Equilibrium Strategies in Two-Sided Mate Choice Problem with Age Preferences*

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Abstract In the paper the two-sided mate choice model of Alpern, Katrantzi and Ramsey (2010) is considered. In the model the individuals from two groups (males and females) want to form a couple. It is assumed that the total number of unmated males is greater than the total number of unmated females and the maximum age of males (m) is greater than the maximum age of females (n). There is steady state distribution for the age of individuals. The aim of each individual is to form a couple with individual of minimum age. We derive analytically the equilibrium threshold strategies and investigate players' payoffs for the case $n = 3$ and large m.

Keywords: mutual mate choice, equilibrium, threshold strategy.

1. Introduction

In the paper the two-sided mate choice model of Alpern, Katrantzi and Ramsey (2010) (Alpern et al., 2010) is considered. The problem is following. The individuals from two groups (males and females) want to form a long-term relationship with a member of the other group, i.e. to form a couple. Each group has steady state distribution for the age of individuals. In the model males and females can form a couple during m and n periods respectively. It is assumed that the total number of unmated males is greater than the total number of unmated females and $m \geq n$. The discrete time game is considered. In the game unmated individuals from different groups randomly meet each other in each period. If they accept each other, they form a couple and leave the game, otherwise they go into the next period unmated and older. It is assumed that individuals of both sexes enter the game at age 1 and stay until they are mated or males (females) pass the age m (n). The initial ratio of age 1 males to age 1 females is given. The payoff of mated player is the number of future joint periods with selected partner. Payoff of a male age i and a female age j if they accept each other is equal to $\min\{m - i + 1, n - j + 1\}$. The aim of each player is to maximize his/her expected payoff. In each period players use threshold strategies: to accept exactly those partners who give them at least the same payoff as the expected payoff from the next period.

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In the literature such problems are called also marriage problems or job search problems. We use here the terminology of "mate choice problem". In papers (Alpern and Reyniers, 1999; Alpern and Reyniers, 2005; Mazalov and Falko, 2008) the mutual mate choice problems with homotypic and common preferences are investigated. In (Alpern et al., 2013) a continuous time model with age preferences is considered. Other two-sided mate choice models were considered in papers (Gale and Shapley, 1962; Kalick and Hamilton, 1986; Roth and Sotomayor, 1992). Alpern, Katrantzi and Ramsey (Alpern et al., 2010) derive properties of equilibrium threshold strategies and analyse the model for small m and n. The case $n = 2$ was considered in paper (Konovalchikova, 2012). In this paper using dynamic programming method we derive analytically the equilibrium threshold strategies and investigate players' payoffs for the case $n = 3$ and large m.

2. Two-Sided Mate Choice Model with Age Preferences

Denote a_i — the number of unmated males of age i relative to the number of females of age 1 and b_j — the number of unmated females of age j relative to the number of females of age 1 ($b_1 = 1$). The vectors of the relative numbers of unmated males and females of each age $a = (a_1, ..., a_m)$, $b = (b_1, ..., b_n)$ remain constant over time.

Denote the ratio of the rates at which males and females enter the adult population by R, $R = \frac{a_1}{b_1} = a_1$.

The total groups of unmated males and females are $A = \sum\limits_{i=1}^{m} a_i$, $B = \sum\limits_{i=1}^{n} b_j$. Denote the total ratio $\frac{A}{B}$ by r and assume that $r > 1$.

Consider the following probabilities:

- $\frac{a_i}{A}$ — the probability a female is matched with a male of age i,
- $\frac{B}{A}$ — the probability a male is matched.
- $\frac{b_j}{B}$ — the probability a male is matched with a female of age j, given that a male is mated.
- $\frac{b_j}{A} = \frac{b_j}{B} \cdot \frac{B}{A}$ — the probability a male is matched with a female of age j.

Denote U_i, $i = 1, ..., m$ — the expected payoff of male of age i and V_j, $j = 1, ..., n$ — the expected payoff of female of age j.

Players use the threshold strategies $F = [f_1, ..., f_m]$ for males and $G = [g_1, ..., g_n]$ for females, where $f_i = k$, $k = 1, ..., n$ — to accept a female of age $1, ..., k$, $g_j = l$, $l = 1, ..., m$ — to accept a male of age $1, ..., l$.

3. Model for $n = 3$ and $m \geq 3$

Consider the two-sided mate choice model with age preferences for the case $n = 3$ and $m \geq 3$.

