Differential Games with Random Duration: A Hybrid Systems Formulation

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Abstract The contribution of this paper is two-fold. First, a new class of differential games with random duration and a composite cumulative distribution function is introduced. Second, it is shown that these games can be well defined within the hybrid systems framework and that the problem of finding the optimal strategy can be posed and solved with the methods of hybrid optimal control theory. An illustrative example is given.

Keywords: games, hybrid, etc.

1. Introduction

Game theory as a branch of mathematics investigates conflict processes controlled by many participants (players). These processes are referred to as games. In this paper we focus on the duration of games. In differential game theory it is common to consider games with a fixed duration (finite time horizon) or games with an infinite time horizon. However, in many real-life applications the duration of a game can not be determined *a priori* but depends on a number of unknown factors and thus is not deterministic any longer.

To take account of this phenomenon, a finite-horizon model with random terminal time is considered. For the first time the class of differential games with random duration was introduced in (Petrosyan and Murzov, 1966) for a particular case of a zero-sum pursuit game with terminal payoffs at random terminal time. Later, the general formulation of the differential games with random duration was given in (Petrosyan and Shevkoplyas, 2003). Section 2. provides a brief overview of these results.

Apparently, Boukas, Haurie and Michel, in (Boukas et al., 1990), were first to consider an optimal control problem with a random stopping time. Apart from that, in the optimal control theory there have also been papers exploring the idea of random terminal time applied to non-game-theoretical problems. In particular, the problem of the consumer's life insurance under condition of the random moment of death was discussed in (Yaari, 1965, Chang, 2004).

In many cases the probability density function of the terminal time may change depending on some conditions, which can be expressed as a function of time and state. Consider, for instance, the example of the development of a mineral deposit. The probability of a breakdown may depend on the development stage. At the initial stage this probability is higher than during the routine mining operation. Therefore one needs to define a composite distribution function for the terminal time as described in Sec. 3. To the best of our knowledge, this formulation has never been considered before despite its obvious practical appeal. In our view, this is caused by the limitations of the generally adopted technique for the computation of optimal strategies.

In non-cooperative differential games players solve the optimal control problem of the payoff maximization. One of the basic techniques for solving the optimal control problem is the Hamilton-Jacobi-Bellman equation (Dockner et al., 2000). However, in the above described case a solution (i.e., a differentiable value function) to the HJB equation may not exist. In this case a generalized solution is sought for (the interested reader is referred to Bardi and Capuzzo-Dolcetta, 1997, Vinter, 2000). An alternative to the HJB equation is the celebrated Pontryagin Maximum Principle (Pontryagin et al., 1963) which was recently generalized to a class of hybrid optimal control problems (see, e.g., Riedinger et al., 2003, Shaikh and Caines, 2007, Azhmyakov et al., 2007). In Sec. 3., we show that the optimization problem for a differential game with random terminal time and composite distribution function can be formulated and solved within the hybrid control systems framework.

Finally, in the last section an application of our theoretical results is presented. We investigate one simple model of non-renewable resource extraction, where the termination time is a random variable with a composite distribution function. Two different switching rules are studied and a qualitative analysis of the obtained results is presented.

2. Differential Game Formulation

Consider an N-person differential game $\Gamma(t_0, x_0)$ starting at the time instant t_0 from the initial state x_0, and with duration $T - t_0$. Here the random variable T with a *cumulative distribution function* (CDF) $F(t)$, $t \in [t_0, \infty)$, is the time instant at which the game $\Gamma(t_0, x_0)$ ends. The CDF $F(t)$ is assumed to be an absolutely continuous nondecreasing function satisfying the following conditions:

C1. $F(t_0) = 0$,

C2. $\lim_{t\to\infty} F(t) = 1$.

Furthermore, there exists an a.e. continuous function $f(t) = F'(t)$, called the probability density function (PDF), such that (Royden, 1988)

$$F(t) = \int_{t_0}^{t} f(\tau) d\tau \quad \forall t \in [t_0, \infty).$$

Let the system dynamics be described by the following ODEs:

$$\dot{x} = g(x, u_1, \ldots, u_N), x \in \mathbb{R}^m,\ u_i \in U \subseteq \text{comp}(\mathbb{R}), \quad x(t_0) = x_0, \tag{1}$$

where $g : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}^m$ is a vector-valued function satisfying the standard existence and uniqueness requirements (see, e.g., Lee and Markus, 1967, Ch. 4).

The instantaneous payoff of the i-th player at the moment τ, $\tau \in [t_0, \infty)$ is defined as $h_i(x(\tau), u_i(\tau))$. Then the expected integral payoff of the player i, where

$i = 1, \ldots, N$ is evaluated by the formula

$$K_i(t_0, x, u) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(x(\tau), u_i(\tau)) d\tau dF(t) = \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(x(\tau), u_i(\tau)) d\tau f(t) dt. \qquad (2)$$

The Pareto optimal strategy in the game $\Gamma(t_0, x_0)$ is defined as the n-tuple of controls $u^*(t) = (u_1^*(t), \ldots, u_n^*(t))$ maximizing the joint expected payoff of players:

$$(u_1^*(t), \ldots, u_n^*(t)) = \operatorname*{argmax}_u \sum_{i=1}^{n} K_i(t_0, x, u). \qquad (3)$$

Hence, the Pareto optimal solution of $\Gamma(t_0, x_0)$ is $(x^*(t), u^*(t))$ and the total optimal payoff $V(x_0)$ is

$$V(x_0, t_0) = \sum_{i=1}^{n} K_i(t_0, x^*, u^*) = \sum_{i=1}^{n} \int_{t_0}^{\infty} \int_{t_0}^{t} h_i(x^*(\tau), u_i^*(\tau)) d\tau f(t) dt. \qquad (4)$$

For the set of subgames $\Gamma(\vartheta, x^*(\vartheta))$, with $\vartheta > t_0$, occurring along the optimal trajectory $x^*(\vartheta)$ one can similarly define the expected total integral payoff in the cooperative game $\Gamma(\vartheta, x^*(\vartheta))$:

$$V(x^*(\vartheta), \vartheta) = \sum_{i=1}^{n} \int_{\vartheta}^{\infty} \int_{\vartheta}^{t} h_i(x^*(\tau), u_i^*(\tau)) d\tau \, dF_\vartheta(t), \qquad (5)$$

where $F_\vartheta(t)$ is a conditional cumulative distribution function defined as

$$F_\vartheta(t) = \frac{F(t) - F(\vartheta)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \qquad (6)$$

and the conditional probability density function has the following form:

$$f_\vartheta(t) = \frac{f(t)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \qquad (7)$$

2.1. Transformation of the Integral Functional

Below, the transformation procedure of the double integral functional (2) and its reduction to a single integral is described. We obtain this result by changing the order of integration; alternative approaches were presented in, e.g., (Boukas et al., 1990, Chang, 2004). In the following, we assume that the expression under the integral sign is such that the order of integration in (2) is immaterial. Note that in general this is not true (see, for example, 1). A detailed account on this issue is presented in (Kostyunin and Shevkoplyas, 2011).

