# A Problem of Purpose Resource Use in Two-Level Control Systems ⋆

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**Abstract** The system including two level players–top and bottom–is considered in the paper. Each of the players have public (purpose) and private (non-purpose) interests. Both players take part of payoff from purpose resource use. The model of resource allocation among the purpose and non-purpose using is investigated for different payoff function classes and for three public gain distribution types. A problem is presented in the form of hierarchical game where the Stackelberg equilibrium is found.

**Keywords:** resource allocation, two-level control system, purpose use, non-purpose resource use, Stackelberg equilibrium.

## 1. Introduction

A wide set of social and economic development problems is solved due to budget financing, which is performed in different forms (grants, subventions, assignments, credits) and always has a strictly purpose character, i.e., allocated funds should be spent only on prescribed needs. Article 289 of the Budget Code of the Russian Federation and Article 15.14 RF Code on Administrative Offences make provisions on responsibility for non-purpose use of budget funds. Nevertheless, non-purpose use of budget financing is widespread and can be considered as a kind of opportunistic behavior corresponding to the private interests of the active agents (Willamson, 1981). Non-purpose use of resources is linked to corruption, especially to "kickbacks", when budget funds are allocated to an agent in exchange for a bribe and only partially used appropriately. They are largely spent on private agent-briber interests.

It is naturally for the resource use problem to be treated in terms of the interest concordance in hierarchical control systems. This allows for a mathematical apparatus of hierarchical game theory (Basar, 1999), of contract theory (Laffont, 2002), information theory of hierarchical systems (Gorelik, 1991), active system theory (Novikov, 2013a) and organizational system theory (Novikov, 2013b). Simultaneously, resource allocation models in hierarchical systems with regard to their misuse are little studied (Germeyer, 1974) and are analyzed in authors' investigation line (Gorbaneva and Ougolnitsky, 2009-2013).

This article is focused on the question how resource allocation among purpose and non-purpose directions is depended on different public and private payoff function classes of distributor and resource recipients.

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## 2. Structure of investigation

We consider a two-level control system which consists of one top level element $A_1$ (resource distributor) and one bottom level element $A_2$ (resource recipient). The top level has some resource amount (which we assume to be a unit). The distributor assigns a part of resources to the recipient for purpose use, and the rest for his own interests. The bottom level assigns in his turn a part of obtained resources for his own interests (non-purpose use), and the rest for the public interests (purpose use). Both levels take part in purpose activity profit and have their payoff functions (Fig. 1).

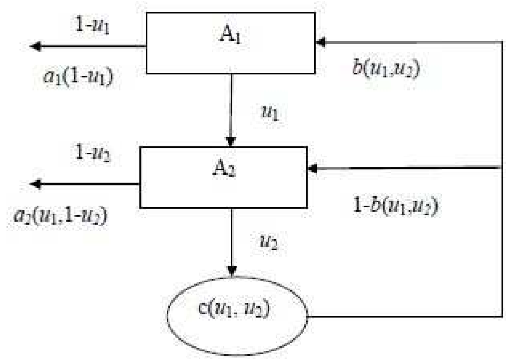


Fig. 1: The structure of modeled system.

The model is built as a hierarchical two-person game in which a Stackelberg equilibrium is sought (Basar, 1999). A payoff function of each player consists of two summands: non-purpose activity profit and a part of the system purpose activity profit. The payoff functions are:

$$g_1(u_1, u_2) = a_1(1 - u_1, u_2) + b(u_1, u_2) \cdot c(u_1, u_2) \to \max_{u_1};$$
$$g_2(u_1, u_2) = a_2(u_1, 1 - u_2) + b(u_1, u_2) \cdot c(u_1, u_2) \to \max_{u_2}.$$

subject to

$$0 \leq u_i \leq 1, i = 1, 2,$$

and conditions on functions a, b and c

$$a_i \geq 0; \frac{\partial a_i}{\partial u_i} \leq 0, \frac{\partial a_i}{\partial u_{j\neq i}} \geq 0, i = 1, 2,$$
$$b_i \geq 0; \frac{\partial b_i}{\partial u_i} \geq 0, i = 1, 2,$$
$$\frac{\partial c}{\partial u_i} \geq 0, i = 1, 2.$$

Here index 1 relates to the top level attributes (a leading player), index 2 relates to the bottom level attributes (a following player);

- $u_i$ is a share of resources assigned by $i$-th level to the purpose use (correspondingly,

$1 - u_i$ remains on non-purpose resource use in private interests);
- $g_i$ is a payoff function of i-th level;
- $a_i$ is a payoff function of i-th level private interest;
- $b_i$ is a share of purpose activity profit obtained by i-th level;
- $c$ is a payoff function of purpose system activity (society, organization).