The expected payoffs of male have the following form

$$V_3 = \sum_{i=1}^{m-1} \frac{a_i}{A} I\{f_i = 3\} + \frac{a_m}{A} \leq 1,$$

$$V_2 = \sum_{i=1}^{m-2} \frac{a_i}{A} 2I\{f_i \geq 2\} + \frac{a_{m-1}}{A} 2 + \frac{a_m}{A} 1 \leq 2,$$

$$V_1 = \sum_{i=1}^{m-2} \frac{a_i}{A} 3 + \frac{a_{m-1}}{A} 2 + \frac{a_m}{A} \max\{1, V_2\},$$

where $I\{C\}$ is indicator of event C.

From this it follows that $g_3 = g_2 = m$. Also $g_1 = m$, if $V_2 < 1$ or $g_1 = m-1$, if $V_2 \geq 1$.

There are three forms of strategies which are presented in the table:

$G_2 = [m, m, m]$	$G_1 = [m-1, m, m]$
I. $F_1 = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$ $k = 1, ..., m-3,\ l = 1, ..., m-3$	II. $F_3 = [\underbrace{2, ..., 2}_{k}, \underbrace{3, ..., 3}_{m-k}]$ $k = 1, ..., m-2$
	III. $F_2 = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$ $k = 1, ..., m-3,\ l = 1, ..., m-3$

Note that for female strategy $G_2 = [m, m, m]$ in the equilibrium male strategy it should be $f_1^* = 1$.

Consider these strategies consequently.

I. Players use strategy profile (F_1, G_2), where $G_2 = [m, m, m]$ (to accept any partner), $F_1 = [f_1, ..., f_m] = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$, $k = 1, ..., m-3$, $l = 1, ..., m-3$.

For equilibrium strategies the male's payoff $V_2 = \sum_{i=1}^{m-2} \frac{a_i}{A} 2I\{f_i \geq 2\} + \frac{a_{m-1}}{A} 2 + \frac{a_m}{A} 1$ must be less than 1. It is equivalent to

$$\sum_{i=1}^{m-1} \left(1 - \frac{1}{r}\right)^{i-1} + \sum_{i=1}^{k} 2 \left(1 - \frac{1}{r}\right)^{i-1} I\{f_i \geq 2\} < 0.$$

Consider the expected payoffs of males

$$
\begin{aligned}
&U_m = \frac{b_1}{A}1 + \frac{b_2}{A}1 + \frac{b_3}{A}1 + \left(1 - \frac{B}{A}\right)0 = \frac{1}{r} < 1,\\
&U_{m-1} = \frac{b_1}{A}2 + \frac{b_2}{A}2 + \frac{b_3}{A}1 + \left(1 - \frac{1}{r}\right)U_m = \frac{2}{r} - \frac{b_3}{A} + \left(1 - \frac{1}{r}\right)U_m < 2,\\
&U_{m-2} = \frac{b_1}{A}3 + \frac{b_2}{A}2 + \frac{b_3}{A}\max\{1, U_{m-1}\} + \left(1 - \frac{1}{r}\right)U_{m-1} < 3,\\
&U_{m-i} = \frac{b_1}{A}3 + \frac{b_2}{A}\max\{2, U_{m-i+1}\} + \frac{b_3}{A}\max\{1, U_{m-i+1}\} + \left(1 - \frac{1}{r}\right)U_{m-i+1} < 3,\\
&i = 3, ..., m-1.
\end{aligned}
\tag{1}
$$

From these expressions it follows that equilibrium strategies are equal to

1) $f^*_{m-1} = f^*_m = 3$,

2)$f^*_{m-2} = 3$, if $U_{m-1} < 1$; $f^*_{m-2} = 2$, if $U_{m-1} \geq 1$,

3) $f^*_{m-i} = 3$, if $U_{m-i+1} < 1$; $f^*_{m-i} = 2$, if $1 \leq U_{m-i+1} < 2$; $f^*_{m-i} = 1$, if $2 \leq U_{m-i+1}$, $i = 3, ..., m-1$.

The equilibrium strategies and the optimal payoffs are presented in the theorem.