From now on, without loss of generality we set $t_0 = 0$.

Consider the integral functional of the i-th player:

$$\int_{0}^{\infty} \int_{0}^{t} h_i(\tau) \, d\tau \, f(t) dt,$$

where $h_i(\tau)$ is a shorthand for $h_i(x(\tau), u_i(\tau))$.

Define function $a(t,\tau)$ as follows:

$$a(t,\tau) = f(t)h_i(\tau)\cdot\chi_{\{\tau\leqslant t\}} = \begin{cases} f(t)h_i(\tau), & \tau \leqslant t; \\ 0, & \tau > t \end{cases}$$

Taking into account the above mentioned assumption, we interchange the variables of integration in the double integral. Then we get:

$$\int_0^\infty dt \int_0^t f(t)h_i(\tau)d\tau = \int_0^\infty dt \int_0^\infty a(t,\tau)d\tau =$$
$$= \int_0^\infty d\tau \int_\tau^\infty f(t)h_i(\tau)dt = \int_0^\infty (1-F(\tau))h_i(\tau)d\tau.$$

In the general case, the expected payoff of the player i in the game $\Gamma(t_0, x_0)$ can be rewritten as:

$$K_i(t_0, x, u) = \int_{t_0}^\infty (1-F(\tau))h_i(x(\tau), u_i(\tau))d\tau. \tag{8}$$

In the same way we get the expression for expected payoff of the player in the subgame $\Gamma(\vartheta, x(\vartheta))$:

$$K_i(\vartheta, x, u) = \frac{1}{1-F(\vartheta)} \int_\vartheta^\infty (1-F(\tau))h_i(x(\tau), u_i(\tau))d\tau. \tag{9}$$

3. Hybrid Formulation of a Differential Game

In this section we give the definition of a hybrid control problem and the associated hybrid optimal control problem. It is shown that the differential game with a composite CDF (CCDF) introduced in Subsection 3.2. fits perfectly in the hybrid framework. Hence, a hybrid differential game as well as a number of particular cases are considered, the respective optimal control problems are defined, and the solution strategies are proposed.

3.1. Hybrid Optimal Control Problem

Below, we give the definition of a hybrid system. For more details, the interested reader is referred to (Riedinger et al., 2003, Shaikh and Caines, 2007), as well as (Azhmyakov et al., 2007).

Definition 1. The hybrid system $\mathcal{HS}$ is defined as a tuple

$$\mathcal{HS} = (Q, X, U, f, \gamma, \Phi, q_0, x_0),$$

where

- $Q = \{1, \ldots, N\}$ is the set of discrete states, $X\mathbb{R}^l$ is the continuous state, $U_q \subset \mathbb{R}^m$, $q \in Q$ are the admissible control sets, which are compact and convex, and

  $$\mathcal{U}_q := \{u(\cdot) \in \mathcal{L}^m_\infty(0, t_f) \ : \ u(t) \in U_q, \text{ a.e. on}[0, t_f]\}$$

  represent the sets of admissible control signals.

- $q_0 \in Q$ and $x_0 \in X$ are the initial conditions.
- $f_q : X \times U \to X$ is the function that associates to each discrete state $q \in Q$ a differential equation of the form

$$\dot{x}(t) = f_q(x(t), u(t)). \tag{10}$$

- $\gamma_{q,q'} : X \to \mathbb{R}^k$ is the function that triggers the change of discrete state. Let $q \in Q$ be the current discrete state and $x(t)$ be the state trajectory evolving according to the respective differential equation (10). The transition to the discrete state $q' \in Q$ occurs at the moment χ when $\gamma_{q,q'}(x(\chi)) = 0$. The set $\Gamma_{q,q'} = \{x \in X | \gamma_{q,q'}(x) = 0\}$ is referred to as the *switching manifold*.
- When the discrete state changes from q to q', the continuous state might change discontinuously. This change is described by the *jump function* $\Phi_{q,q'} : X \to X$. Let χ be the time at which the discrete state changes from q to q', then the continuous state at $t = \chi$ is described as $x(\chi) = \Phi_{q,q'}(x(\chi^-))$, where $x(\chi^-) = \lim\limits_{t \to \chi - 0} x(t)$.

Definition 2. A hybrid trajectory of $\mathcal{HS}$ is a triple $\mathcal{X} = (x, \{q_i\}, \tau)$, where $x(\cdot) : [0, T] \to \mathbb{R}^n$, $\{q_i\}_{i=1,\dots,r}$ is a finite sequence of locations and τ is the corresponding sequence of switching times $0 = t_0 < \cdots < t_r = T$ such that for each $i = 1, \dots, r$ there exists $u_i(\cdot) \in \mathcal{U}_i$ such that:

- $x(0) = x_0$ and $x_i(\cdot) = x(\cdot)|_{[t_{i-1}, t_i)}$ is an absolutely continuous function in $[t_{i-1}, t_i)$ continuously extendable to $[t_{i-1}, t_i]$, $i = 1, \dots, r$.
- $\dot{x}_i(t) = f_{q_i}(x_i(t), u_i(t))$ for almost all $t \in [t_{i-1}, t_i]$, $i = 1, \dots, r$.
- The switching condition $x_i(t_i) \in \Gamma_{q_i, q_{i+1}}$ along with the jump condition $x_{i+1}(t_i) = \Phi_{q_i, q_{i+1}}(x_i(t_i))$ are satisfied for each $i = 1, \dots, r-1$.