Power, linear, exponential and logarithmic functions are considered as functions $a$ and $c$. These functions depend on variables $u_1$, $u_2$ , and they are cumulative ones, i.e. $a_1 = a_1(1 - u_1)$, $a_1 = a_2(u_1(1 - u_2))$, $c = c(u_1u_2)$. In this case a share of resources being is assigned to the public aims.

The relations $a_1 = a_1(1-u_1)$, $a_1 = a_2(u_1(1-u_2))$, reflect the hierarchical structure of the system. The non-purpose activity income of top level does not depend on the part of the funds the bottom level assigned for the public aims but the non-purpose activity income of bottom level depends on the part of the funds the top level gives him.

Three income types of purpose income distribution $b$ are considered:

1) uniform one, in which the shares in purpose activity income are the same for both players, in particular, if $n = 2$

$$b_i = \frac{1}{2}, i = 1, 2,$$

2) proportional one, in which the shares in income are proportional to the shares assigned to the public aims by the corresponding level, i.e.

$$b_1 = \frac{u_1}{u_1 + u_2},$$
$$b_2 = \frac{u_2}{u_1 + u_2};$$

3) constant one, in which:

$$b_1 = b,$$
$$b_2 = 1 - b;$$

The player strategy is a share $u_i$ of available resources assigned to the public aims. The top-level player $u_1$defines and informs the bottom level about it. Then the second player chooses the optimal value $u_2$ knowing the strategy of the first player. The investigation aim is to study how the relation of functions $a_1$, $a_2$, $b_1$, $b_1$, $c$ effects on the game solution (Stackelberg equilibrium).

The next functions are taken as a non-purpose payoff function:
- power with an exponent less than one ($a(x) = ax^{\alpha}$, $\alpha < 1$, $a > 0$),
- linear ($a(x) = ax$, a particular case of power function with an exponent equaled to one),
- power with an exponent greater than one, ($a(x) = ax^k$, $k > 1$, $a > 0$);
- exponential ($a(x) = a(1 - \exp -\lambda x)$, $\lambda > 0$, $a > 0$);
- logarithmic ($a(x) = alog(1 + x)$, $a > 0$).

As a rule, functions are chosen with constraints $\frac{\partial a}{\partial x} \geq 0$, $\frac{\partial^2 a}{\partial x^2} \leq 0$. The first condition is satisfied by all functions, the second condition is not satisfied only by function $a(x) = ax^k$, $k > 1$. The first and the second functions are production functions. The last two functions are not production ones since the property of scaling production returns does not hold.

Similarly, the next functions are taken as a purpose payoff function:
- power with an exponent less than one ($c(x) = cx^{\alpha}$, $\alpha < 1$, $c > 0$);
- linear ($c(x) = cx$);
- power with an exponent greater than one, ($c(x) = cx^k$, $k > 1$, $c > 0$);
- exponential ($c(x) = c(1 - \exp -\lambda x)$, $\lambda > 0$, $c > 0$);
- logarithmic ($c(x) = clog(1 + x)$, $c > 0$).

Thirteen of twenty five possible combinations are solved analytically:
1) combinations of similar functions, when $a$ and $c$ are either power, or exponential, or logarithmic ones;
2) combinations of any non-purpose use function and linear purpose use function;
3) combinations of linear non-purpose use function and any purpose use function.

Six of the rest cases are investigated numerically.

