinates (z_1, z_2 in our case).

When some experience had been accumulated in numerical study of level sets with narrow throats in the case of problem from works (Shinar et al., 1984; Shinar and Zarkh, 1996; Melikyan and Shinar, 2000), the author decided to construct another examples with narrow throats in the framework of games with the dynamics of the generalized L.S.Pontryagin's test example.

The most interesting results of constructing level sets of the value function for the generalized L.S.Pontryagin's test example are when at least one of the sets P and Q is not a round (since the level sets of the payoff, which is distance between objects at the termination instant, are rounds, we need something that destroys uniformity of the sets). So, let us take the sets P and Q as ellipses with center at the origin and main axes parallel to the coordinate axes. Then the players' vectograms $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ for all instants are ellipses homothetic to the ellipses P and Q respectively.

As it becomes clear from the previous example, a narrow throat appears when there is a change of advantage of players. Namely, at the initial period of the backward time the second player should be stronger to contract t-section of level sets. Then the advantage should pass to the first player to allow him to expand the sections. The easiest way to obtain such a change of advantage is to assign an oscillating dynamics to one or both players.

The most illustrative way to study the passages of the advantage is to investigate tubes of vectograms, that is the sets $\mathcal{P} = \big\{(t,u) : u \in \mathcal{P}(t)\big\}$, $\mathcal{Q} = \big\{(t,v) : v \in \mathcal{Q}(t)\big\}$. If one of the tubes includes the other in some period of time, then in this period the corresponding player has complete advantage.

Example 2. The dynamics is the following:

$$\begin{aligned} &\ddot{\mathbf{x}} + 2\,\dot{\mathbf{x}} = u, \\ &\ddot{\mathbf{y}} + 0.2\,\dot{\mathbf{y}} + \mathbf{y} = v, \qquad \mathbf{x}, \mathbf{y} \in R^2, \quad u \in P, \quad v \in Q. \end{aligned}$$

Here, the first player controls an inertial point in the plane. The second object is a two-dimensional oscillator. Both objects have a friction proportional to their velocities. The controls are constrained by the ellipses

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.8^2} + \frac{u_2^2}{0.4^2} \le 1 \right\}, \qquad Q = \left\{ v \in R^2 : \frac{v_1^2}{1.5^2} + \frac{v_2^2}{1.05^2} \le 1 \right\}.$$

The tubes of vectograms appearing in this example are shown in Fig. 4a. Since the dynamics of the second player describes an oscillating system, the advantage passes from one player to another several times. At the beginning of the backward time, the second player has the advantage, but later after a number of passes, the advantage comes to the first player. An enlarged fragment of the tubes can be seen in Fig. 4 b.

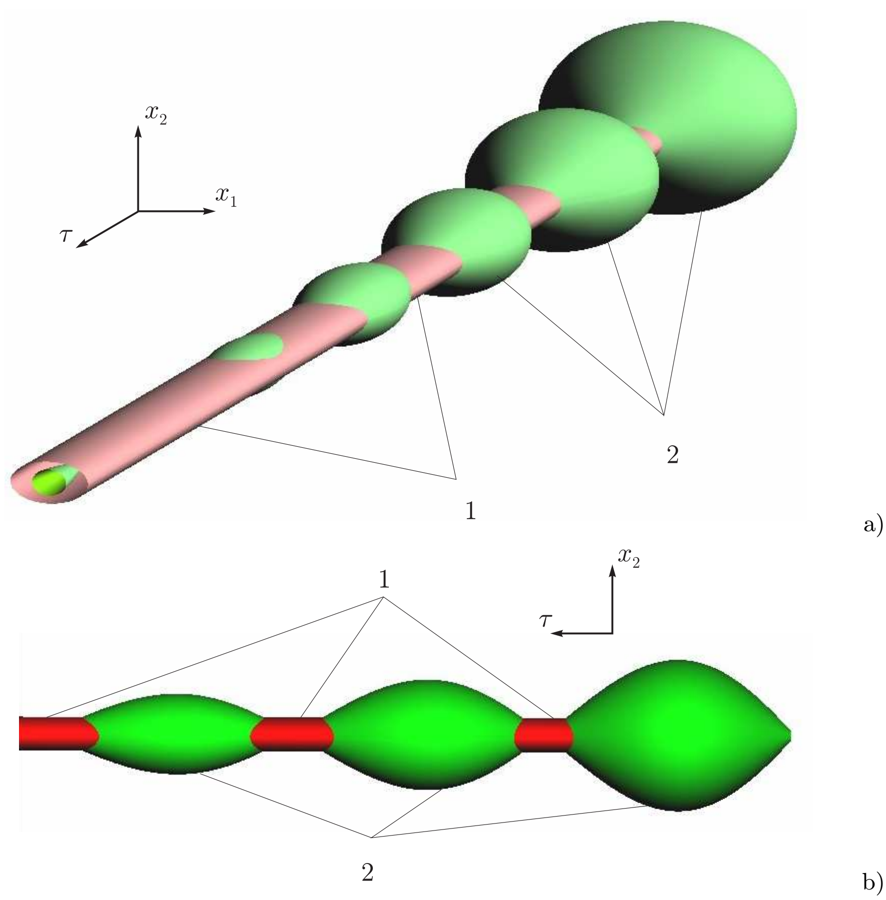

Fig. 4: Example 2. Two views of the vectogram tubes. Number 1 denotes the tube of the first player (the set $\mathcal{P}$), number 2 corresponds to the second player's vectogram tube (the set $\mathcal{Q}$)

Fig. 5 shows a level set W_c for $c = 2.45098$. This level set breaks (that is, is finite in time and has empty t-sections from some instant of the backward time). Before the break, orientation of elongation of the t-sections of $W_c(t)$ changes. Namely, before the last contraction of the tube, the sections are elongated vertically, and after it the elongation is horizontal. As in the example in the previous subsection, this change is due to delicate interaction of the vectogram tubes $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ in the time interval of the narrow throat.

