On Uniqueness of Coalitional Equilibria

Michael Finus,[1] **Pierre von Mouche**[2] **and Bianca Rundshagen**[3]

[1] *University of Bath, Department of Economics,*
Bath BA2 7AY, United Kingdom
Email: m.finus@bath.ac.uk
[2] *Wageningen Universiteit,*
Hollandseweg 1, 6700 EW, Wageningen, The Netherlands
E-mail: pvmouche@yahoo.fr
[3] *Universität Hagen, Department of Economics,*
Universitätsstrasse 11, 58097 Hagen, Germany
E-mail: bianca.rundshagen@fernuni-hagen.de

Abstract For the so-called 'new approach' of coalition formation it is important that coalitional equilibria are unique. Uniqueness comes down to existence and to semi-uniqueness, i.e. there exists at most one equilibrium. Although conditions for existence are not problematic, conditions for semi-uniqueness are. We provide semi-uniqueness conditions by deriving a new equilibrium semi-uniqueness result for games in strategic form with higher dimensional action sets. The result applies in particular to Cournot-like games.

Key words: Coalition formation, Cournot oligopoly, equilibrium (semi-)uniqueness, game in strategic form, public good.

1. Introduction

The analysis of coalition formation – in particular in the context of externalities – has become an important topic in economics. Examples do not only include firms that coordinate their output or prices in oligopolistic markets (cartels), jointly invest in research assets (R&D-agreements) or completely merge (joint ventures), but also countries that coordinate their tariffs (trade agreements and customs unions) or their environmental policy (international environmental agreements).

Our article contributes to the so-called 'new approach' of coalition formation (see for instance Yi(1997) and Bloch(2003) for an extensive overview). The goal of this approach is to determine equilibrium coalition structures. As the approach consists of modelling coalition formation as a 2-stage game with simultaneous actions in each of both stages, it is important that for each possible coalition structure coalitional equilibria, i.e. equilibria in the second stage, are unique.[1]

So for the new approach it is important to have results that guarantee uniqueness of coalitional equilibria. Conditions should be such that they can be easily checked for the base games that appeared so far in these models, like Cournot and public good games. As far as we know, general uniqueness results for coalitional equilibria

[1] Roughly speaking, in the first stage, the players choose a membership action which via a given member-ship rule leads to a coalition structure. In the second stage, the players are the coalitions in this coalition structure. Each of these coalitions chooses a 'physical' action for each of its members in a base game. See e.g. Finus and Rundshagen(2009) for more. Also see Bartl(2012).

of such games are not present in the literature. There it is just assumed that one deals with a situation where coalitional equilibria are unique or that one deals with a simple concrete example where uniqueness explicitly can be shown. Developing an abstract general uniqueness result is the main objective of this article.

As shown in Section 3., a general equilibrium existence theorem guarantees for various common cases existence of coalitional equilibria. So existence is not a real issue, but equilibrium semi-uniqueness is. Especially as for coalitional equilibria one has to leave the comfortable usual setting of one dimensional action sets. Indeed: a coalition is formally treated as a meta-player whose action set is the Cartesian product of the action sets of the players in this coalition.

In order to obtain our semi-uniqueness result for coalitional equilibria we develop a semi-uniqueness result for Nash equilibria of games in strategic form with higher dimensional action sets. This result, Theorem 1, can be considered as a variant of a result in Folmer and von Mouche(2004) to higher dimensions. It can handle various aggregative[2] base-games with one-dimensional action sets. We identify a class of such games which contains Cournot and public good games and give with Corollary 3 a result that guarantees that for each possible coalition structure there exists a unique coalitional equilibrium.

2. Coalitional equilibria

In this section, we fix the setting and notations and formally define the notion of coalitional equilibrium.