## 3. Analytical investigation of different model classes

We consider the case when $a_1(u_1, u_2) = 1-u_1$, $a_2(u_1, u_2) = u_1(1-u_2)$ ,$c2(u_1, u_2) = u_1u_2$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$ .
Then:

$$g_1(u_1, u_2) = 1 - u_1 + \frac{u_1u_2}{2} \to \max_{u_1}, \tag{1}$$

$$g_2(u_1, u_2) = u_1 - \frac{u_1u_2}{2} \to \max_{u_2} \tag{2}$$

This is the game with constant sum. Function $g_2$ decreases in $u_2$, therefore the optimal value ${u_2}^* = 0$, at which $g_1(u_1, 0) = 1 - u_i$. Function $g_1$ decreases on $u_1$, therefore the value ${u_1}^* = 0$ is optimal.

So, Stackelberg equilibrium in the game is $ST_1 = \{(0; 0)\}$ , while the player gains are $g_1 = a_1$, $g_2 = 0$ , i.e. both players use strategy of egoism (assign all available resources for private aims), but the top level gets maximum, while the bottom level gets zero.

Consider the case when $a_1 = a_1(1 - u_1)$, $a_2 = a_2u_1(1 - u_2)$, $c = (u_1u_2)^k$ is the production power function.

There may be two fundamentally different cases:
1) $k = 1$ (linear resource use function);

Then, $g_1(u_1, u_2) = a_1(1-u_1)+b_1u_1u_2$, $g_2(u_1, u_2) = a_2u_1(1-u_2)+b_2u_1u_2$. We find optimal strategy of the bottom level:

$$\frac{\partial g_2}{\partial u_2} = (b_2 - a_2)u_1,$$

$$u_2^* = \begin{cases} 1, b_2 > a_2, \\ 0, b_2 < a_2. \end{cases}$$

The top level optimizes his gain function:

$$g_1(u_1, u_2^*) = \begin{cases} a_1(1 - u_1) + b_1u_1u_2, & b_2 > a_2, \\ a_1(1 - u_1), & b_2 < a_2. \end{cases}$$

$$\frac{\partial g_1}{\partial u_1} = \begin{cases} b_1 - a_1, & b_2 > a_2, \\ -a_1, & b_2 < a_2. \end{cases}$$

Thus, (Fig. 2),

$$u_1^* = \begin{cases} 1, (b_2 > a_2) \wedge (b_1 > a_1), \\ 0, (b_2 < a_2) \vee (b_1 < a_1). \end{cases}$$

If $b_2 > a_2$ and $b_1 > a_1$ then both players apply altruistic strategy ($u_1{}^* = u_2{}^* = 1$), and $g_1 = b_1$, $g_2 = b_2$. In other cases the leading player behaves egoistically ($u_1{}^* = 0$), then $g_1 = a_1$, $g_2 = 0$.

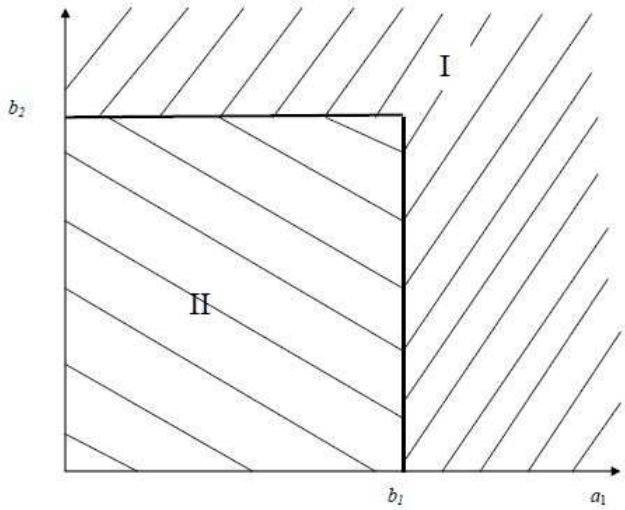


Fig. 2: Game outcomes (3.1)-(3.2).