If to increase the value of c, the length in time of the level sets grows jump-like. The level set for $c = 2.45100$ can be seen in Fig. 6. In Fig. 7, its enlarged fragment is given, which is near the narrow throat at $\tau = 11.95$. This value of c can be regarded as critical: for $c < 2.45100$ level sets break, for $c \geq 2.45100$ they are infinite in time. More exact reconstruction of the level sets corresponding to values c close to the critical one needs a very accurate computations.

This example was computed in the time interval $\tau \in [0; 20]$. The time step is $\Delta = 0.05$. The round level sets $\mathcal{M}_c$ of the payoff function and the ellipses P and Q were approximated by 100-gons.

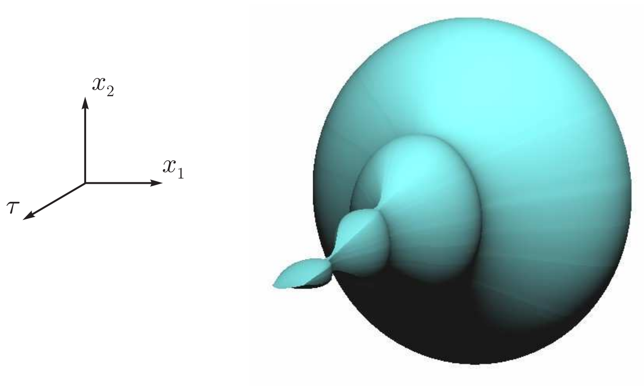

Fig. 5: Example 2. A broken level set close to the critical one, $c = 2.45098$

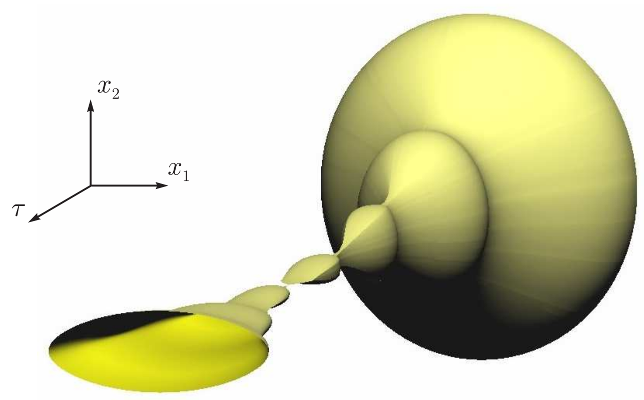

Fig. 6: Example 2. A general view of the level set with a narrow throat, $c = 2.45100$

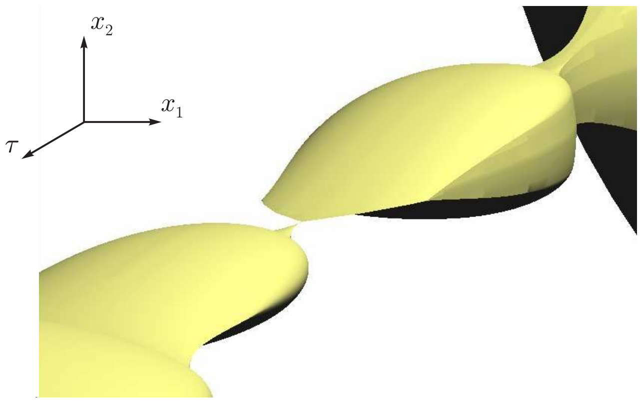

Fig. 7: A large view of the narrow throat

Example 3. To get an example with a level set of the value function with more than one narrow throat, we should choose players' dynamics to provide multiple passage of advantage in such a way that each of players has it for a quite long time (to allow the second player contract t-section almost to nothing). The most reasonable way to get such a situation is to put an oscillating dynamics to both players.

Let the dynamics be the following:

$$\begin{aligned} &\ddot{\mathbf{x}} - 0.025\,\dot{\mathbf{x}} + 1.3\,\mathbf{x} = u, \\ &\ddot{\mathbf{y}} + \mathbf{y} = v, \qquad \mathbf{x}, \mathbf{y} \in R^2, \quad u \in P, \quad v \in Q. \end{aligned}$$

Constraints for the players' controls are equal ellipses:

$$P = Q = \left\{ v \in R^2 : \frac{v_1^2}{1.5^2} + \frac{v_2^2}{1.05^2} \le 1 \right\}.$$

Since the sets P and Q constraining the players' controls are equal, then at any instant the players' vectograms $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are homothetic.

In Figs. 8a and 8b, the tubes of players' vectograms are shown. The difference of these figures is that in Fig. 8b the second player's tube is transparent.

Fig. 9 contains a general view of the level set W_c for $c = 1.2$. In Fig. 10, there is an enlarged fragment of the set near the first (in the backward time) narrow throat. The instants of the backward time of the most thin parts of the set are $\tau_1 = 5.65$ and $\tau_2 = 8.50$.

The level set has been computed in the time interval $\tau \in [0; 16]$. The time step is $\Delta = 0.05$. Near the narrow throats, the time step was ten times smaller: $\Delta' = 0.005$. Again, the approximating polygons for the constraints for the controls and for the payoff level set have 100 vertices.