Using the introduced notation we can state a hybrid optimal control problem and characterize an optimal solution to this problem. Let the overall performance of $\mathcal{HS}$ be evaluated by the following functional criterion:

$$J(x_0, q_0, u) = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} L_{q_i}(x_i(t), u_i(t), t) dt, \tag{11}$$

where $L_{q_i} : X \times U \times \mathbb{R}_{\geqslant 0}$, $q_i \in Q$, are twice continuously differentiable functions. Assume that the sequence of discrete states q^* is given. Then the necessary conditions for a solution (x^*, q^*, τ, u^*) to $\mathcal{HS}$ to minimize (11) is given by the following theorem.

Theorem 1 (Riedinger et al., 2003). *If $u^*(t)$ and $(x^*(t), q^*(t), \tau)$ are the optimal control and the corresponding hybrid trajectory for $\mathcal{HS}$, then there exists a piecewise absolutely continuous curve $p^*(t)$ and a constant $p_0^* \geqslant 0$, $(p^*, p_0^*) \neq (0, \mathbf{0})$ such that*

- *The tuple $(x^*(t), q^*(t), p^*(t), u^*(t), \tau)$ satisfies the associated Hamiltonian system*

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H_{q_i}}{\partial p}(x^*(t), p^*(t), u^*(t)), \\ \dot{p}(t) &= -\frac{\partial H_{q_i}}{\partial x}(x^*(t), p^*(t), u^*(t)), \\ & t \in [t_{i-1}, t_i],\ i = 1, \dots, r \end{aligned} \tag{12}$$

where

$$H_{q_i}(x^*(t), p^*(t), u^*(t)) = \\ = p_0^* L_{q_i}(x_i(t), u_i(t), t) + p^*(t) f_{q_i}(x_i(t), u_i(t)).$$

– *At any time $t \in [t_{i-1}, t_i)$, the following maximization condition holds:*

$$H_{q_i}(x^*(t), p^*(t), u^*(t)) = \sup_{u(t) \in U} H_{q_i}(x^*(t), p^*(t), u(t)). \tag{13}$$

– *At the switching time t_i, there exists a vector $\pi \in \mathbb{R}^n$ such that the following transversality conditions are satisfied:*

$$\begin{aligned} p^*(t_i^-) &= \sum_{k=1}^n p_k(t_i) \frac{\partial \Phi^k_{q_i, q_{i+1}}}{\partial x_j}(t_i^-) + \sum_{k=1}^n \pi_k^i \frac{\partial \gamma^k_{i,i+1}}{\partial x_j}(t_i^-), \\ H_{q_{i-1}}(t_i^-) &= H_{q_i}(t_i) - \sum_{k=1}^n p_k(t_i) \frac{\partial \Phi^k_{i,i+1}}{\partial t}(t_i^-) - \\ &\quad - \sum_{k=1}^n \pi_k^i \frac{\partial \gamma^k_{i,i+1}}{\partial t}(t_i^-) \end{aligned} \tag{14}$$

3.2. Composite Cumulative Distribution Function

Let t_0 be the initial time, $F_i(t)$, $i = 1, \ldots, N$ be a set of CDFs characterizing different modes of operation and satisfying, along with **C1** and **C2**, the following property:

C3. The CDFs $F_i(t)$ are assumed to be absolutely continuous *nondecreasing* functions such that each CDF converges to 1 asymptotically, i.e., $F_i(t) < 1 \; \forall t < \infty$.

Furthermore, let $\tau = \{\tau_i\}$ s.t. $t_0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = \infty$ be an ordered sequence of time instants at which the switches between individual CDFs occur.

The composite CDF $F_\sigma(t)$ is defined as follows:

$$F_\sigma(t) = \begin{cases} F_1(t), & t \in [\tau_0, \tau_1), \\ \alpha_i(\tau_i) F_{i+1}(t) + \beta_i(\tau_i), & t \in [\tau_i, \tau_{i+1}), \\ & 1 \leqslant i \leqslant N-1, \end{cases} \tag{15}$$

where $\alpha_i(\tau_i) = \frac{F_\sigma(\tau_i^-) - 1}{F_{i+1}(\tau_i) - 1}$, and $\beta_i(\tau_i) = 1 - \frac{F_\sigma(\tau_i^-) - 1}{F_{i+1}(\tau_i) - 1}$. Here, $F_\sigma(\tau_i^-)$ is defined as the left limit of $F_\sigma(t)$ at $t = \tau_i^-$, i.e., $F_\sigma(\tau_i^-) = \lim_{t \to (\tau_i - 0)} F_\sigma(t)$.

The composite PDF is defined as $f_\sigma(t) = F'_\sigma(t)$ and has the following form:

$$f_\sigma(t) = \begin{cases} f_1(t), & t \in [\tau_0, \tau_1), \\ \alpha_i(\tau_i) f_{i+1}(t), & t \in [\tau_i, \tau_{i+1}), \\ & 1 \leqslant i \leqslant N-1. \end{cases} \tag{16}$$

Proposition 1. *Given a set of CDFs $F_i(t)$, $1 \leqslant i \leqslant N$, such that* **C1**-**C3** *hold for each $F_i(t)$. Then the composite CDF F_σ defined by (15) satisfies* **C1**-**C3**.

Proof. See Appendix.

From Lemma 1 it follows that $f_\sigma(t)$ has well-defined finite left and right limits at points τ_i, $1 \leqslant i \leqslant N-1$,

$$f_\sigma(\tau_i^-) = \lim_{t \to (\tau_i - 0)} f_\sigma(t), \quad f_\sigma(\tau_i^+) = \lim_{t \to (\tau_i + 0)} f_\sigma(t),$$

which are not necessarily equal, and is continuous otherwise.

