# Stationary State in a Multistage Auction Model*

**Aleksei Y. Kondratev**

*Institute of Applied Mathematical Research,*
*Karelian Research Center of RAS,*
*Pushkinskaya str. 11, Petrozavodsk, 185910, Russia*
*E-mail:* kondratev@krc.karelia.ru

**Abstract** We consider a game-theoretic multistage bargaining model with incomplete information related with deals between buyers and sellers. A player (buyer or seller) has private information about his reserved price. Reserved prices are random variables with known probability distributions. Each player declares a price which depends on his reserved price. If the bid price is above the ask price, the good is sold for the average of two prices. Otherwise, there is no deal. We investigate model with infinite time horizon and permanent distribution of reserved prices on each stage. Two types of Nash-Bayes equilibrium are derived. One of them is a threshold form, another one is a solution of a system of integro-differential equations.

**Keywords:** multistage auction model, Nash equilibrium, integro-differential equations for equilibrium, threshold strategies.

## 1. Introduction

In (Mazalov and Kondratyev, 2012; Mazalov and Kondratyev, 2013) there was considered bargaining model with incomplete information, where a buyer and a seller have an opportunity to make a deal at only one stage. In (Mazalov et al., 2012) there was proposed auction model with finite number of steps. We fix time horizon $n$. A seller and a buyer meet each other at random. Reserved prices $s$ and $b$ are independent random variables on interval $[0,1]$ with density functions $f(s)$ and $g(b)$ accordingly. Seller asks $S_k = S_k(s) \geq s$, buyer bids $B_k = B_k(b) \leq b$ on step $k = 1, 2, \dots n$ We have a deal on the average of the two prices $(S_k(s) + B_k(b))/2$ if $B_k \geq S_k$. If there is no deal then agents go to the next step $k+1$. Differential equations for equilibrium strategies were found. In this paper we generalize and research this auction model for case of infinite time horizon.

Let $\delta$ be discount factor. Consider a game with infinite time horizon. Let reserved prices of sellers and buyers $s$ and $b$ at the stage $i = 1, 2, \dots$ have density functions $f_i(s), s \in [0,1]$ and $g_i(b), b \in [0,1]$ accordingly. At the stage $i$ players use strategies $S_i(s)$ and $B_i(b)$. If there was a deal then the buyer $b$ and the seller $s$ get outcome $\delta^{i-1}(b - \frac{B(b)+S(s)}{2})$ and $\delta^{i-1}(\frac{B(b)+S(s)}{2} - s)$ accordingly and in this case they do not move to the next stage. Additionally let fix count of new agents appears in the market at the each stage. We will study this model assuming that when $i \to \infty$ and if agents act optimal then $f_i(s)$ and $g_i(b)$ tend to the limit density distribution $f(s)$ and $g(b)$. Hence we research stationary state on the market, when distributions

* This research is supported by Russian Fund for Basic Research (project 13-01-91158-*ГФЕН*_ a, project 13-01-00033-a), the Division of Mathematical Sciences of RAS (the program "Algebraic and Combinatorial Methods of Mathematical Cybernetics and New Information System") and Strategic Development Program of PetrSU.

$f(s)$ and $g(b)$ are the same at each stage, i.e. new agents replace making a bargain players.

## 2. Integro-differential equations for Nash equilibrium

To find optimal strategies we count them as functions of reserved prices accordingly $S = S(s)$ and $B = B(b)$. Let they are differentiable and strictly increasing. Then inverse functions (differentiable and strictly increasing too) are $U = B^{-1}$ and $V = S^{-1}$, i.e. accordingly $s = V(S)$ and $b = U(B)$. There is a deal, if $B \geqslant S$. If there is a deal then we have a deal price $(S(s) + B(b))/2$. Pay functions are (1) and (2). Fix buyer's strategy $B(b)$ and derive the best response of the seller $s$.