2.1. Games in strategic form

A game in strategic form Γ is an ordered 3-tuple

$$\Gamma = (I, (X_i)_{i \in I}, (f_i)_{i \in I}),$$

where I is a non-empty finite set, every X_i is a non-empty set and every f_i is a function

$$f_i : \prod_{j \in I} X_j \to \mathbb{R}.$$

The elements of I are called *players*, X_i is called the *action set* of player i, the elements of X_i are called *actions* of player i, f_i is called the *payoff function* of player i and the elements of $\prod_{j \in I} X_j$, being by I indexed families $(x_j)_{j \in I}$ with $x_j \in X_j$, are called *action profiles.*

For $i \in I$, let

$$\hat{\imath} := I \setminus \{i\}.$$

For $i \in I$ and $\mathbf{z} = (z_j)_{j \in \hat{\imath}} \in \prod_{j \in \hat{\imath}} X_j$, the *conditional payoff function* $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ is defined by

$$f_i^{(\mathbf{z})}(x_i) \ := \ f_i(x_i; \mathbf{z});$$

here $(x_i; \mathbf{z})$ is the by I indexed family with x_i for the element with index i and z_j for the element with index $j \neq i$. An action profile $\mathbf{x} = (x_j)_{j \in I} \in \prod_{j \in I} X_j$ is a

[2] I.e. games where the payoff function of each player i is a function of his own action x_i and of a weighted sum $\sum_l \gamma_l x_l$ of all actions.

(Nash) equilibrium if, for all $i \in N$, writing again $\mathbf{x} = (x_i; \mathbf{z})$, x_i is a maximiser of $f_i^{(\mathbf{z})}$. We denote by

$$E$$

the set of equilibria of Γ.

We need some further notations for the sequel. For $C \subseteq I$, let

$$\mathbf{X}_C := \prod_{j \in C} X_j.$$

So an element $\boldsymbol{\xi}_C$ of $\mathbf{X}_C$ is a by C indexed family $(\xi_{C;i})_{i \in C}$ with $\xi_{C;i} \in X_i$; for $i \in N$, we identify $\mathbf{X}_{\{i\}}$ with X_i. And an element of element of $\prod_{C \in \mathcal{C}} \mathbf{X}_C$ is a by $\mathcal{C}$ indexed family

$$\boldsymbol{\xi} = (\boldsymbol{\xi}_C)_{C \in \mathcal{C}} = ((\xi_{C;i})_{i \in C})_{C \in \mathcal{C}}.$$

2.2. Notion of coalitional equilibrium

Suppose given a game in strategic form $\Gamma = (I, (X_i)_{i \in I}, (f_i)_{i \in I})$.

A *coalition* is a subset of I and a *coalition structure* of I is a partition of I, i.e. a set with as elements non-empty disjoint coalitions whose union is I.

Given a coalition structure $\mathcal{C}$, we denote for $i \in I$ by C_i the unique element of $\mathcal{C}$ with

$$i \in C_i$$

and define the mapping $J^{\mathcal{C}} : \prod_{C \in \mathcal{C}} \mathbf{X}_C \to \prod_{j \in I} X_j$ by

$$J^{\mathcal{C}}((\boldsymbol{\xi}_C)_{C \in \mathcal{C}}) := (\xi_{C_j;j})_{j \in I}.$$

For a subset D of I the function $f_D : \prod_{C \in \mathcal{C}} \mathbf{X}_C \to \mathbb{R}$ is defined by

$$f_D := \sum_{i \in D} f_i \circ J^{\mathcal{C}}.$$

Having these notations, the next definition formalizes the intended notion of coalitional equilibrium (with base game Γ) as outlined in section 1..

Definition 1. Given a game in strategic form $\Gamma = (I, (X_i)_{i \in I}, (f_i)_{i \in I})$ and a coalition structure $\mathcal{C}$ of I, the (*with $\mathcal{C}$ associated*) game in strategic form $\Gamma_{\mathcal{C}}$ is defined by

$$\Gamma_{\mathcal{C}} := (\mathcal{C}, (\mathbf{X}_C)_{C \in \mathcal{C}}, (f_C)_{C \in \mathcal{C}}). \ \diamond$$

A Nash equilibrium of $\Gamma_{\mathcal{C}}$ also is called a *coalitional equilibrium* of Γ; more precisely we speak of a $\mathcal{C}$-equilibrium of Γ. We also will refer to the elements of $\mathcal{C}$ as *meta-players*. The action sets $\mathbf{X}_C$ of $\Gamma_{\mathcal{C}}$ are typically more dimensional. Note that if

$$\mathcal{C} = \{\{1\}, \{2\}, \ldots \{n\}\},$$

then $\Gamma_{\mathcal{C}} = \Gamma$ and a $\mathcal{C}$-equilibrium of Γ is nothing else than a Nash equilibrium of Γ. And if

$$\mathcal{C} = \{I\},$$

a $\mathcal{C}$-equilibrium is nothing else than a maximizer of the total payoff function $\sum_{i \in I} f_i$.