Note again that despite the players' vectograms are homothetic, the t-sections of the level set and the vectograms are not. Absence of this homothety leads to complicated shape of the t-sections of the level set of the value function and, therefore, complicated shape of narrow throats.

Example 4. The dynamics of this example is described by the relations

$$\begin{aligned} &\ddot{\mathbf{x}} + 0.025\,\dot{\mathbf{x}} + 1.2\,\mathbf{x} = u, \\ &\ddot{\mathbf{y}} + 0.01\,\dot{\mathbf{y}} + 0.85\,\mathbf{y} = v, \qquad \mathbf{x}, \mathbf{y} \in R^2, \quad u \in P, \quad v \in Q. \end{aligned}$$

The constraints are taken as follows:

$$P = \left\{ u \in R^2 : \frac{u_1^2}{2.0^2} + \frac{u_2^2}{1.3^2} \le 1 \right\}, \qquad Q = \left\{ v \in R^2 : \frac{v_1^2}{1.5^2} + \frac{v_2^2}{1.05^2} \le 1 \right\}.$$

The vectograms appearing in this game are given in Fig. 11. The level set W_c corresponding to $c = 0.397$ is shown in Fig. 12a. One can see three narrow throats. An enlarged view of the middle one (which is the narrowest among them) can be seen in Fig. 12b.

3. Games with Non-Convex Payoff Function

In the previous section, we demonstrate examples where number of narrow throats is more than one. One can think that examples of this kind are artificial and, therefore,

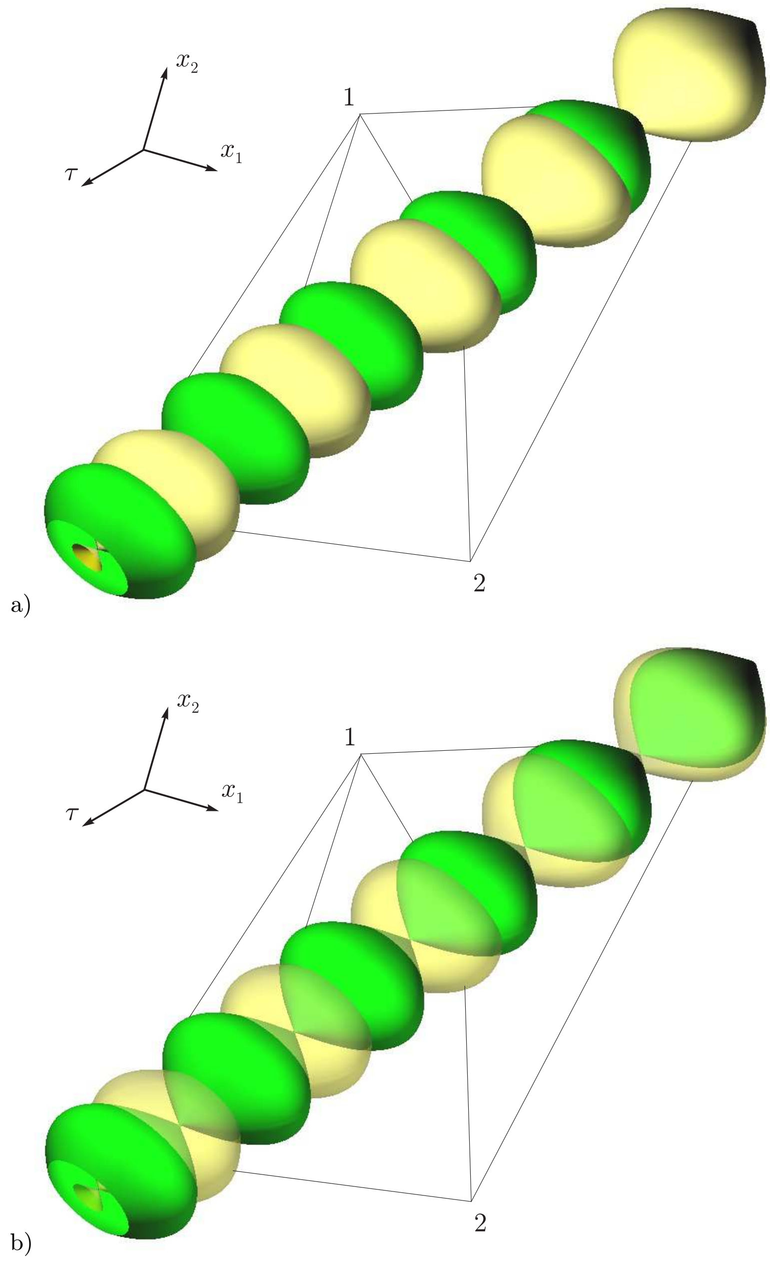

Fig. 8: Example 3. A general view of the vectogram tubes. Number 1 denotes the tube of the first player (the set $\mathcal{P}$), number 2 corresponds to the second player's vectogram tube (the set $\mathcal{Q}$). In subfigure b), the tube of the second player is transparent