The optimization problem (3) for the CCDF (15) can be written taking into account the transformation (8):

$$
\begin{aligned}
u^*(t) &= \underset{u}{\operatorname{argmax}} \sum_{i=1}^{n} K_i(x_0, t_0, u_1, \ldots, u_n) = \\
&= \underset{u}{\operatorname{argmax}} \sum_{i=1}^{n} \int_{\tau_0}^{\tau_N} (1 - F_\sigma(\tau)) h_i(x(\tau), u_i(\tau)) d\tau.
\end{aligned} \tag{17}
$$

3.3. Hybrid Differential Game

The optimization problem (1), (17) can hardly be solved in a straightforward way due to the special structure of the composite CDF $F_\sigma(t)$. However, this problem can be readily formulated as a hybrid optimal control problem. We know that $F_\sigma(t)$ is defined by a number of elementary CDFs, (15), and the switching instants τ_i, $i = 1, \ldots, N$. There are two types of switching instants τ_i corresponding to

a) Time-dependent switches;
b) State-dependent switches.

In the first case, the sequence τ is given; the remaining degrees of freedom are the values of the state at the switching times τ_i, i.e., $x(\tau_i)$. In the second case, the switching times τ_i are determined as the solutions to the equations $\gamma_i(x^i(\tau_i^-)) = 0$, i.e., the regime changes as the state crosses the *switching manifold* defined by the map $\gamma : \mathbb{R}^{n\times l} \to \mathbb{R}^k$. We assume that the sequence of operation modes (i.e., discrete states) is fixed *a priori*. Therefore, there is no need in performing any combinatorial optimization and the problem of determining the optimal strategy can be completely formulated within the framework of hybrid optimal control as shown below.

Time-dependent case To apply the results of Theorem 1 the problem (1), (17) has to be modified. Namely, we extend the system (1) by one differential equation modelling the CDF F_σ. Thus, on each interval $[\tau_{i-1}, \tau_i)$ the differential equations the payoff function are written as

$$
\begin{aligned}
&\dot{x} = g(x, u), \\
&\dot{x}_\sigma = \bar{f}_i(t), \\
&K_i(t_0, x, u) = \int_{\tau_{i-1}}^{\tau_i} (1 - x_\sigma(t)) h(x(t), u(t)) dt,
\end{aligned} \tag{18}
$$

where $h(x(t), u(t)) = \sum_{i=1}^{n} h_i(x_i(t), u_i(t))$ is the total instantaneous payoff, and

$$
\bar{f}_i = f_\sigma\big|_{[\tau_{i-1}, \tau_i)}.
$$

Note that since the switching times are fixed *a priory*, functions $\bar{f}_i$ are well defined. The respective Hamiltonian functions are

$$
H_i(x_t, u, p_0, p_t) = p_0(1 - x_\sigma) h(x, u) + \langle p, g(x, u) \rangle + p_\sigma \bar{f}_i(t),
$$

where $p_0 = -1$, $x_t(t) = [x(t), x_\sigma(t)]'$, and $p_t(t) = [p(t), p_\sigma(t)]'$. Solving the Hamiltonian equations (12) together with the maximization condition (13) one obtains a solution to (18). To solve (12), a number of boundary conditions has to be defined. First, these are initial and end point conditions $x(\tau_0) = x_0,\ x(\infty) = 0$, and $x_\sigma(\tau_0) = 0,\ x_\sigma(\infty) = 1$. Second, there are constraints imposed on the state and adjoint variables at switching times τ_i, $i = 1, \dots, N-1$:

$$\begin{aligned} &x(\tau_i^-) = x(\tau_i),\ p(\tau_i^-) = p(\tau_i),\\ &x_\sigma(\tau_i^-) = x_\sigma(\tau_i),\ p_\sigma(\tau_i^-) = p_\sigma(\tau_i),\\ &H_{i-1}(x_t(\tau_i^-), p_t(\tau_i^-)) = H_i(x_t(\tau_i), p_t(\tau_i)). \end{aligned} \tag{19}$$

With these conditions the problem becomes well-defined. We note that the right-hand sides of the differential equations in (18) depend on t. Therefore, on each interval $[\tau_{i-1}, \tau_i)$, an additional condition $H(x(\tau_i), u^*(\tau_i), p(\tau_i)) = 0$ has to be added. For details on the time-variant Maximum Principle see, e.g., (Pontryagin et al., 1963, Ch. 1).

State-dependent case This case is slightly more involved compared to the previous one. The problem is that the switching instants τ_i are defined from the solution of the switching condition $\gamma_{i,i+1} = x(\tau_i) - \tilde{x}_i = 0$ and, thus, not defined *a priori*. Looking at (16) one can notice that the composite PDF depends on τ_i which means that the functions $\bar{f}_i(t)$ in (18) are not well-defined. Therefore, the equations (18) are modified as shown below

$$\begin{aligned} &\dot{x} = g(x, u),\\ &\dot{x}_\sigma = x_\alpha f_i(t),\\ &\dot{x}_\alpha = 0, \qquad x_\alpha(\tau_0) = 1\\ &K_i(t_0, x, u) = \int_{\tau_{i-1}}^{\tau_i} (1 - x_\sigma(t)) h(x(t), u(t)) dt. \end{aligned} \tag{20}$$

with Hamiltonian functions modified accordingly

$$H_i(x_t, u, p_t) = p_0(1 - x_\sigma) h(x, u) + \langle p, g(x, u)\rangle + p_\sigma x_\alpha f_i(t).$$

The particularity of this model is that along with the mentioned switching condition $\gamma_{i,i+1} = x(t) - \tilde{x}_i$, there is a jump function associated with x_α. When a switching between discrete states occurs, the state changes discontinuously according to the jump function

$$\begin{aligned} [x(\tau_i), x_\sigma(\tau_i), x_\alpha(\tau_i)] &= \Phi_{i,i+1}(x, x_\sigma) =\\ &= \left[x(\tau_i^-), x_\sigma(\tau_i^-), \tfrac{x_\sigma(\tau_i^-) - 1}{F_{i+1}(\tau_i) - 1}\right]. \end{aligned}$$

The intermediate conditions (19) have to be rewritten to take into account the switching and jump functions:

$$\begin{aligned} &x(\tau_i) = x(\tau_i^-), \qquad p(\tau_i^-) = p(\tau_i) + \pi,\\ &x_\sigma(\tau_i) = x_\sigma(\tau_i^-), \quad p_\sigma(\tau_i^-) = \tfrac{p_\sigma(\tau_i)}{F_{i+1}(\tau_i) - 1},\\ &x_\alpha(\tau_i) = \tfrac{x_\sigma(\tau_i^-) - 1}{F_{i+1}(\tau_i) - 1},\ p_\alpha(\tau_i^-) = 0. \end{aligned}$$

Furthermore, the Hamiltonian function is not continuous any longer since the jump function is time-variant. The condition on the Hamiltonian at switching instants τ_i

is hence

$$H_{q_{i-1}}(\tau_i^-) = H_{q_i}(\tau_i) + p_\alpha(\tau_i)\frac{(x_\sigma(\tau_i)-1)f_{i+1}(\tau_i)}{(F_{i+1}(\tau_i)-1)^2}(t_i^-).$$

The end point conditions remain unchanged. With all conditions imposed, the optimization problem (20) becomes well-defined and can be solved using standard procedures as illustrated in the following section.