Condition $B(b) \geqslant S$ is equivalent to $b \geqslant U(S)$. Outcome of the seller equals

$$H_s(S) = \int\limits_{U(S)}^{1} \left(\frac{B(b) + S}{2} - s\right) g(b)db + \delta G(U(S))H_s(S),$$

$$H_s(S) = \frac{1}{1 - \delta G(U(S))} \int\limits_{U(S)}^{1} \left(\frac{B(b) + S}{2} - s\right) g(b)db. \tag{1}$$

Differentiating (1) with respect to $S$, we have first order condition

$$\frac{\partial H_s(S)}{\partial S} = \frac{1}{(1 - \delta G(U(S)))^2}\Big[\Big(\frac{1 - G(U(S))}{2} - (S - s)g(U(S))U'(S)\Big)\cdot$$
$$\cdot (1 - \delta G(U(S))) + \int\limits_{U(S)}^{1} \left(\frac{B(b) + S}{2} - s\right) g(b)db \cdot \delta \cdot g(U(S))U'(S)\Big],$$

and so we get integro-differential equation for equilibrium strategies (inverse functions) $U(B), V(S)$

$$\Big(\frac{1 - G(U(S))}{2} - (S - V(S))g(U(S))U'(S)\Big)(1 - \delta G(U(S)))+$$
$$\delta \cdot g(U(S))U'(S)\Big(\big(\frac{S}{2} - V(S)\big)(1 - G(U(S))) + \frac{1}{2}\int\limits_{U(S)}^{1} B(b)g(b)db\Big) = 0.$$

The same way let $S(s)$ be seller's strategy. We find the best response of the buyer $b$. His outcome is

$$H_b(B) = \int\limits_{0}^{V(B)} \left(b - \frac{S(s) + B}{2}\right) f(s)ds + \delta(1 - F(V(B)))H_b(B),$$

$$H_b(B) = \frac{1}{1 - \delta + \delta F(V(B))} \int\limits_{0}^{V(B)} \left(b - \frac{S(s) + B}{2}\right) f(s)ds. \tag{2}$$

Differentiating (2) with respect to $B$, we have first order condition

$$\frac{\partial H_b(B)}{\partial B} = \frac{1}{(1-\delta+\delta F(V(B)))^2}\Big[\Big((b-B)f(V(B))V'(B)-\frac{F(V(B))}{2}\Big)\cdot$$
$$\cdot\,(1-\delta+\delta F(V(B)))-\int\limits_0^{V(B)}\Big(b-\frac{S(s)+B}{2}\Big)f(s)ds\cdot\delta\cdot f(V(B))V'(B)\Big],$$

and so we get the second integro-differential equation for equilibrium strategies (inverse functions) $U(B), V(S)$

$$\Big((U(B)-B)f(V(B))V'(B)-\frac{F(V(B))}{2}\Big)\cdot(1-\delta+\delta F(V(B)))-$$
$$\delta\cdot f(V(B))V'(B)\cdot\Big((U(B)-\frac{B}{2})F(V(B))-\frac{1}{2}\int\limits_0^{V(B)}S(s)f(s)ds\Big)=0.$$

Now we have the system of equations for Nash equilibrium

$$\frac{\partial U}{\partial t}=\frac{(1-G(U))(1-\delta G(U))}{2g(U)\left[(t-V)(1-\delta G(U))-\delta(\frac{t}{2}-V)(1-G(U))-\frac{\delta}{2}\int\limits_U^1 B(b)g(b)db\right]}, \tag{3}$$

$$\frac{\partial V}{\partial t}=\frac{F(V)(1-\delta+\delta F(V))}{2f(V)\left[(U-t)(1-\delta+\delta F(V))-\delta(U-\frac{t}{2})F(V)+\frac{\delta}{2}\int\limits_0^V S(s)f(s)ds\right]}. \tag{4}$$