2) $0 < k < 1$ (power resource use function).
Then, $g_1(u_1, u_2) = a_1(1 - u_1) + b_1(u_1u_2)^k$, $g_2(u_1, u_2) = a_1u_1(1 - u_2) + b_2(u_1u_2)^k$. We find the bottom level optimal strategy:

$$\frac{\partial g_2}{\partial u_2} = -a_2u_1 + kb_2(u_1u_2)^{k-1} = 0,$$

$$u_2{}^* = \frac{(\frac{a_2}{kb_2})^{\frac{1}{k-1}}}{u_1}.$$

The top level optimizes his payoff function:

$$g_1(u_1, u_2{}^*) = b_1(\frac{a_2}{kb_2})^{\frac{k}{k-1}} + a_1(1 - u_1).$$

Since function $g_1$ decreases on $u_1$, then $u_1{}^* = 0$.
We consider the case when the payoff function from non-purpose activity is linear, the payoff function from purpose activity is logarithmic, and a share of the purpose activity profit is constant for both levels:

$$a_1(u_1, u_2) = a_1(1 - u_1), a_2 = a_2u_1(1 - u_2),$$
$$c = c\log_2(1 + u_1u_2), b_1 = b, b_2 = 1 - b.$$

Then gain functions are

$$g_1(u_1,u_2)=a_1(1-u_1)+bc\log_2(1+u_1u_2)\to\max_{u_1}, \tag{3}$$

$$g_2(u_1,u_2)=a_2u_1(1-u_2)+(1-b)c\log_2(1+u_1u_2)\to\max_{u_2}, \tag{4}$$

subject to

$$0\le u_i\le 1, i=1,2.$$

Find the Stackelberg equilibrium. We divide this process into two phases and describe in detail now.
1) First, we solve a bottom level optimization problem. Suppose the value $u_1$ is known. We find the derivative of $g_2$ with respect to $u_2$ and equate it to zero:

$$\frac{\partial g_2}{\partial u_2}(u_1,u_2)=-a_2u_1+\frac{(1-b)cu_1}{(1+u_1u_2)\ln 2}=0.$$

We solve the equation. The case $u_1=0$ has no practical interest, therefore we can divide both parts of equation by $u_1$ and express $u_2$: ${u_2}^*=\frac{1}{u_1}\left(\frac{(1-b)c}{a_2\ln 2}-1\right)$. Finding the second derivative of the function $g_2$ with respect to $u_2$, we see that the point ${u_2}^*$ is a maximum point:

$$\frac{\partial^2 g_2}{\partial {u_2}^2}(u_1,u_2)=-\frac{(1-b)c{u_1}^2}{(1+u_1u_2)^2\ln 2}<0.$$

Taking into account the restriction on $u_2$, note that the optimal strategy of the bottom level player is

$$u_2^*=\begin{cases}0, & a_2\ln 2\ge(1-b)c,\\ \frac{1}{u_1}\left(\frac{(1-b)c}{a_2\ln 2}-1\right), & 0<\frac{1}{u_1}\left(\frac{(1-b)c}{a_2\ln 2}-1\right)<1,\\ 1, & \frac{(1-b)c}{a_2\ln 2}\ge 1+u_1,\end{cases}$$

2) Solve a top level problem if the bottom level answer is known. Consider three cases: a) ${u_2}^*=0$ .
In this case $g_1(u_1,0)=a_1(1-u_1)+bc\log_2 1=a_1(1-u_1)$. Since $g_1$ decreases in $u_1$, the top level optimal strategy is ${u_1}^*=0$, i.e. if top level knows that the bottom level assigns all available resources for the private aims, then he gives no resources to the bottom level and assigns the resources for his private aims.
b) ${u_2}^*=\frac{1}{u_1}\left(\frac{(1-b)c}{a_2\ln 2}-1\right)$.
Then, $g_1(u_1,{u_2}^*)=a_1(1-u_1)+bc\log_2\frac{(1-b)c}{a_2\ln 2}$.
Here, similar to the previous case, the function $g_1$ decreases with respect to $u_1$. Note that the bottom level chooses his strategy so that the constant value of resources is assigned for the public aims. Hence, the more resource is given to the bottom level by the top one, the more may be spent on the bottom level private aims (as the difference between resources, which were given by the top level, and constant value $u_1u_2=\frac{(1-b)c}{a_2\ln 2}-1$, which were assigned for the public aims by the bottom level). And conversely, the less resource is given to the bottom level by the top one, the less may be spent on bottom level private aims. Hence, taking into account