4. Example

To illustrate the presented approach we consider a simple example of finding a Pareto optimal solution in the game of resource extraction with N players and two operation modes. Note that despite its obvious simplicity, this example can demonstrare rather non-trivial behaviour.

The two CDFs are $F_1(t) = 1 - \exp(-\lambda_1 t)$ and $F_2(t) = 1 - \exp(-\lambda_2 t)$ with $\lambda_1, \lambda_2 > 0$ and the switching time τ. The resulting CCDF $F_\sigma(t)$ is defined as

$$F_\sigma(t) = \begin{cases} 1 - \exp(-\lambda_1 t), & t \in [0,\tau), \\ 1 - \frac{\exp(-\lambda_1\tau)}{\exp(-\lambda_2\tau)}\exp(-\lambda_2 t), & t \in [\tau,\infty). \end{cases} \tag{21}$$

We consider two exponential CDF with rate parameters $\lambda = 0.01$ and $\lambda = 0.1$. The corresponding CDFs are shown in Fig. 1.

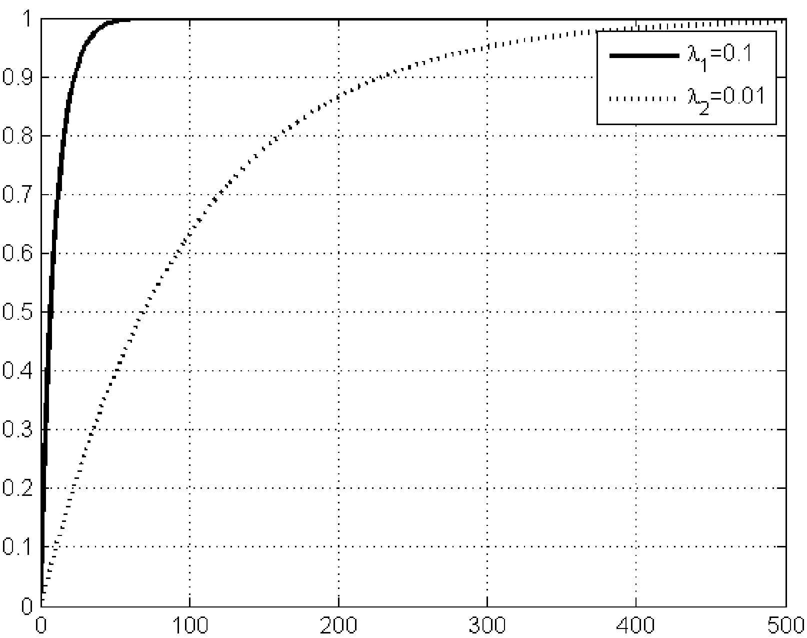

Fig. 1: Two exponential distributions

The system dynamics is described by a first order DE:

$$\dot{x}(t) = -\sum_{i=1}^{N} u_i(t), \quad x(0) = x_0,\ x(\infty) = 0,\ u_i(\cdot) \in [0, u_{max}], \tag{22}$$

where $u(t)$ is the rate of extraction. The initial amount of resource is set to $x(0)=100$ and $x(\infty)$ is routinely defined as $x(\infty) = \lim_{t\to\infty} x(t)$.

The instantaneous payoff function is chosen as $h_i(x(t), u(t)) = \ln(u_i(t))$. The optimal control problem is thus defined to be

$$\min \sum_{i=1}^{N} K_i(x,u) = -\int_0^{\infty} (1 - F_\sigma(s)) \sum_{i=1}^{N} \ln(u_i(s)) ds. \tag{23}$$

Before proceeding to the hybrid formulation, we present the solution to the optimal control problem (23) defined over a single interval. This result is of independent interest, as this class of optimization problems is fairly common for a wide range of resource extraction applications (see, e.g., Dockner et al., 2000).

4.1. Optimal Solution to a Single Mode Optimal Control Problem

Consider a more general version of the problem (22), (23) on the interval $[t_0, t_f] \subset [0, \infty) \cup \{\infty\}$ with the boundary conditions $x(t_0) = x_0$, $x(t_f) = x_f$, $x_0 > x_f$. Moreover, we assume that there is one single (non-composite) CDF $F(t)$ such that $F(t_f) = 1$. The Hamiltonian is written as

$$H = -\psi \sum_{i=1}^{N} u_i(t) + \psi_0 \left(1 - F(t)\right) \sum_{i=1}^{N} \ln(u_i(t)), \quad \psi_0 = 1.$$

The differential equation for the adjoint variable ψ is

$$\dot{\psi} = -\frac{\partial H}{\partial x} = 0,$$

whence we conclude that $\psi(t) = \psi^* = const$ for all t.

The optimal controls u_i^* are found from the first order extremality condition $\frac{\partial H}{\partial u_i} = 0$:

$$u_i^*(\psi, t) = \frac{1}{\psi^*}(1 - F(t)).$$

Moreover, u_i^* maximize H as follows from $\frac{\partial^2 H}{\partial u_i^2} = -\psi_0(1 - F(t))\frac{1}{u_i^2} < 0$.