Functions $U$ and $V$ must satisfy $U(a)=a$, $U(c)=1$, $V(a)=0$, $V(c)=c$. From (3) and (4) it is easy to find

$$U'(a)=\frac{(1-G(a))(1-\delta G(a))}{2g(a)\left[a(1-\delta G(a))-\frac{\delta}{2}a(1-G(a)-\frac{\delta}{2}\int\limits_a^1 B(b)g(b)db\right]}, \tag{5}$$

$$V'(c)=\frac{F(c)(1-\delta+\delta F(c))}{2f(c)\left[(1-c)(1-\delta+\delta F(c))-\delta(1-\frac{c}{2})F(c)+\frac{\delta}{2}\int\limits_0^c S(s)f(s)ds\right]}. \tag{6}$$

To figure out marginal prices $a$ and $c$ assume that there exist finite derivative $V'(a)>0$ and density $f(0)>0$. Using L'Hopital's rule we derive

$$V'(a)=\lim_{t\to a}\frac{f(V)V'(1-\delta+\delta F(V))+\delta F(V)f(V)V'}{2f(V)[(U'-1)(1-\delta+\delta F(V))+(U-t)\delta f(V)V'-\delta(U'-\frac{1}{2})F(V)}$$
$$\frac{}{-\delta(U-\frac{t}{2})f(V)V'+\frac{1}{2}\delta tf(V)V']}=\frac{f(0)V'(a)(1-\delta)}{2f(0)(U'(a)-1)(1-\delta)}=\frac{V'(a)}{2(U'(a)-1)},$$

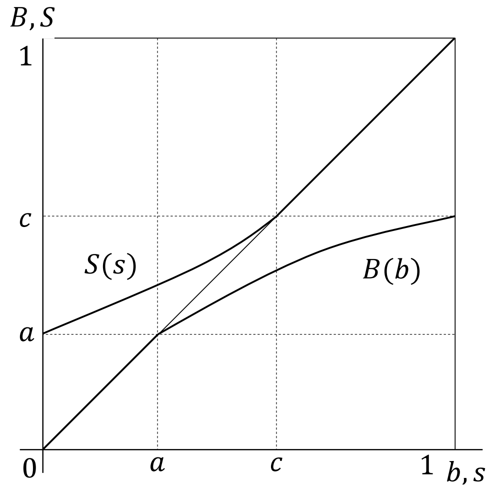
and so $U'(a)=1.5$.

The same way assume that there exist finite derivative $U'(c) > 0$ and density $g(1) > 0$. Using L'Hopital's rule we derive

$$U'(c) = \lim_{t \to c} \frac{-g(U)U'(1-\delta G(U)) - \delta(1-G(U))g(U)U'}{2g(U)[(1-V')(1-\delta G(U)) - (t-V)\delta g(U)U' - \delta(\frac{1}{2}-V')(1-G(U)) + \delta(\frac{t}{2}-V)g(U)U' + \frac{1}{2}\delta t g(U)U']} = \frac{-g(1)U'(c)(1-\delta)}{2g(1)(1-V'(c))(1-\delta)} = \frac{-U'(c)}{2(1-V'(c))},$$

and so $V'(c) = 1.5$.

So we find necessary condition for differentiable strictly increasing strategies to be Nash equilibrium. The case of $\delta = 0$ leads to single-stage auction model researched in (Mazalov and Kondratyev, 2012).