the decreasing of function in $u_1$, it is profitable for the top level to assign as little as possible resource for the public aims, hence the bottom level assigns as little as possible for the public aims. So, it is profitable for the bottom level to assign for the public aims as much resources as the bottom level assigns for the public aims, namely ${u_1}^* = \frac{(1-b)c}{a_2 \ln 2} - 1$, thereby causing the lower level to spend all the resources on public aims, i.e. $u_2 = 1$.
c) ${u_2}^* = 1$.
In this case $g_1(u_1, 1) = a_1(1 - u_1) + bc \log_2(1 + u_1)$. Maximize this function taking into account the restriction $0 \le u_1 \le 1$.
From the first order conditions

$$\frac{\partial g_1}{\partial u_1}(u_1, 1) = -a_1 + \frac{bc}{(1 + u_1) \ln 2} = 0.$$

we obtain:

$${u_1}^* = \frac{bc}{a_1 \ln 2} - 1.$$

Finding the second derivative of $g_1$ with respect to $u_1$, we can see that the point ${u_1}^*$ is a point of maximum:

$$\frac{\partial^2 g_1}{\partial {u_1}^2}(u_1, {u_2}^*(u_1)) = -\frac{bc}{(1 + u_1)^2 \ln 2} < 0.$$

Taking into account the restriction on u1, the optimal strategy of the bottom level is

$$u_1^* = \begin{cases} 0, & a_1 \ln 2 \ge bc, \\ \frac{bc}{a_1 \ln 2} - 1, & 0 < \frac{bc}{a_1 \ln 2} - 1 < 1, \\ 1, & \frac{bc}{a_1 \ln 2} - 1 \ge u_1, \end{cases}$$

So, the Stackelberg equilibrium is

$$\bar{u} = \begin{cases} (0;0), & \left(a_2 > \frac{(1-b)c}{\ln 2}\right) \textit{ or } \left(a_1 > \frac{bc}{\ln 2}\right), \\ (1;1), & \left(a_2 < \frac{(1-b)c}{2\ln 2}\right) \textit{ and } \left(a_1 < \frac{bc}{2\ln 2}\right), \\ \left(\frac{bc}{a_1 \ln 2} - 1; 1\right), & \left(a_2 < \frac{a_1(1-b)}{b}\right) \textit{ and } \left(\frac{bc}{2\ln 2} < a_1 < \frac{bc}{\ln 2}\right), \\ \left(\frac{(1-b)c}{a_2 \ln 2} - 1; 1\right), & \left(a_2 > \frac{a_1(1-b)}{b}\right) \textit{ and } \left(\frac{(1-b)c}{2\ln 2} < a_2 < \frac{(1-b)c}{\ln 2}\right). \end{cases}$$

As can be seen from this formula, if assigning of some resource part for the public aims is profitable for the bottom level then the top level can enforce the bottom level to assign all the resources for the public aims. I.e., the bottom level assigns all the resources either only for public aims or only for private aims.
Consider each branch of the Stackelberg equilibrium:
I. $u = (0; 0)$ if $a_2 > \frac{(1-b)c}{\ln 2}$ or $a_1 > \frac{bc}{\ln 2}$ (Fig.2). In this case for one or two of the players the private activity gives much more profit than the public activity. It is not profitable for this player to assign the resources for the public aims, but then another player either has no incentive to assign resources to the public aims (for the top level) or has no resources (for the bottom level). The players' gains are

$$g_1 = a_1, g_2 = 0.$$

II. $u = (1;1)$ if $a_2 < \frac{(1-b)c}{2\ln 2}$ and $a_1 < \frac{bc}{2\ln 2}$ (Fig.3). In this case for both players the public activity gives much more profit than the private activity, therefore each of them assigns all the resources for the public aims. The players' gains are

$$g_1 = bc, g_2 = (1-b)c.$$

III. $u = (\frac{bc}{a_1 \ln 2} - 1; 1)$ if $a_2 < \frac{a_1(1-b)}{b}$ and $\frac{bc}{2\ln 2} < a_1 < \frac{bc}{\ln 2}$ (Fig.3). In this case for the top level it is profitable to assign only a part of resources for the public aims (since the both activities profits are comparable) while for the bottom level it is profitable to assign all the resources for the public aims. The players' gains are

$$g_1 = 2a_1 - \frac{bc}{\ln 2} + bc \log_2\left(\frac{bc}{a_1 \ln 2}\right), g_2 = (1-b)c\log_2\left(\frac{bc}{a_1 \ln 2}\right).$$