The value of ψ^* is determined from the boundary condition $x(t_f) = x_f$. Solving (22) and taking into account this condition we find ψ^* as

$$\psi^* = \frac{N}{x_0 - x_f} \int_{t_0}^{t_f} (1 - F(t)) dt$$

and hence, the optimal controls take the following form:

$$u_i^*(t) = \frac{x_0 - x_f}{N \int_{t_0}^{t_f} (1 - F(\tau)) d\tau} (1 - F(t)). \tag{24}$$

The state $x(t)$ of the system (22) with the control (24) is

$$x^*(t) = x_0 - \int_{t_0}^{t} \frac{x_0 - x_f}{N \int_{t_0}^{t_f} (1 - F(\tau))d\tau} (1 - F(s))ds.$$

Note that the optimal control $u^*(t)$ exists if the integral in the denominator converges, i.e. $\int_{t_0}^{t_f} (1 - F(\tau))d\tau < \infty$ (which might not be the case if $t_f = \infty$). Taking into account the Bellman optimality principle, the optimal controls $u_i^*(t)$ can be expressed as functions of the current state:

$$u_i^*(t, x(t)) = \frac{x(t) - x_f}{N \int_{t}^{t_f} (1 - F(\tau))d\tau} (1 - F(t)). \tag{25}$$

Hence, from (9) and (25), the value function $V(t, x(t))$ is given by

$$V(t, x(t)) = -\frac{I(t)}{1 - F(t)} \ln\left(\frac{x(t) - x_f}{I(t)}\right) - \frac{1}{1 - F(t)} \int_{t}^{t_f} (1 - F(s)) \ln(1 - F(s))ds, \tag{26}$$

where $I(t) = N \int_{t}^{t_f} (1 - F(\tau))d\tau$.

Finally, in the framework of the resource extraction problem one may need to compute the expectation of the state $x(t)$ at the end of the exploration process:

$$\mathbf{E}(x(t)) = \int_{t_0}^{t_f} f(t) \left[x_0 - \int_{t_0}^{t} \frac{x_0 - x_f}{N \int_{t_0}^{t_f} (1 - F(\tau))d\tau} (1 - F(s))ds \right] dt =$$

$$= x_f + (x_0 - x_f) \frac{\int_{t_0}^{t_f} F(t)(1 - F(t))dt}{N \int_{t_0}^{t_f} (1 - F(\tau))d\tau}.$$

In the following, we will assume $N = 1$ to simplify the notation.

4.2. Time-Dependent Case

We assume that the switching time τ is fixed and equal to τ_s and the state at time τ_s is $x(\tau_s) = x_s$. Hence, the optimal control problem can be decomposed into two problems, on the intervals $I_1 = [0, \tau_s)$ and $I_2 = [\tau_s, \infty)$.

The optimal control on the first interval $[0, \tau)$ is

$$u^*(t) = \frac{(x_0 - x_s)}{\int_{0}^{\tau_s} (1 - F_\sigma(s))ds} (1 - F_\sigma(t)) = \frac{(x_0 - x_s)\lambda_1}{(1 - \exp(-\lambda_1 \tau_s))} \exp(-\lambda_1 t), \quad t \in I_1.$$

In the same way we define the optimal control on the second interval:

$$u^*(t) = \frac{x_s}{\int_{\tau_s}^{\infty} (1 - F_\sigma(s))ds} (1 - F_\sigma(t)) = \frac{x_s \lambda_2}{\exp(-\lambda_2 \tau_s)} \exp(-\lambda_2 t), \quad t \in I_2.$$

Both expressions contain the unknown switching state x_s. Solving the optimal control problem, x_s is found as a function of the switching time τ_s:

$$x_s = \frac{\lambda_1 x_0}{\lambda_2 \exp(\lambda_1 \tau_s) - (\lambda_2 - \lambda_1)}.$$

In Fig. 2, the dependence of the switching state x_s on the switching time τ_s is shown for two sequences of operation modes.

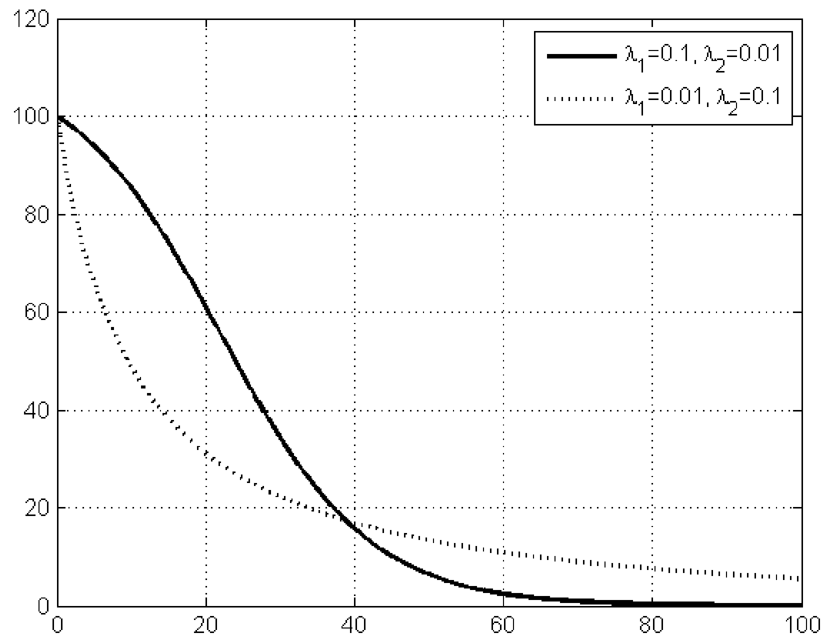

Fig. 2: Dependence of the optimal switching state $x_s^* = x(\tau_s)$ on the switching time τ_s. The continuous line corresponds to the case $\lambda_1 = 0.01, \ \lambda_2 = 0.1$, the dotted one – $\lambda_1 = 0.1, \ \lambda_2 = 0.01$

Informally, one can describe these two cases as the "safe" mode first ($\lambda_1 = 0.01, \ \lambda_2 = 0.1$), and the "dangerous" mode first ($\lambda_1 = 0.1, \ \lambda_2 = 0.01$). The second case is of particular interest. It turns out that for a small τ_s the optimal strategy is to preferrably extract during the "safe" mode. However, as τ_s grows, the risk that the system breaks down grows and so, the expected gain in the payoff is compensated by the risk of an abrupt interruption of the game. As τ_s reaches a certain value the optimal strategy becomes to extact as much as possible during the "dangerous" phase as there is only a slight hope that the process will "survive" until the switching time τ_s. Interesting to note that the switching time at which the optimal strategy changes is determined from the equation

$$\lambda_1^2 \left(\exp(\lambda_2 \tau_s) - 1\right) - \lambda_2^2 \left(\exp(\lambda_1 \tau_s) - 1\right) = 0.$$

4.3. State-Dependent Case

In the second case, we assume that the switching time τ is determined from the condition $x(\tau) = ax_0$, $a \in [0,1]$, where the parameter a describes the extent of exploration at which the regime changes (i.e., a switching occurs).