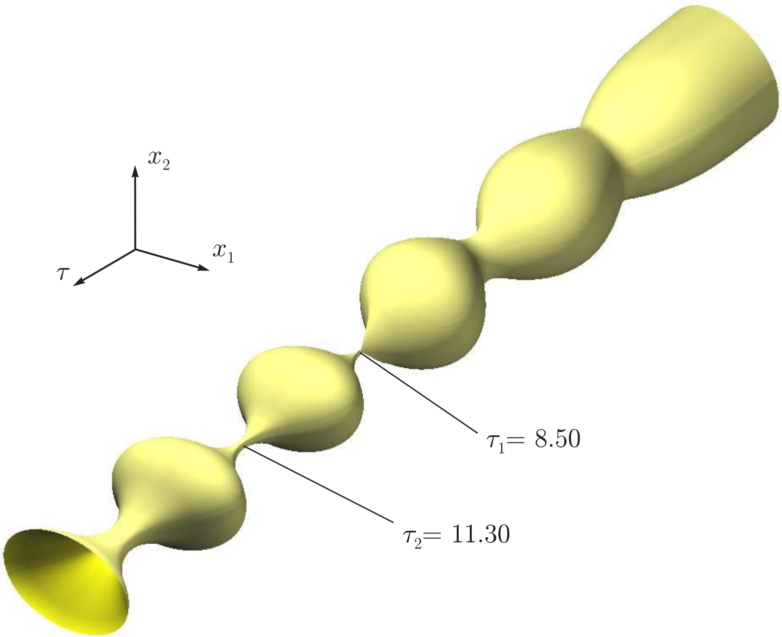

Fig. 9: Example 3. A general view of the level set with two narrow throats, $c = 1.2$

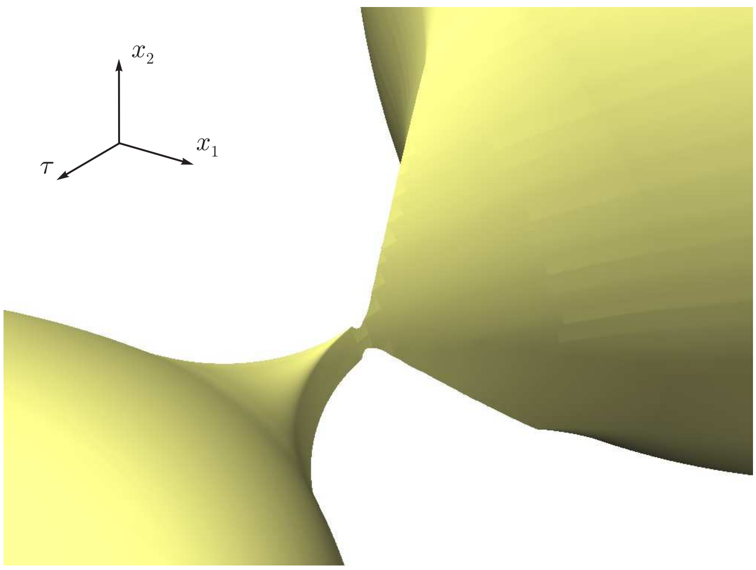

Fig. 10: An enlarged view of the first (in the backward time) narrow throat

rare. During last few years, the authors investigate differential games arising from consideration of pursuit problems in near space or in upper atmosphere. Descriptions of dynamics of the objects involved in the pursuit were taken from works by J. Shinar and his pupils. Games of this type also bring examples having level sets with, at least, two narrow throats.

3.1. Problem Formulation

Consider a game

$$\begin{aligned}
&\dot{\mathbf{z}}_{P_1} = A_{P_1}(t)\mathbf{z}_{P_1} + B_{P_1}(t)u_1,\\
&\dot{\mathbf{z}}_{P_2} = A_{P_2}(t)\mathbf{z}_{P_2} + B_{P_2}(t)u_2,\\
&\dot{\mathbf{z}}_{E} = A_{E}(t)\mathbf{z}_{E} + B_{E}(t)v,\\
&\mathbf{z}_{P_1} \in R^{n_1},\ \mathbf{z}_{P_2} \in R^{n_2},\ \mathbf{z}_E \in R^m,\ |u_i| \le \mu_i,\ |v| \le \nu
\end{aligned} \tag{12}$$

with three objects moving in a straight line. The objects P_1 and P_2 described by the phase vectors $\mathbf{z}_{P_1}$ and $\mathbf{z}_{P_2}$ are the pursuers. The object E with the phase vector $\mathbf{z}_E$ is the evader. The first components z_{P_1}, z_{P_2}, and z_E of the vectors $\mathbf{z}_{P_1}$, $\mathbf{z}_{P_2}$, and $\mathbf{z}_E$ respectively are the one-dimensional geometric coordinates of the objects.

Two instants T_1 and T_2 are prescribed. At the instant T_1, the pursuer P_1 terminates its pursuit, and the distance between him and the evader E is measured: $r_1(T_1) = \bigl|z_{P_1}(T_1) - z_E(T_1)\bigr|$. Similarly, the second pursuer P_2 stops to pursue at the instant T_2, when the distance $r_2(T_2) = \bigl|z_{P_2}(T_2) - z_E(T_2)\bigr|$ is measured.

The payoff is the minimum of these distances: $\varphi = \min\bigl\{r_1(T_1), r_2(T_2)\bigr\}$. The first player that consists of the pursuers and governs the controls u_1, u_2 minimizes the value of payoff φ. The second player, which is identified with the evader E, maximizes the payoff. All controls are scalar and have bounded absolute value.