3. Existence of coalitional equilibria

General equilibrium existence and semi-uniqueness results for games in strategic form have immediate counterparts regarding coalitional equilibria if they allow for higher-dimensional action sets.

A powerful standard existence result in Tan et al.(1995)Tan, Yu, and Yuan leads to the following sufficient conditions for the game $\Gamma_{\mathcal{C}} := (\mathcal{C}, (\mathbf{X}_C)_{C \in \mathcal{C}}, (f_C)_{C \in \mathcal{C}})$ to have a Nash equilibrium:

I. each action set $\mathbf{X}_C$ is a compact convex subset of a Hausdorff topological linear space;
II. each payoff function f_C is upper-semi-continuous;
III. every f_C is lower-semi-continuous in the variable related to $\mathbf{X}_{\hat{C}}$;
IV. every f_C is quasi-concave in $\boldsymbol{\xi}_C \in \mathbf{X}_C$.

It may be useful to note that if each function f_i is quasi-concave in (its own action) x_i, this does not necessarily imply that IV holds. Even assuming that each function f_i is concave in each variable is not sufficient.[3]

A natural question is to ask for simple sufficient conditions such that for each coalition structure $\mathcal{C}$ a $\mathcal{C}$-equilibrium exists. As can be easily verified by the above existence result, such conditions are for instance: each action set X_i is a segment of $\mathbb{R}$, each payoff function f_i is continuous and concave.

4. A higher dimensional equilibrium semi-uniqueness result

In this section we consider a game in strategic form $\Gamma = (I, (X_i)_{i \in I}, (f_i)_{i \in I})$ where each player $i \in I$ has action set $X_i = \prod_{j \in M_i} X_{i;j}$ with M_i a non-empty set and the $X_{i;j}$ proper intervals of $\mathbb{R}$.

Theorem 1. *For $i \in I$ let $E_i := \{\mathbf{e}_i \mid \mathbf{e} \in E\}$ and $E_{i;j} := \{e_{i;j} \mid \mathbf{e} \in E\}$ $(j \in M_i)$. Suppose the following conditions I-III hold.*

I. The partial derivatives $\frac{\partial f_i}{\partial x_{i;j}}$ $(i \in I,\ j \in M_i)$ exist at every $\mathbf{e} \in E$ as an element of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.
II. There exist functions

$$\Phi_i : E_i \to \mathbb{R}\ (i \in I),\ \ \Theta_i : \{(\Phi_l(\mathbf{e}_l))_{l \in I} \mid \mathbf{e} \in E\} \to \mathbb{R}\ (i \in I),$$

and, with $\Psi_i : E \to \mathbb{R}$ $(i \in I)$ defined by $\Psi_i(\mathbf{e}) := \Theta_i((\Phi_l(\mathbf{e}_l))_{l \in I})$, functions

$$\mathcal{T}_{i;j} : E_{i;j} \times \Phi_i(E_i) \times \Psi_i(E) \to \overline{\mathbb{R}}\ (i \in I,\ j \in M_i),$$

such that for all $i \in I$ and $j \in M_i$
a. $\frac{\partial f_i}{\partial x_{i;j}}(\mathbf{e}) = \mathcal{T}_{i;j}(e_{i;j}, \Phi_i(\mathbf{e}_i), \Psi_i(\mathbf{e}))$ $(\mathbf{e} \in E)$;
b. $\mathcal{T}_{i;j}$ is decreasing in each of its three variables, and strictly decreasing in the first or second.

[3] It is worth noting that the sum of quasi-concave functions may fail to be quasi-concave and a function that is concave in each of its variables may fail to be concave.