Theorem 1. *If players use the equilibrium strategy profile* (F^*_1, G^*_2)*, where* $G^*_2 = [m, m, m]$*,* $F^*_1 = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$*, for certain values of* k *and* l *(*$k = 1, ..., m-3$*,* $l = 1, ..., m-3$*) then the males' optimal payoffs are equal to*

$$
\begin{cases}
U_m = 1 - z,\\
U_{m-1} = 2 - z^2 - z,\\
U_{m-i} = 3 - z^{i+1} - z^i - z^{i-1},\ i = 2, ..., m-1,
\end{cases}
\tag{2}
$$

the equilibrium age distributions are equal to

$$
\begin{aligned}
&a = (R, Rz, Rz^2, ..., Rz^{m-1});\ b = (1, 0, 0),\\
&R = \frac{1}{(1-z)(1 + z + z^2 + ... + z^{m-1})},\\
&A = r = 1/(1-z),
\end{aligned}
$$

where $z = 1 - 1/r$.

Proof. Let the players use strategy profile (F_1, G_2), where $G_2 = [m, m, m]$, $F_1 = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$, $k = 1, ..., m-3$, $l = 1, ..., m-3$.

Then the age distributions are equal to

$b = (1, 0, 0)$;

$a = (R, a_1(1-\frac{1}{r}), ..., a_{m-1}(1-\frac{1}{r}))$ or $a = (R, R(1-\frac{1}{r}), R(1-\frac{1}{r})^2, ..., R(1-\frac{1}{r})^{m-1})$.

Taking into account that $Br = A$, where $A = \sum\limits_{i=1}^{m} a_i$, $B = \sum\limits_{i=1}^{n} b_j$ we get

$$
R = \frac{r}{1 + (1 - 1/r) + (1 - 1/r)^2 + ... + (1 - 1/r)^{m-1}}.
$$

Then we can rewrite the expected male's payoffs (1) in the following recurrent form ($z = 1 - 1/r$):

$$U_m = \frac{1}{r} = 1 - z,$$

$$U_{m-1} = \frac{2}{r} + \left(1 - \frac{1}{r}\right) U_m = 2(1-z) + zU_m,$$

$$U_{m-i} = \frac{3}{r} + \left(1 - \frac{1}{r}\right) U_{m-i+1} = 3(1-z) + zU_{m-i+1},\ i = 2, ..., m-1.$$

Substituting each expression into the next one we get

$$U_m = 1 - z,$$

$$U_{m-1} = 2(1-z) + zU_m = (1-z)(2+z),$$

$$U_{m-2} = 3(1-z) + zU_{i+1} = (1-z)\left(3 + z^2 + 2z\right),$$

$$U_{m-i} = 3(1-z) + zU_{m-i+1} = (1-z)\left(3\sum_{j=0}^{i-1} z^j - z^{i-1} + z^i\right),\ i = 3, ..., m-1.$$

Simplifying the payoffs we obtain (2).

For the equilibrium females' strategy $G_2^* = [m, m, m]$ the equilibrium males' strategy F_1^* can be obtained from the system

$$\begin{cases} V_2 < 1, \\ U_{m-1} < 1, \\ ... \\ U_{k+l+2} < 1, \\ 1 \leq U_{k+l+1} < 2, \\ ... \\ 1 \leq U_{k+2} < 2, \\ U_{k+1} \geq 2, \\ ... \\ U_2 \geq 2 \end{cases}$$

for different value of r.

Example 1. For $m = 4$ and $r = 2$, we obtain $a = \left(\frac{16}{15}, \frac{8}{15}, \frac{4}{15}, \frac{2}{15}\right)$, $b = (1, 0, 0)$, $F_1^* = [1, 2, 3, 3]$, $G_2^* = [4, 4, 4]$.

Example 2. $F_1^* = [1, ..., 1, 2, 3, 3]$ for $r \in (1; 2.191]$ and $m \geq 4$, where $r^* = 2.191$ is the solution of the equation $2r^3 - 6r^2 + 4r - 1 = 0$

$F_1^* = [1, ..., 1, 2, 2, 3, 3]$ for $r \in [2.191; 2.618]$ and $m \geq 6$, where ãäå $r_1^* = 2.191$ is the solution of the equation $2r^3 - 6r^2 + 4r - 1 = 0$, and $r_2^* = 2.618$ is the solution of the equation $r^2 - 3r + 1 = 0$,

$F_1^* = [1, ..., 1, 2, 3, 3, 3]$ for $r \in [2.618; 3.14]$ and $m \geq 6$,

$F_1^* = [1, ..., 1, 2, 2, 3, 3, 3]$ for $r \in [3.14; 4.079]$ and $m \geq 7$.