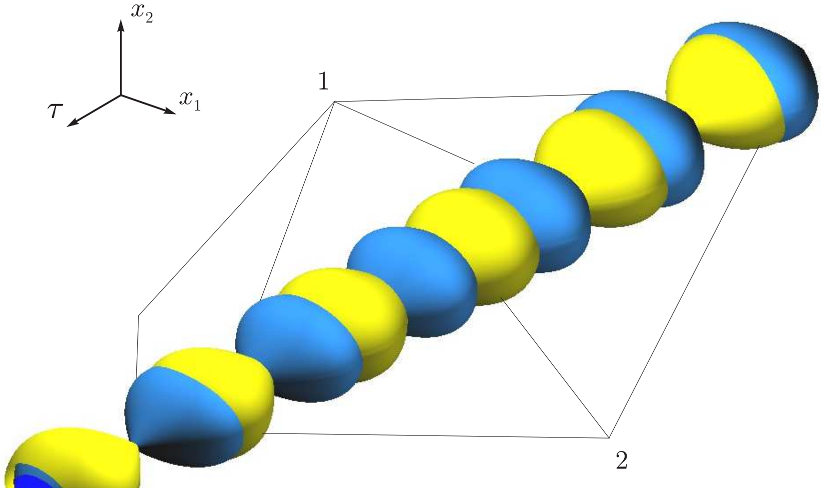

Fig. 11: Example 4. A large view of the vectogram tubes. Number 1 denotes the tube of the first player (the set $\mathcal{P}$), number 2 corresponds to the second player's vectogram tube (the set $\mathcal{Q}$)

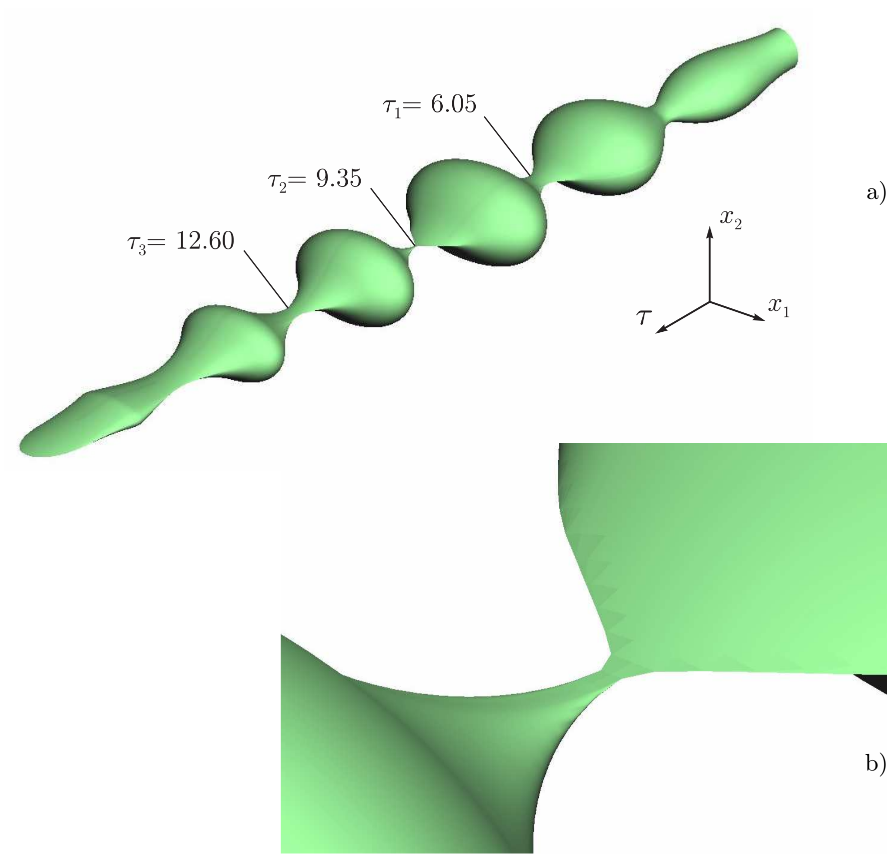

Fig. 12: Example 4. a) A general view of a level set with three narrow throats, $c = 0.397$; b) An enlarged view of the narrowest of the throats (the middle one)

3.2. Equivalent Differential Game

Let us pass from system (12) with separated objects to two relative dynamics. To do this, introduce new phase vectors $y^{(1)} \in R^{n_1+n_E-1}$ and $y^{(2)} \in R^{n_2+n_E-1}$ such that

$$y_1^{(1)} = z_{P_1} - z_E, \quad y_1^{(2)} = z_{P_2} - z_E.$$

The rest components $y_i^{(1)}$, $i = 2, \ldots, n_1 + n_E - 1$, of the vector $y^{(1)}$ equal components of the vectors $\mathbf{z}_{P_1}$ and $\mathbf{z}_E$ other than z_{P_1} and z_E. In the same way, the rest components $y_i^{(2)}$, $i = 2, \ldots, n_2 + n_E - 1$, of the vector $y^{(2)}$ are the components of the vectors $\mathbf{z}_{P_2}$ and $\mathbf{z}_E$ other than z_{P_2} and z_E. Due to linearity of dynamics (12), the new dynamics consisting of the two relative ones, is linear too:

$$\begin{array}{l} \dot{y}^{(1)} = \mathcal{A}_1(t)y^{(1)} + \mathcal{B}_1(t)u_1 + \mathcal{C}_1(t)v, \quad t \in [t_0; T_1], \\ \dot{y}^{(2)} = \mathcal{A}_2(t)y^{(2)} + \mathcal{B}_2(t)u_2 + \mathcal{C}_2(t)v, \quad t \in [t_0; T_2], \\ y^{(1)} \in R^{n_1+n_E-1}, \ y^{(2)} \in R^{n_2+n_E-1}, \\ |u_i| \le \mu_i, \ |v| \le \nu, \ \varphi = \min\bigl(\bigl|y_1^{(1)}(T_1)\bigr|, \bigl|y_1^{(2)}(T_2)\bigr|\bigr). \end{array} \tag{13}$$

The payoff function depends now on the first components of the phase vectors of the individual games. An *individual game* of the pursuer P_i against the evader E is the game with the dynamics of the vector $y^{(i)}$ and the payoff $\left|y_1^{(i)}(T_i)\right|$. The dynamics of the individual games are linked only through the control of the evader.