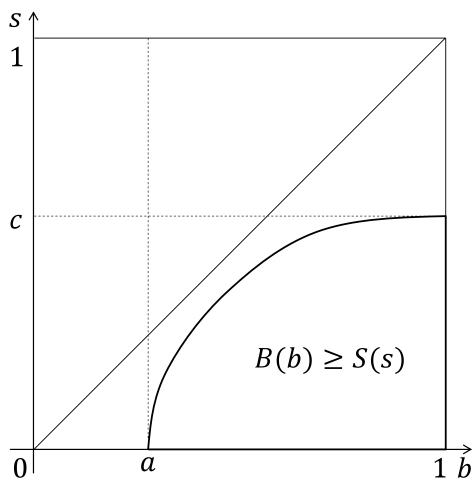



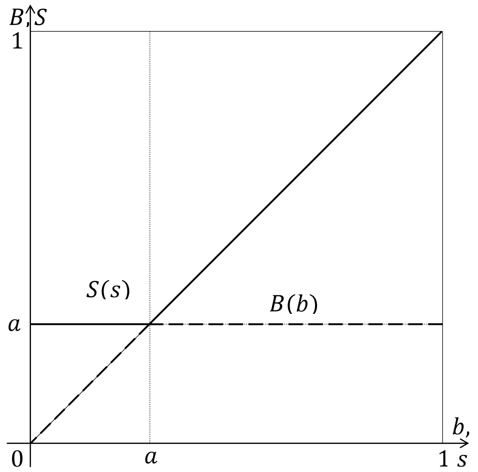

Fig. 1: Equilibrium strategies

Fig. 2: Deal area (for theorem 1)

**Theorem 1.** *If density functions $g(b)$ and $f(s)$ are continuous on $[0, 1]$, $0 < f(0) < +\infty$, $0 < g(1) < +\infty$. Derivatives $0 < S'(0), B'(1) < +\infty$ exist. Then differentiable and strictly increasing strategies $S(s)$ on $[0, c]$ and $B(b)$ on $[a, 1]$ are Nash equilibrium, if they satisfy (3),(4) on interval $(a, c)$, with respect to boundary conditions $U(a) = a$, $U(c) = 1$, $V(a) = 0$, $V(c) = c$. Marginal prices $a$ and $c$ must be derived from equations $U'(a) = 1.5$, $V'(c) = 1.5$, using (5), (6).*

## 3. Nash equilibrium with threshold strategies

We derive necessary and sufficient condition for threshold strategies to be Nash equilibrium in the underlying



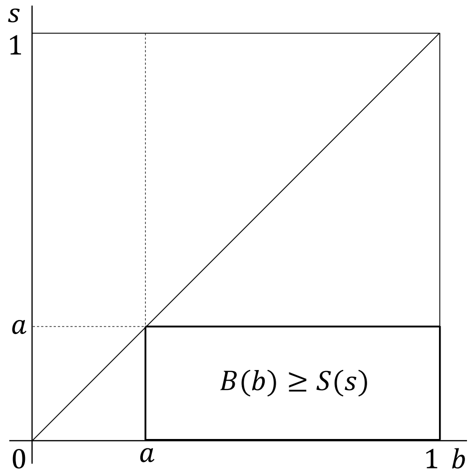




Fig. 3: Threshold strategies

Fig. 4: Deal area (for theorem 2)

**Theorem 2.** *If strategies $S(s), B(b)$ are threshold with price of a deal $a \in [0,1]$, i.e. $S(s) = \max\{a, s\}, B(b) = \min\{a, b\}$. Then it is Nash equilibrium if and only if*
*(∗) $H_{s=0}(S)$ on $[0,a]$ has a maximum for $S = a$,*
*(∗∗) $H_{b=1}(B)$ on $[a,1]$ has a maximum for $B = a$.*

*Proof.* The deal is made if seller's reserved price $s \in [0,a]$ and ask price $S \in [s,a]$, $S \le B(b)$. Outcome of the seller (1) equals

$$H_s(S) = \frac{1}{1-\delta G(S)} \left[ \int_S^a \left( \frac{b+S}{2} - s \right) dG(b) + \int_a^1 \left( \frac{a+S}{2} - s \right) dG(b) \right] =$$

$$= \frac{1}{1-\delta G(S)} \left[ \left( \frac{S}{2} - s \right) (1 - G(S)) + \frac{1}{2} \int_S^a b dG(b) + \frac{a}{2}(1 - G(a)) \right]. \quad (7)$$

It is easy to check that

$$H_s(S) = H_{s=0}(S) + \frac{(1-\delta)s}{\delta(1-\delta G(S))} - \frac{s}{\delta},$$

and in respect that $G(S)$ is increasing, from (∗) we have a result that for any $s \in [0,a]$ seller's outcome has maximum point $S = a$.