IV. $u = (\frac{(1-b)c}{a_2 \ln 2} - 1; 1)$ if $a_2 > \frac{a_1(1-b)}{b}$ and $\frac{(1-b)c}{2\ln 2} < a_2 < \frac{(1-b)c}{\ln 2}$ (Fig.3). In this case for both players it is profitable to assign a part of the resources for the public aims, since the both activities profits are comparable. The bottom level is going to assign a fixed value of resources for the public aims and to leave the rest for the private aims. But the top level gives only this fixed value of resources to the bottom level thereby he enforces the bottom level to assign all the resources for the public aims. The players' payoffs are

$$g_1 = 2a_1 - \frac{a_1(1-b)c}{a_2 \ln 2} + bc\log_2\left(\frac{(1-b)c}{a_2 \ln 2}\right), g_2 = (1-b)c\log_2\left(\frac{(1-b)c}{a_2 \ln 2}\right).$$

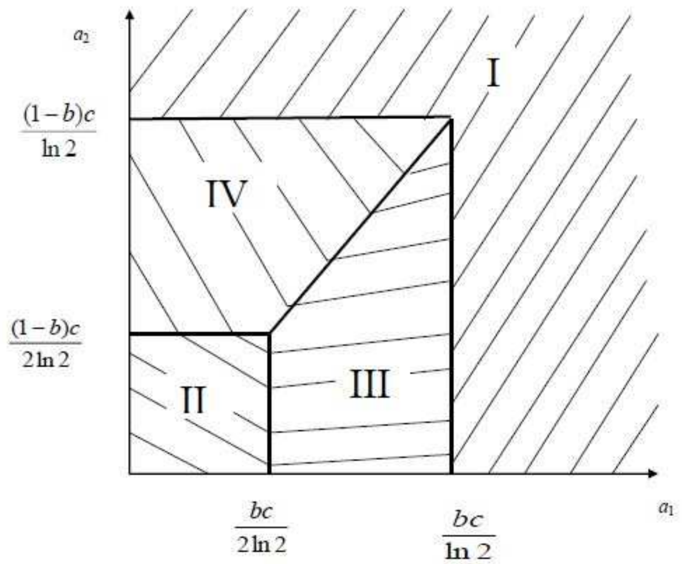


Fig. 3: Game outcomes (3.3)-(3.4)

Finally, we consider the case when purpose and non-purpose activity functions are power with an exponent less than one:

$$a_1 = a_1(1-u_1)^{\alpha}, a_2 = a_2\left(u_1(1-u_2)\right)^{\alpha},$$
$$c = (u_1 u_2)^{\alpha}, b_1 = b, b_2 = 1-b.$$

Then gain functions are

$$g_1(u_1, u_2) = a_1(1 - u_1)^\alpha + bc(u_1 u_2)^\alpha \to \max_{u_1}, \tag{5}$$

$$g_2(u_1, u_2) = a_2 \left(u_1(1 - u_2)\right)^\alpha + (1 - b)c(u_1 u_2)^\alpha \to \max_{u_2}, \tag{6}$$

The Stackelberg equilibrium is (Fig. 4):

$$\bar{u} = \left( \frac{\sqrt[1-\alpha]{b(1-b)^{\frac{\alpha}{1-\alpha}} c^{\frac{1}{1-\alpha}}}}{\sqrt[1-\alpha]{a_1\left(\sqrt[1-\alpha]{(1-b)c} + \sqrt[1-\alpha]{a_2}\right)^\alpha} + \sqrt[1-\alpha]{b(1-b)^{\frac{\alpha}{1-\alpha}} c^{\frac{1}{1-\alpha}}}} ; \frac{\sqrt[1-\alpha]{(1-b)c}}{\sqrt[1-\alpha]{(1-b)c} + \sqrt[1-\alpha]{a_2}} \right)$$

We omit the players' payoffs in this case.

All the thirteen considered cases can be grouped together on the number of outcomes of the game:

1) One outcome, when public and private payoff functions are power with an exponent less than one. In this case for both players it is profitable to assign a part of resources for the public aims, and another part for the private aims.