II. Consider the case when the female's strategy is $G_1 = [m-1, m, m]$ ($V_2 \geq 1$). The expected males' payoffs are equal to

$$U_m = \frac{b_1}{A}0 + \frac{b_2}{A}1 + \frac{b_3}{A}1 + \left(1 - \frac{B}{A}\right)0 = \frac{1}{r} - \frac{b_1}{A} < 1,$$

$$U_{m-1} = \frac{b_1}{A}2 + \frac{b_2}{A}2 + \frac{b_3}{A}1 + \left(1 - \frac{1}{r}\right)U_m = \frac{2}{r} - \frac{b_3}{A} + \left(1 - \frac{1}{r}\right)U_m < 2$$

$$U_{m-2} = \frac{b_1}{A}3 + \frac{b_2}{A}2 + \frac{b_3}{A}\max\{1, U_{m-1}\} + \left(1 - \frac{1}{r}\right)U_{m-1} < 3,$$

$$U_{m-i} = \frac{b_1}{A}3 + \frac{b_2}{A}\max\{2, U_{m-i+1}\} + \frac{b_3}{A}\max\{1, U_{m-i+1}\} + \left(1 - \frac{1}{r}\right)U_{m-i+1} < 3,$$
$i = 3, ..., m-1.$

It follows that $f^*_{m-1} = f^*_m = 3$, and $f^*_i = \{1, 2, 3\}$, $i = 1, ..., m-2$ depending on the values of r.

Consider the case when the male's strategy is $F_3 = [\underbrace{2, ..., 2}_{k}, \underbrace{3, ..., 3}_{m-k}]$, $k = 1, ..., m-2$.

Theorem 2. *If players use the equilibrium strategy profile* (F_3^*, G_1^*)*, where* $G_1^* = [m-1, m, m]$*,* $F_3^* = [\underbrace{2, ..., 2}_{k}, \underbrace{3, ..., 3}_{m-k}]$*, for certain values of* k *(*$k = 1, ..., m-2$*) then the males' optimal payoffs are equal to*

$$U_m = 1 - z - \frac{1}{A},$$

$$U_{m-1} = 2(1-z) + zU_m,$$

$$U_{m-i} = 3 - \frac{a_m}{A^2(1-z)} - \left(1 - \frac{a_m}{A^2(1-z)}\right)z^{i-1} - \left(1 + \frac{1}{A}\right)z^i - z^{i+1},$$
$i = 2, ..., m-2,$

the equilibrium age distributions are equal to

$$a = \left(R, Rz, Rz^2, ..., Rz^{m-1}\right), b = \left(1, \frac{z^{m-1}}{\sum\limits_{i=0}^{m-1} z^i}, 0\right).$$

$$R = \frac{1 + z + z^2 + ... + z^{m-2} + 2z^{m-1}}{(1-z)(1 + z + z^2 + ... + z^{m-1})^2},$$

$$A = R\sum_{i=0}^{m-1} z^i,$$

where $z = 1 - 1/r$.

Proof. The distributions for the age of males and females have form

$a = \left(R, Rz, Rz^2, ..., Rz^{m-1}\right)$, $b = \left(1, \frac{a_m}{A}, 0\right)$, where $z = 1 - 1/r$.
The ratio of the rates at which males and females enter the adult population has form

$$R = \frac{1 + z + z^2 + ... + z^{m-2} + 2z^{m-1}}{(1-z)(1 + z + z^2 + ... + z^{m-1})^2}, \text{ where } z = 1 - 1/r.$$

We have that $V_2 = 2 - \frac{a_m}{A} \geq 1$.