We can hold the same reasoning for buyers. The deal is made if buyer's reserved price $b \in [a,1]$, and a bid price $B \in [a,b]$, $B \geqslant S(s)$. From (2) we find his outcome

$$H_b(B) \quad = \quad \frac{1}{1-\delta+\delta F(B)}\left[\left(b-\frac{B}{2}\right)F(B)-\frac{1}{2}\int\limits_a^B sdF(s)-\frac{a}{2}F(a)\right]. \quad (8)$$

Note that

$$H_b(B) = H_{b=1}(B) - \frac{(1-b)}{\delta} + \frac{(1-b)(1-\delta)}{\delta(1-\delta+\delta F(B))},$$

and in respect that $F(B)$ is increasing, from $(**)$ we get that for any $b \in [a,1]$ buyer's outcome has a maximal value for $B = a$.

**Theorem 3.** *If distribution functions $F(s), G(b)$ have piecewise-continuous and limited density functions $f(s) \le L$ on $[a,1]$ and $g(b) \le M$ on $[0,a]$, then in theorem 2 for $(*)$ it is sufficient that*

$$\delta \ge 1 - \frac{(1-G(a))^2}{2aM}, \quad (9)$$

*and condition $(**)$ is true if*

$$\delta \ge 1 - \frac{F^2(a)}{2(1-a)L}. \quad (10)$$

*Proof.* At the points of continuity for $g(b)$, differentiating the (7), we derive that

$$H'_{s=0}(S) = \frac{1}{2(1-\delta G(S))^2}\Big[1-G(S)-2Sg(S)-\delta G(S)+\delta G^2(S)+$$
$$+\delta Sg(S)G(S)+\delta g(S)\int\limits_S^a bg(b)db+\delta ag(S)-\delta aG(a)g(S)+\delta Sg(S)\Big]. \quad (11)$$

Using that

$$g(S)\int\limits_S^a bg(b)db \ge g(S)\int\limits_S^a Sg(b)db = Sg(S)(G(a)-G(S)),$$

substituting $\delta = 1-(1-\delta)$, it is easy to check that in (11) expression in square brackets is not less than

$$(1-G(S))^2 + g(S)(a-S)(1-G(a)) - (1-\delta)(-G(S)+G^2(S)+$$
$$+(S-a)g(S)G(a)+(a+S)g(S)) \ge$$

further as $S \le a$ and (9) it results that

$$\ge (1-G(S))^2 - (1-\delta)(a+S)g(S)) \ge (1-G(a))^2 - (1-\delta)2aM \ge 0.$$

Hence we prove that derivative $H'_{s=0}(S)$ is nonnegative on the interval $[0,a]$, so it leads to $(*)$.

At the points of continuity for $f(s)$, differentiating the (8), we find that

$$H'_{b=1}(B) = \frac{1}{2(1-\delta+\delta F(B))^2}\Big[-(1-\delta)F(B) + 2(1-\delta)f(B) - \delta F^2(B) +$$
$$- \delta B f(B)F(B) + \delta f(B)\int\limits_a^B sf(s)ds + \delta a F(a)f(B) - 2(1-\delta)Bf(B)\Big]. \quad (12)$$

Noting that

$$f(B)\int\limits_a^B sf(s)ds \leq f(B)\int\limits_a^B Bf(s)ds = Bf(B)(F(B)-F(a)),$$

substituting $\delta = 1-(1-\delta)$, we calculate that in (12) expression in square brackets is not less than

$$2(1-\delta)(1-B)f(B) - (1-\delta)(F(B)-F^2(B)) - F^2(B) - \delta(B-a)f(B)F(a) \leq$$

further as $B \geq a$ and (10) we get that

$$\leq 2(1-\delta)(1-B)f(B) - F^2(B) \leq 2(1-\delta)(1-a)L - F^2(a) \leq 0.$$

We prove that derivative $H'_{b=1}(B)$ not positive on $[a,1]$, and this fact implies $(**)$.