In each individual game, let us pass to the forecasted geometric coordinates in the same way as it is done from game (1) to (2). In the game of the pursuer P_i, $i = 1, 2$, against the evader E, the passage is provided by the matrix $X_1^{(i)}(T_i, t)$ constructed from the first row of the fundamental Cauchy matrix $X^{(i)}(T_i, t)$ that corresponds to the matrix $\mathcal{A}_i$. The variable changes are defined by the formulas $x_1(t) = X_1^{(1)}(T_1, t)y^{(1)}$, $x_2(t) = X_1^{(2)}(T_2, t)y^{(2)}$. Note that $x_1(T_1) = y_1^{(1)}(T_1)$ and $x_2(T_2) = y_1^{(2)}(T_2)$.

Dynamics of the individual games is the following:

$$\begin{array}{ll} \dot{x}_1 = d_1(t)u_1 + e_1(t)v, & t \in [t_0; T_1], \\ \dot{x}_2 = d_2(t)u_2 + e_2(t)v, & t \in [t_0; T_2], \\ x_1, x_2 \in R, |u_i| \leq \mu_i, \ |v| \leq \nu. & \end{array} \tag{14}$$

Here, $d_i(t)$ and $e_i(t)$ are scalar functions:

$$d_i(t) = X_1^{(i)}(T_i, t)\mathcal{B}_i(t), \ e_i(t) = X_1^{(i)}(T_i, t)\mathcal{C}_i(t), \quad i = 1, 2.$$

In the joint game of the pursuers against the evader, the payoff function is

$$\varphi = \min\{\left|x_1(T_1)\right|, \left|x_2(T_2)\right|\}.$$

3.3. Numerical Construction of Level Sets

Numerical constructions for the taken formulation are more complicated due to the following circumstances.

At first, for problem (2), any level set of the payoff function is plunged into the phase space at the instant T. But for the new formulation, level sets of the payoff can consist of two parts at two different (generally speaking) instants T_1 and T_2. At second, level sets of the payoff in problem (2) compact. But now the components of level sets corresponding to a constant c are infinite strips $\mathcal{M}_c^{(1)} = \{x : |x_1| \leq c\}$ at the instant T_1 (that is an infinite strip along the axis x_2) and $\mathcal{M}_c^{(2)} = \{x : |x_2| \leq c\}$ at the instant T_2 (an infinite strip along the axis x_1). Presentation of infinite objects in a computational program is a quite difficult problem. At third, a realization of procedure (3) for problem (2) is oriented on work with convex sets. In problem (14), we need to proceed non-convex time sections.

An algorithm taking into account these considerations and also based on procedure (3) can be formulated as follows.

For definiteness, let us assume that $T_2 \leq T_1$. The opposite case is considered in the same way.

For numerical constructions, fix a time grid in the interval $[t_0; T_1]$. It should include the instant T_2. At the instant T_1, the set $\mathcal{M}_c^{(1)}$ is taken as the start for the procedure (3). In the case of numerical constructions, the infinite strip is cut becoming a rectangle with a quite large size along the axis x_2. Then, in the interval $(T_2, T_1]$, the procedure (3) is applied with the set $D(t)P$ taken as a segment $[-|d_1(t)|\mu_1; |d_1(t)|\mu_1] \times \{0\}$ (by this, we ignore the action of the second pursuer). When the construction are made up to the instant T_2, we unite the obtained t-section $W_c(T_2 + 0)$ with the set $\mathcal{M}_c^{(2)}$ (also cut to a finite size in the case of numerical constructions). Thus, we get the set $W_c(T_2)$, which is the start value for the further iterations. In the time interval $[t_0; T_2]$, the rectangle vectogram $[-|d_1(t)|\mu_1; |d_1(t)|\mu_1] \times [-|d_2(t)|\mu_2; |d_2(t)|\mu_2]$ of the first player is taken; now, actions of both players are involved. The vectogram of the second player (of the evader) equals $\big[\big(-|e_1(t)|\nu, -|e_2(t)|\nu\big); \big(|e_1(t)|\nu, |e_2(t)|\nu\big)\big]$ for all time instants from the grid.

If $T_2 = T_1 = T$, then the start set $\mathcal{M}_c$ at the instant T is union of the strips $\mathcal{M}_c^{(1)}$ and $\mathcal{M}_c^{(2)}$ (possibly, cut).

As it was mentioned above, the necessary realization of procedure (3) should be able to process non-convex sets. Namely, we need operations of Minkowski sum and difference, which first operand is not convex (the second one is convex because it is computed as a convex vectogram multiplied by the time step). A helpful fact is that both operations, sum and difference, can be fulfilled by one operation, namely, sum. Indeed, it is true that

$$A \overset{*}{-} B = (A' - B)'.$$

Here, the prime denotes set complement. The authors worked out an algorithm for construction Minkowski sum when the first operand is a union of a number of simple-connected closed polygonal sets (possibly, non-convex), or is a complement to such a polygon (in other words, is an infinite closed set with a number of polygonal holes).