The expected payoffs have form

$$U_m = \frac{1}{r} - \frac{1}{A},$$

$$U_{m-1} = \frac{2}{r} + \left(1 - \frac{1}{r}\right) U_m,$$

$$U_{m-i} = \frac{3}{r} + \left(1 - \frac{1}{r}\right) U_{m-i+1} - \frac{a_m}{A^2}, \; i = 2, ..., m-2$$

or

$$U_m = 1 - z - \frac{1}{A},$$

$$U_{m-1} = 2(1-z) + zU_m,$$

$$\begin{aligned} U_{m-i} &= 3(1-z) + zU_{m-i+1} - \frac{a_m}{A^2} = \\ &= 3 - \frac{a_m}{A^2(1-z)} - \left(1 - \frac{a_m}{A^2(1-z)}\right) z^{i-1} - \left(1 + \frac{1}{A}\right) z^i - z^{i+1}, \\ & i = 2, ..., m-2, \text{ where } z = 1 - 1/r \end{aligned}$$

For the equilibrium female's strategy $G_1^* = [m-1, m, m]$ the equilibrium males' strategies F_3^* can be obtained from the system

$$\begin{cases} U_{m-1} < 1, \\ ... \\ U_{k+1} < 1, \\ 1 < U_k < 2, \\ ... \\ 1 < U_2 < 2 \end{cases}$$

for different value of r.

III. Finally, consider the case when the female's strategy is $G_1 = [m-1, m, m]$ ($V_2 \geq 1$) and the male's strategy is $F_2 = [\underbrace{1, ..., 1}_{k}, \underbrace{2, ..., 2}_{l}, \underbrace{3, ..., 3}_{m-k-l}]$, $k = 1, ..., m-3$, $l = 1, ..., m-3$.

Then the expected payoff of female of age 2 is equal to

$V_2 = 2 - \frac{a_m}{A} - 2\sum_{i=1}^{k} \frac{a_i}{A}$ and it must be greater then or equal to 1.

Then the distributions for the age of males and females have forms

$$a = (a_1, ..., a_m);\ b = \left(1,\ \frac{a_m}{A},\ \frac{a_m}{A} \sum_{i=1}^{k} \frac{a_i}{A}\right),$$

where

$$a_1 = R,\ a_i = a_{i-1}(1 - 1/A),\ i = 2, ..., k+1,$$
$$a_j = a_{j-1}\left(\frac{b_3}{A} + 1 - \frac{1}{r}\right),\ j = k+2, ..., k+l+1,$$
$$a_s = a_{s-1}\left(1 - \frac{1}{r}\right),\ s = k+l+2, ..., m.$$

The expected payoffs of males are equal to

$$U_m = \frac{1}{r} - \frac{1}{A},$$
$$U_{m-1} = \frac{2}{r} - \frac{b_3}{A} + \left(1 - \frac{1}{r}\right) U_m,$$
$$U_i = \frac{3}{r} - \frac{b_2}{A} - 2\frac{b_3}{A} + \left(1 - \frac{1}{r}\right) U_{i+1},\ i = k+l+1, ..., m-2,$$
$$U_j = \frac{3}{r} - \frac{b_2}{A} - 3\frac{b_3}{A} + \left(1 - \frac{1}{r} + \frac{b_3}{A}\right) U_{j+1},\ j = k+1, ..., k+l,$$
$$U_s = \frac{3}{r} - 3\frac{b_2}{A} - 3\frac{b_3}{A} + \left(1 - \frac{1}{r} + \frac{b_2}{A} + \frac{b_3}{A}\right) U_{s+1},\ s = 2, ..., k+1.$$

For the equilibrium females' strategy $G_1^* = [m-1, m, m]$ the equilibrium males' strategy F_2^* can be obtained from the system

$$\begin{cases} V_2 \geq 1, \\ U_{m-1} < 1, \\ ... \\ U_{k+l+2} < 1, \\ 1 \leq U_{k+l+1} < 2, \\ ... \\ 1 \leq U_{k+2} < 2, \\ U_{k+1} \geq 2, \\ ... \\ U_2 \geq 2. \end{cases}$$

In Table 1. the numerical results for the optimal threshold strategies are given for different values of r.

Table 1: Equilibrium strategy for $m = 5$ for different values of r.

Equilibrium	$r = \frac{A}{B}$	R
$([1, 1, 2, 3, 3], [5, 5, 5])$	$(1, 2.191]$	$(1, 1.049]$
$([1, 2, 3, 3, 3], [4, 5, 5])$	$[2.016, 2.79]$	$[1.081, 1.191]$
$([2, 2, 3, 3, 3], [4, 5, 5])$	$[2.85, 4.517]$	$[1.209, 1.560]$
$([2, 3, 3, 3, 3], [4, 5, 5])$	$[4.517, 6.87]$	$[1.560, 2.097]$
$([3, 3, 3, 3, 3], [4, 5, 5])$	$[6.87, +\infty)$	$[2.097, +\infty)$

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