2) Two outcomes (0; 0) and (1;1) (Fig. 2), when:

a. The private payoff function is power with an exponent less than one and the public payoff function is linear;

b. The public and private payoff functions are either linear or power with an exponent greater than one in any combinations.

3) Three outcomes, when private activity function is linear and public payoff function is power with an exponent less than one. In this case for one of the player it is profitable to assign all the resources for the public aims.

4) Four outcomes (Fig. 3), when one of the functions (either private or public payoff) is linear and another function is logarithmic.

5) Five outcomes (Fig. 4), when a. Public and private payoff functions are linear or exponential in any combinations except the case when both the functions are linear. b. Public and private payoff functions are logarithmic.

## 4. Numerical investigation of different model classes

We use a numerical investigation for a few cases that could not be solved analytically. At first we consider a case when the purpose activity function is exponential and the non-purpose activity function is power with an exponent less than one, purpose activity profit share is constant for the players:

$$a_1 = a_1(1 - u_1)^\alpha, a_2 = a_2 \left(u_1(1 - u_2)\right)^\alpha,$$
$$c = c(1 - e^{-\lambda u_1 u_2}), b_1 = b, b_2 = 1 - b.$$

In this case the payoff functions are

$$g_1(u_1, u_2) = a_1(1 - u_1)^\alpha + bc(1 - e^{-\lambda u_1 u_2}) \to \max_{u_1}, \tag{7}$$

$$g_2(u_1, u_2) = a_2 \left(u_1(1 - u_2)\right)^\alpha + (1 - b)c(1 - e^{-\lambda u_1 u_2}) \to \max_{u_2}, \tag{8}$$

subject to

$$0 \le u_i \le 1, i = 1, 2.$$

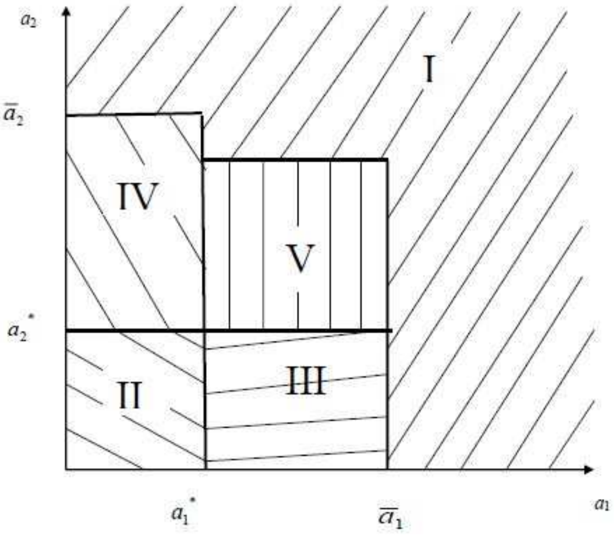


Fig. 4: One of the possible cases of the considered game with five outcomes

To find the bottom level optimal strategy we calculate the derivative of $g_2$ with respect to $u_2$ and equate it to zero:

$$\frac{\partial g_2}{\partial u_2}(u_1, u_2) = -\frac{a_2 \alpha {u_1}^{\alpha}}{(1-u_2)^{1-\alpha}} + \lambda u_1 (1-b) c e^{-\lambda u_1 u_2} = 0. \tag{9}$$

Prove that the bisection method may be applied for solving this equation. Note that the second derivative of $g_2$ with respect to $u_2$ is negative,

$$\frac{\partial^2 g_2}{\partial {u_2}^2}(u_1, u_2) = \frac{a_2 \alpha (1-\alpha) {u_1}^{\alpha}}{(1-u_2)^{2-\alpha}} - \lambda^2 u_1^2 (1-b) c e^{-\lambda u_1 u_2} < 0,$$

therefore, the function $\frac{\partial g_2}{\partial u_2}(u_1, u_2)$ is monotone.
Then find signs of $\frac{\partial g_2}{\partial u_2}(u_1, u_2)$ at the endpoints of [0,1].

$$\frac{\partial g_2}{\partial u_2}(u_1, 0) = -a_2 \alpha {u_1}^{\alpha} + \lambda u_1 (1-b) c, \tag{10}$$

$$\frac{\partial g_2}{\partial u_2}(u_1, u_2) \to_{u_2 \to 1_-} -\frac{a_2 \alpha {u_1}^{\alpha}}{0_+} + \lambda u_1 (1-b) c e^{-\lambda u_1 u_2} \to_{u_2 \to 1_-} -\infty. \tag{11}$$