As in the previous case, the optimal control problem can be decomposed into two problems, on the intervals $[0,\tau)$ and $[\tau,\infty)$.

Now consider the first interval $[0,\tau)$. The optimal control is

$$u^*(t) = \frac{x_0 - ax_0}{\int_0^\tau (1 - F_\sigma(s))ds}(1 - F_\sigma(t)) = \frac{x_0(1-a)\lambda_1}{(1 - \exp(-\lambda_1 \tau))} \exp(-\lambda_1 t).$$

The optimal control on the second interval is

$$u^*(t) = \frac{ax_0}{\int_\tau^\infty (1 - F_\sigma(s))ds}(1 - F_\sigma(t)) = \frac{ax_0\lambda_2}{\exp(-\lambda_2 \tau)} \exp(-\lambda_2 t).$$

The remaining step is to determine the value of the optimal switching time τ^*, which is equal to

$$\tau^* = \frac{\ln\left(1 + \frac{\lambda_1 x_0(1-a)}{e^{\frac{\lambda_2 - \lambda_1 + \lambda_1 \ln(a\lambda_2 x_0)}{\lambda_2}}}\right)}{\lambda_1}.$$

The limit case $(\alpha = 0)$ looks as follows:

$$\tau_s = \frac{\ln\left(\frac{\lambda \, \mathrm{x0}}{\mathrm{e}} + 1\right)}{\lambda}$$

We compute the optimal switching time for different values of a and for the two different sequences of modes. We remind that the parameter a determines the state, and implicitly the time instant, at which the switching between two modes occur. The resulting dependencies are shown in Fig. 3, 4.

5. Conclusions

A new class of differential games with random duration and a composite cumulative distribution function has been introduced. It has been shown that these games can be well defined within the hybrid systems framework and that the problem of finding the optimal strategy can be posed and solved with the methods of hybrid optimal control theory. An illustrative example along with a qualitative analysis of the results have been presented.

The further work on the topic will be devoted to the analysis of the cooperative behaviour in this class of differential games. In particular, we will study the impact which the change of mode may have of the coalition agreement of the players.

Appendix

Proof of Proposition 1

Property **C1** is satisfied since it is satisfied for the function $F_1(t)$:

$$F_\sigma(t_0) = F_1(t_0) = 0$$

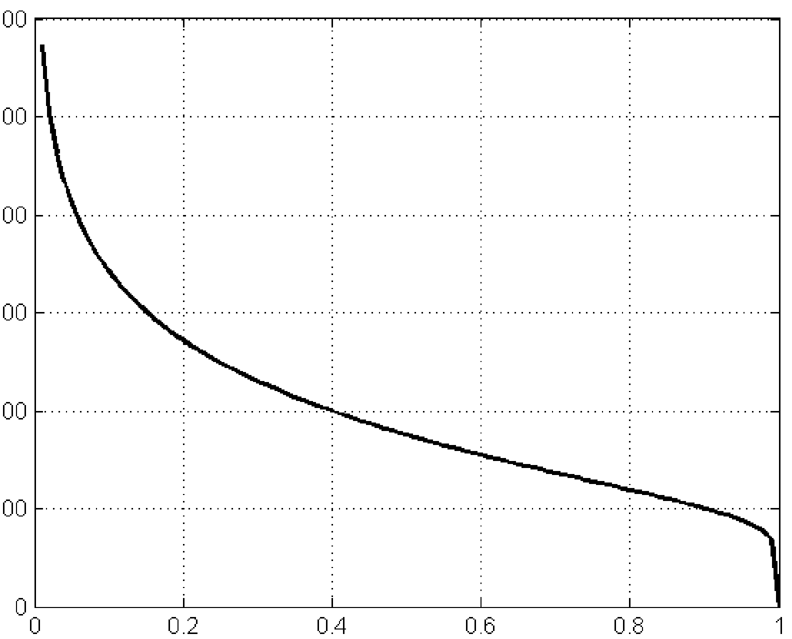

Fig. 3: Dependence of the switching time τ_s on the parameter a for $\lambda_1 = 0.1; \lambda_2 = 0.01$

Property **C2** follows from $\lim_{t\to\infty} F_N(t) = 1$ and from the definition of $\alpha_i(\tau_i)$ and $\beta_i(\tau_i)$:

$$\lim_{t\to\infty} F_\sigma(t) = \alpha_{N-1}(\tau_{N-1}) \lim_{t\to\infty} F_N(t) + \beta_{N-1}(\tau_{N-1}) = \\ = \frac{F_\sigma(\tau_{N-1}^-)-1}{F_N(\tau_{N-1})-1} \cdot 1 + 1 - \frac{F_\sigma(\tau_{N-1}^-)-1}{F_N(\tau_{N-1})-1} = 1,$$

where τ_{N-1} is a fixed switching time.