3.4. Variants of Servomechanism Dynamics

1. A First Order Link. In work (Le Ménec, 2011), a pursuit problem is formulated that includes two pursuers and one evader. Each object has a three-dimensional phase variable: one-dimensional coordinate, velocity, and acceleration. The acceleration is affected by the control through a link of the first order:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = (u - z_3)/l. \tag{15}$$

Dynamics of the pursuer in problem (4) is similar, but now the geometric coordinate is one-dimensional. As above, l is the time constant describing the inertiality of the servomechanisms.

2. Damped Oscillating Control Contour. In work (Shinar et al., 2013), a game is considered, in which one of the objects has a damped oscillating control contour:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = -\omega^2 z_3 - \zeta z_4 + u. \tag{16}$$

Here, ω is the natural frequency of the contour, ζ is the damping coefficient.

3. Tail/Canard Air Rudders. When considering an objects moving in the atmosphere, it is important to take into account position of its rudders with respect to the center of mass. A corresponding model is set forth in (Shima, 2005):

$$\ddot{z} = a + du, \quad \dot{a} = \big((1 - d)u - a\big)/l. \tag{17}$$

The parameter d is defined by the position of the rudder. A positive (negative) value corresponds to the situation when the rudder is located in front of (behind) the center of mass. The first situation is called *canard control scheme*, the second one is called *tail control scheme*. The absolute value of d describes now far from the center of mass the rudder is. The parameter l again is the time constant.

4. Dual Tail/Canard Scheme. As a development of model (17), work (Shima and Golan, 2006) suggests a dynamics of an objects, which has both canard and tail rudders:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3 + d_c u_c + d_t u_t, \quad \dot{z}_3 = \big((1-d_c)u_c + (1-d_t)u_t - z_3\big)/l. \qquad (18)$$

Here, the constant $d_c > 0$ describes the capabilities of the canard rudder (then index c means "canard" here), the constant $d_t < 0$ corresponds to the tail rudder (the index t means "tail"). The time constant l regarded to be common for inertiality of both rudders.

In this model, one can see two scalar controls u_c and u_t (or one vector control $u = (u_c, u_t)^\top$ taken from a rectangle). Therefore, formally this model does not belong to class (12). But the procedures for construction level sets of the value function suggested by the authors can be applied to such a dynamics with double scalar control. The difference is that formula (3) includes now two summands connected to two controls of the first player.

3.5. Examples

In this subsection, we assume that the evader has dynamics of type (15).

Example 5. Let both pursuers have the same dynamics of type (15). The parameters of the game are

$$\mu_1 = \mu_2 = 1.5,\ \nu = 1.0,\ l_{P_1} = l_{P_2} = 1/0.25,\ l_E = 1/1.0,\ T_1 = T_2 = 15.$$

The level set of the value function corresponding to $c = 1.32$ is shown in Fig. 13.

In similar problems studied in detail by the authors (Ganebny et al., 2012; Kumkov et al., 2013), the advantage of a player in an individual game, can be detected analytically by analyzing the parameters $\eta_i = \mu_i/\nu$ and $\varepsilon_i = l_E/l_{P_i}$. When $\eta_i > 1$, $\eta_i\varepsilon_i > 1$, the ith pursuer has advantage over the evader. Vice versa, if $\eta_i < 1$, $\eta_i\varepsilon_i < 1$, the advantage belongs to the evader. If the parameters do not obey one of these conditions, then there is a situation of changing advantage of the ith pursuer over the evader in time.

For the example, the data are such that both pursuers are weaker then the evader at the beginning of the backward time (near the target set, which is located in the plane $t = T_1 = T_2$). Due to this, at the beginning of the backward time, the

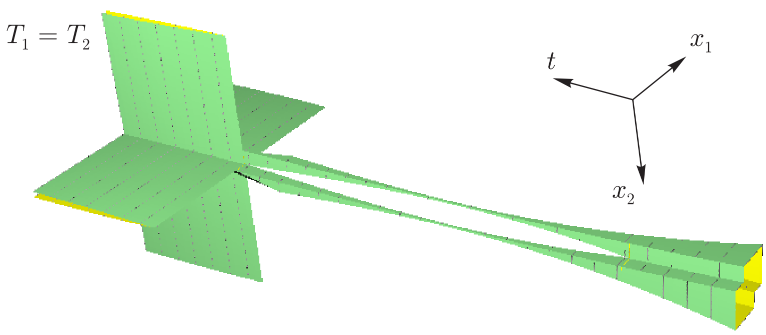

Fig. 13: Example 5. Narrow throats in a problem with three objects having dynamics (15)

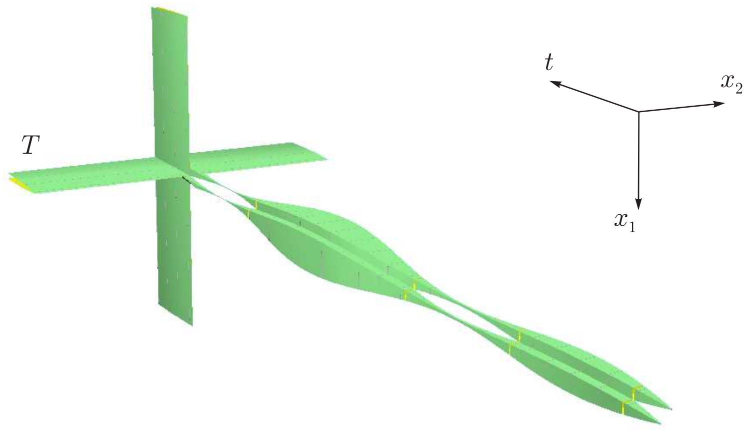

Fig. 14: Example 6. A level set for example 2 with two narrow throats

t-sections start to contract. In Fig. 13, an instant can be distinguished when the infinite strips (the rectangles elongated along the corresponding axes) degenerate to a line due to this contraction and disappear. After this instant, the t-sections consist of two finite disconnected parts that correspond to zones of joint capture. If the position of the system is in such a zone, then the evader escaping from one pursuer is captured (with the given miss) by the another one. These parts continue to contract until an instant when the pursuers get the advantage. Further, the contraction turns to expansion, and at some instant growing parts joins into one simple-connected set that continue to grow infinitely.