If (10) is positive, then the equation may be solved by the bisection method, and the solution obtained is a maximum point since the second derivative is negative. If (10) is negative, then bisection method is not applied, but the left part of equation is monotone then it is negative at the segment [0, 1], hence, function $g_2$ decreases, then the maximum point is $u_2 = 0$.
That is,

$$u_2^* = \begin{cases} 0, & -a_2 \alpha {u_1}^{\alpha} + \lambda u_1 (1-b) c < 0, \\ \in (0;1), & -a_2 \alpha {u_1}^{\alpha} + \lambda u_1 (1-b) c > 0, \end{cases}$$

The top level can use this information to enforce the bottom level to choose non-zero strategy. For the bottom level to choose the positive strategy $u_2 > 0$, it is necessary to satisfy the condition $-a_2 \alpha {u_1}^{\alpha} + \lambda u_1 (1-b) c > 0$. When the inequality have been solved for the variable $u_1$, we obtain $u_1 > \sqrt[1-\alpha]{\frac{a_2\alpha}{\lambda(1-b)c}}$

For the bottom level not to spend all the resources on private aims, it is recommended for the top level to choose the strategy $u_1 > \sqrt[1-\alpha]{\frac{a_2\alpha}{\lambda(1-b)c}}$. But he can do it only if $\sqrt[1-\alpha]{\frac{a_2\alpha}{\lambda(1-b)c}} < 1$, which is equivalent to $a_2 < \left(\frac{a_2\alpha}{\lambda(1-b)c}\right)^{1-\alpha}$.

If the top level cannot use this strategy or this strategy is not profitable for him then the bottom level choose the strategy $u_2 = 0$. Find then the optimal top level behavior and his payoff

$$g_1(u_1, 0) = a_1(1 - u_1)^{\alpha}.$$

As can be seen, the function $g_1$ decreases in $u_1$, therefore, $u_1 = 0$.

Draw some conclusions:

I. If $a_2 > \left(\frac{a_2\alpha}{\lambda(1-b)c}\right)^{1-\alpha}$ then the top level cannot effect on the bottom one, in this case $u_2 = 0$, and therefore $u_1 = 0$. This occurs when the capacity of the bottom level of non-purpose activity is significantly more than production capacity of purpose activity.

II. If $a_2 < \left(\frac{a_2\alpha}{\lambda(1-b)c}\right)^{1-\alpha}$ then the top level can enforce the bottom level to spend some part of resources on the public aims assigning $u_1 > \sqrt[1-\alpha]{\frac{a_2\alpha}{\lambda(1-b)c}}$. This occurs when the capacity of the bottom level of purpose activity is significantly more than production capacity of non-purpose activity.

## 5. Conclusion

In this paper a problem of non-purpose resource use is treated in terms of analysis of control mechanism properties providing the concordance of interests in hierarchical (two-level) control systems. The interests of players are described by their payoff functions including two summands: purpose and non-purpose resource use profits. Different classes of these functions are considered. The top level subject (resource distributor) is treated as a leading player and the bottom level (resource recipient) subject is treated as a following player. This leads to the Stackelberg equilibrium concept. Performed analytical and numerical investigation permits to make the next conclusions.

In the case when the payoff functions for purpose and non-purpose activities are power with an exponent less than one it is profitable to assign only a part of resources for the public aims and another part of them for the private aims for both players.

In the case when one of the payoff functions for purpose or non-purpose activities is power with an exponent greater than one and another of them is either linear or power with an exponent greater than one it is profitable to assign all the resources for only public aims ("egoism" strategy) or for only private aims ("altruism" strategy).

In other cases the next situations may occur:

A) if the effect of the private activities of a player is much more than effect of the public activity then for a player the "egoism" strategy is profitable;

B) if the effect of the private activities of a player is much less than effect of the public activity then for a player the "altruism" strategy is profitable;

C) if the effects of the private and public activities of a player are comparable then for any player it is profitable to assign only a part of resources for the public aims and the other part for the private aims.

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