Threshold strategies and deal area are on pic. 3 and pic. 4. Theorem 3 shows that for any price $a \in (0,1)$, with limited density functions $f(s), g(b)$ and discount factor $\delta$ close to 1 then Nash equilibrium with threshold strategies exists.

*Example 1.* Let consider a situation of uniform distribution for reserved prices on the interval $[0,1]$, i. e. $F(s)=s$, $G(b)=b$. As $f(s)=1$, $g(b)=1$, so in the theorem 3 we can set $L=M=1$. By (9), (10) we find that if $\delta \geq \max\{1-\frac{(1-a)^2}{2a}, 1-\frac{a^2}{2(1-a)}\}$ then threshold strategies with price $a \in (0,1)$ are Nash equilibrium. For $a=\frac{1}{2}$ we get sufficient condition (by using theorem 3) $\delta \geq \frac{3}{4}$.

Now we calculate explicit minimal value for discount factor $\delta$ when it is Nash equilibrium with threshold at price $a=\frac{1}{2}$. Derivative of outcome (11) is

$$H'_{s=0}(S) = \frac{1}{(1-\delta S)^2}\left[\frac{3}{4}\delta S^2 - \frac{3}{2}S + \frac{1}{2} - \frac{1}{4}\delta a^2 + \frac{1}{2}\delta a\right],$$

solving appropriate inequality we derive that for

$$S \leq \frac{3-\sqrt{9-6\delta+3\delta^2a^2-6\delta^2 a}}{3\delta}$$

derivative of seller's outcome is nonnegative. Hence, we have necessary and sufficient condition

$$a \leq \frac{3-\sqrt{9-6\delta+3\delta^2a^2-6\delta^2 a}}{3\delta}.$$

From where we find

$$a \in \left[0, \frac{3-\delta-\sqrt{\delta^2-10\delta+9}}{2\delta}\right].$$

The same reasoning for buyers leads to condition

$$a \in \left[\frac{3\delta - 3 + \sqrt{\delta^2 - 10\delta + 9}}{2\delta}, 1\right],$$

and finally we find that for $\delta \geq \frac{2}{3}$ threshold strategies with price $a = \frac{1}{2}$ are Nash equilibrium in multistage auction model.

## 4. Conclusion

We offer multistage double closed auction model. Distribution of reserved prices are common knowledge. On every stage pairs of agents with different reserved prices are randomly selected. After then they decide to make a deal or no deal. In classical version (Chatterjee and Samuelson, 1983) it is single-stage process. In our model if there is no deal then agents move to the next stage. Outcome is discounted.

We find Nash equilibrium in the model. Strategies are functions of reserved prices. Assuming the existence of stationary state for distribution of reserved prices from stage to stage, we research criteria for strategies to be Nash equilibrium. In theorem 1 we prove criteria for equilibrium in class of strictly increasing strategies, and in theorem 2 in class of threshold strategies.

## References

Chatterjee, K. and Samuelson, W. (1983). *Bargaining under incomplete informationg.* Operations Research, **31(5)**, 835–851.

Mazalov, V. V. and Kondratyev, A. Y. (2012). *Bargaining model with incomplete information.* Vestnik St. Petersburg University. Ser. 10, **1**, 33-Ũ40.

Mazalov, V. V. and Kondratev, A. Y. (2013). *Threshold strategies equilibrium in bargaining model.* Game theory and applications, **5(2)**, 46–63.

Mazalov, V. V., Mentcher, A. E. and Tokareva, J. S. (2012). *Negotiations. Mathematical theory.* Lan. Saint-Petersburg. 304 P.

Mazalov, V. V. and Tokareva, J. S. (2011). *Equilibrium in bargaining model with non-uniform distribution for reservation prices.* Game theory and applications. **3(2)**, 37–49.

Myerson, R. and Satterthwait, M. A. (1983) *Efficient mechanisms for Bilateral Trading.* Journal of Economic Theory, **29**, 265–281.

Myerson, R. (1984) *Two-Person Bargaining Problems with Incomplete Information.* Econometrica, **52**, 461–487.