Since the parameters of both pursuers coincide and the time lengths of both individual games are equal, the dynamics of the coordinates x_1 and x_2 are the same. Therefore, the evolution of t-sections is the same along both coordinate axes. As it will be seen from the following examples, this is not true if the pursuers' parameters or game lengths are different.

Example 6. Let both pursuers be equal again, but now they have dynamics (16) with oscillating control contour. The parameters are the following:

$$\mu_1 = \mu_2 = 0.3,\ \nu = 1.3,\ \omega_{P_1} = \omega_{P_2} = 0.5,$$
$$\zeta_{P_1} = \zeta_{P_2} = 0.0025,\ l_E = 1.0,\ T = T_1 = T_2 = 30.$$

The level set of the value function corresponding to $c = 1.6$ can be seen in Fig. 14.

In this problem, due to fundamental difference of the pursuers' and evader's dynamics, it is difficult to get analytically the conditions of advantage of one or other player. Thus, the example is constructed on the base of the evolution of the players' vectograms obtained numerically. Presence of two narrow throats is connected to repeat of a period such that at the beginning the advantage belongs to the evader and at the end it comes to pursuers. The repeat is provided by the oscillating type of the pursuers' dynamics. More throats can be obtained by putting to the evader an oscillating dynamics too.

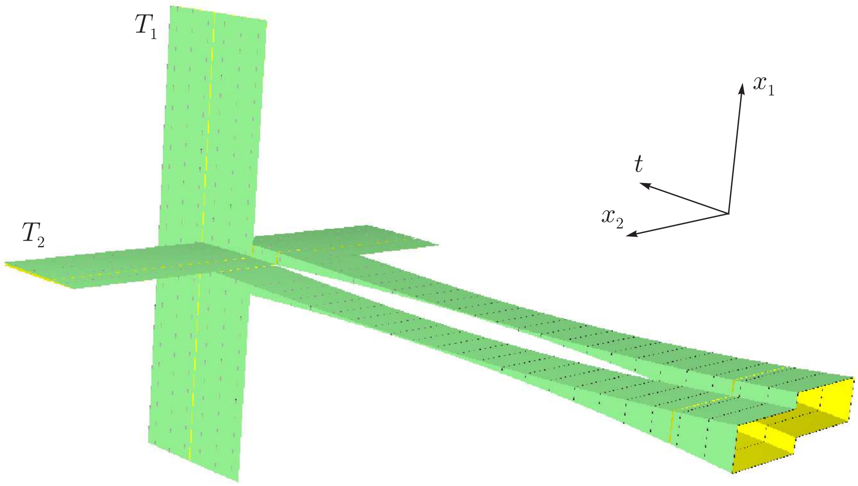

Fig. 15: Example 7. The level set $W_{0.525}$ for the pursuers' dynamics of type (18); the pursuers have different parameters of the dynamics

Example 7. Consider now a pair of pursuers both having dynamics (18). Let some dynamics parameters be different:

$$a_{P_1,\max} = 1.05,\ a_{P_2,\max} = 1.15,\ l_{P_1} = l_{P_2} = 1/0.18807,$$
$$d_{c,1} = d_{c,2} = 0.605,\ d_{t,1} = d_{t,2} = -0.5,\ \alpha_1 = 0.9,\ \alpha_2 = 0.8,$$
$$a_{E,\max} = 0.95,\ d_E = 0.157980, l_E = 1.0,\ T_1 = 32,\ T_2 = 29.$$

The value α_i defines distribution of the control resource $a_{P_i,\max}$ of the ith pursuer over the rudders:

$$|u_c| \le \alpha \cdot a_{P,\max},\ |u_t| \le \beta \cdot a_{P,\max}; \quad \alpha,\, \beta \ge 0,\ \alpha + \beta = 1.$$

The level set W_c that correspond to $c = 0.525$ is given in Fig. 15. One can see that due to difference of the pursuers' dynamics the contraction of the set is different along the axes x_1 and x_2: degeneration of the infinite strips happens at different instants. Moreover, the finite parts remaining after degeneration of infinite strips have sufficiently different sizes along the two coordinate axes.

4. Conclusion

A level set (Lebesgue set) of the value function corresponding to some value c can be regarded as a solvability set of a game problem with the payoff equal to c. For differential games with fixed termination instants, a level set of the value function is a tube in the space *time* $\times$ *phase space* along the time axis. It is very important to establish the law of evolution of time sections of the tubes in time. For example, if a tube corresponding to some c has a small length in time, then it means that the zone of guaranteed capture with the miss not greater than c is small too. If a solvability set has a narrow throat, that is, a period of time where its t-sections are close to degeneration (to loss of interior), then one should analyze accurately the possibility of practical application of the control law based on such a tube. In the paper, it is shown that the presence of narrow throats is not rare both in model differential games and practical pursuit problems.

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