To show that Property **C3** holds true for F_σ, we first show that F_σ is continuous. This follows from the equality of left and right limits at $t = \tau_i$:

$$\lim_{t\to\tau_i+} F_\sigma = \lim_{t\to\tau_i+} \left(\frac{F_\sigma(\tau_i^-)-1}{F_{i+1}(\tau_i)-1} F_{i+1}(t) + 1 - \frac{F_\sigma(\tau_i^-)-1}{F_{i+1}(\tau_i)-1} \right) = \\ = \frac{F_\sigma(\tau_i^-)-1}{F_{i+1}(\tau_i)-1} F_{i+1}(\tau_i) + 1 - \frac{F_\sigma(\tau_i^-)-1}{F_{i+1}(\tau_i)-1} = F_\sigma(\tau_i^-) = \lim_{t\to\tau_i-} F_\sigma$$

Next, to demonstrate that the function $F_\sigma(t)$ is non-decreasing we consider two cases:

i) $t_1,\ t_2 \in [\tau_i, \tau_{i+1})$, $i = 0, \ldots, N-1$. Then, $F_\sigma(t_1) \leqslant F_\sigma(t_2)$ as $F_\sigma(t)$ is proportional to $F_{i+1}(t)$ on $[\tau_i, \tau_{i+1})$ and $F_{i+1}(t)$ is non-decreasing.

ii) $t_1 \in [\tau_i, \tau_{i+1})$, $t_2 \in [\tau_j, \tau_{j+1})$, $i, j = 0, \ldots, N-1$, $i < j$. Taking into account the continuity property, we have

$$F_\sigma(t_1) \leqslant F_\sigma(\tau_{i+1}) \leqslant \ldots \leqslant F_\sigma(\tau_j) \leqslant F_\sigma(t_2).$$

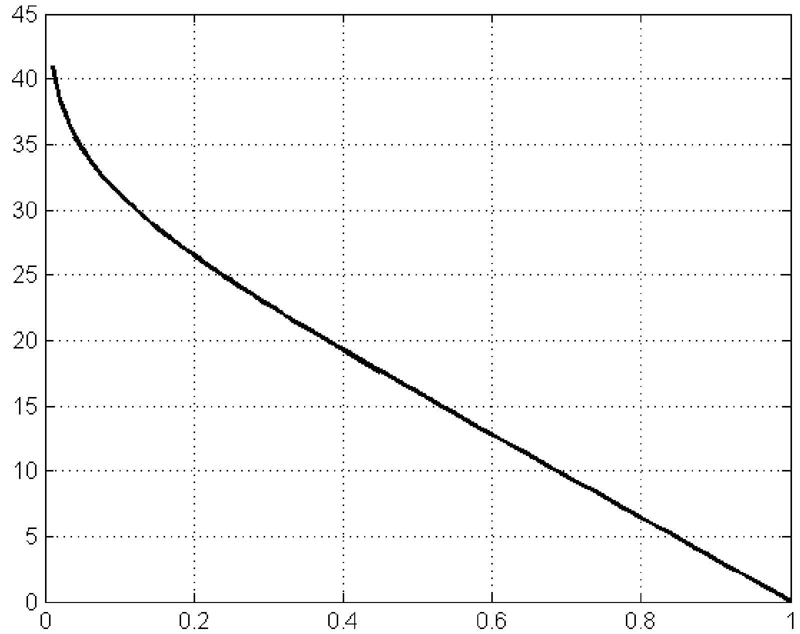

Fig. 4: Dependence of the switching time τ_s on the parameter a for $\lambda_1 = 0.01; \lambda_2 = 0.1$

Thus, the function $F_\sigma(t)$ is non-decreasing.

Finally, we show that $F_\sigma(t)$ is absolutely continuous. This is equivalent to the following requirement (Royden, 1988): $\forall\varepsilon > 0,\ \exists\delta > 0$ such that for any finite set of non-intersecting intervals (x_k, y_k) from $[t_0, \infty)$, the inequality $\sum |y_k - x_k| \leqslant \delta$ implies $\sum |F_\sigma(y_k) - F_\sigma(x_k)| \leqslant \varepsilon$.

We use the fact that the functions $F_i(t)$, $i = 1, \ldots, N$ are absolutely continuous. Then, for any $i = 1, \ldots, N$ and for any $\varepsilon_i = \frac{\varepsilon}{2N} > 0$, there exists $\delta_i > 0$ such that for any finite set of non-intersecting intervals $(x_{(i,k)}, y_{(i,k)})$ from $[\tau_{i-1}, \tau_i]$, satisfying $\sum\limits_k |y_{(i,k)} - x_{(i,k)}| \leqslant \delta_i$, holds

$$\sum_k |F_i(y_{(i,k)}) - F_i(x_{(i,k)})| \leqslant \varepsilon_i. \tag{27}$$

Let $\delta = \min(\delta_i, (\tau_j - \tau_{j-1}))$, $i, j = 1, \ldots, N$. For an arbitrary finite set of non-intersecting intervals (x_k, y_k), satisfying $\sum\limits_k |y_k - x_k| \leqslant \delta$ there are two possible variants:

i) Intervals (x_k, y_k) are proper subsets of the partition intervals $[\tau_i, \tau_{i+1}]$. Then, using the absolute continuity property of F_i and summing over all partition intervals we get

$$\sum_k |F_\sigma(y_k) - F_\sigma(x_k)| = \sum_{i=1}^N \sum_k |F_i(y_k) - F_i(x_k)| < N\varepsilon_i = \varepsilon,$$

whereas the following convention is employed: $|F_i(a) - F_i(b)| = 0$, if $(a,b) \cap [\tau_i, \tau_{i+1}] = \emptyset$.

ii) Some intervals of the finite set (x_k, y_k) may include switching instants $\tau_{i(k)}$. According to the definition of δ, an interval (x_k, y_k) can intersect with at most two partition intervals. Therefore, one can represent (x_k, y_k) as a union of two intervals $(x_k, \tau_{i(k)}) \subset (\tau_{i(k)-1}, \tau_{i(k)})$, and $(\tau_{i(k)}, y_k) \subset (\tau_{i(k)}, \tau_{i(k)+1})$. In this way, we can subdivide the sum $\sum |y_k - x_k|$ into two: $\sum |y_k - x_k| = \sum |y_k - \tau_{i(k)}| + \sum |x_k - \tau_{i(k)}| < \delta$. Summing over all partition intervals and using the triangle inequality we get

$$\begin{aligned}
&\sum_k |F_\sigma(y_k) - F_\sigma(x_k)| = \sum_{i=1}^{N} \sum_k |F_i(y_k) - F_i(x_k)| \leqslant \\
&\leqslant \sum_{i=1}^{N} \sum_k \big(|F_i(y_k) - F_i(\tau_{i(k)})| + |F_i(\tau_{i(k)}) - F_i(x_k)|\big) < \\
&< 2N\varepsilon_i = \varepsilon,
\end{aligned}$$

where we make use of the same convention as in item *i)*.

The condition (27) is met and thus, F_σ is absolutely continuous. This concludes the proof.

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