III. a. *For all $i \in I$: Φ_i and Θ_i are increasing.*[4]
b. *For all $\mathbf{a}, \mathbf{b} \in E$: $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$ or $\Psi_i(\mathbf{b}) \geq \Psi_i(\mathbf{a})$ $(i \in I)$.*

1. *For all $\mathbf{a}, \mathbf{b} \in E$: $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$ and even $\Phi_i(\mathbf{a}_i) = \Phi_i(\mathbf{b}_i)$ $(i \in I)$.*
2. *If every $\mathcal{T}_{i;j}$ is strictly decreasing in the first variable, then $\#E \leq 1$.* $\diamond$

Proof. 1. Suppose $\mathbf{a}, \mathbf{b} \in E$.

Step 1: $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$ $\Rightarrow$ $\Phi_i(\mathbf{a}_i) \leq \Phi_i(\mathbf{b}_i)$ $(i \in I)$.
Proof: by contradiction assume $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$ and for some $m \in I$

$$\Phi_m(\mathbf{a}_m) > \Phi_m(\mathbf{b}_m).$$

With J the set of elements $j \in M_m$ for which $a_{m;j}$ is a left boundary point of $X_{m;j}$ or $b_{m;j}$ is a right boundary point of $X_{m;j}$, we have

$$a_{m;j} \leq b_{m;j} \ (j \in J).$$

Now suppose $j \in M_m \setminus J$. Because $\mathbf{a}$ is an equilibrium and $a_{m;j}$ is not a left boundary point of $X_{m;j}$, it follows by condition I that $D_{m;j} f_m(\mathbf{a}) \geq 0$. And, by the same arguments, $D_{m;j} f_m(\mathbf{b}) \leq 0$. So by condition IIa we have

$$\mathcal{T}_{m;j}(a_{m;j}, \Phi_m(\mathbf{a}_m), \Psi_m(\mathbf{a})) \geq 0 \geq \mathcal{T}_{m;j}(b_{m;j}, \Phi_m(\mathbf{b}_m), \Psi_m(\mathbf{b})). \quad (1)$$

As $\Psi_m(\mathbf{a}) \geq \Psi_m(\mathbf{b})$ and $\Phi_m(\mathbf{a}_m) > \Phi_m(\mathbf{b}_m)$, condition IIb implies

$$\mathcal{T}_{m;j}(a_{m;j}, \Phi_m(\mathbf{a}_m), \Psi_m(\mathbf{a})) \leq \mathcal{T}_{m;j}(a_{m;j}, \Phi_m(\mathbf{b}_m), \Psi_m(\mathbf{b})), \quad (2)$$

with strict inequality if $\mathcal{T}_{m;j}$ is strictly decreasing in the second variable. (1) and (2) imply

$$\mathcal{T}_{m;j}(a_{m;j}, \Phi_m(\mathbf{b}_m), \Psi_m(\mathbf{b})) \geq \mathcal{T}_{m;j}(b_{m;j}, \Phi_m(\mathbf{b}_m), \Psi_m(\mathbf{b})),$$

with strict inequality if $\mathcal{T}_{m;j}$ is strictly decreasing in the second variable. As $\mathcal{T}_{m;j}$ is decreasing, and strictly decreasing in the first or second variable, it follows that $a_{m;j} \leq b_{m;j}$. Hence, we proved

$$a_{m;j} \leq b_{m;j} \ (j \in M_m), \text{ i.e. } \mathbf{a}_m \leq \mathbf{b}_m.$$

By condition IIIa this implies $\Phi_m(\mathbf{a}_m) \leq \Phi_m(\mathbf{b}_m)$, a contradiction.

Step 2: $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$ $\Rightarrow$ $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$
Proof: suppose $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$. By Step 1: $\Phi_i(\mathbf{a}_i) \leq \Phi_i(\mathbf{b}_i)$ $(i \in I)$. This implies, as Θ_i is increasing, $\Psi_i(\mathbf{a}) \leq \Psi_i(\mathbf{b})$. Thus $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$.

Step 3: $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$.

[4] Given a finite product $\prod_{r \in J} Z_r$ of subsets of $\mathbb{R}$ the relation $\geq$ (and its dual $\leq$) on $\prod_{r \in J} Z_r$ is defined by: $(a_r)_{r \in J} \geq (b_r)_{r \in J}$ means $a_r \geq b_r$ $(r \in J)$. And a function $f : \prod_{r \in J} Z_r \to \mathbb{R}$ is called *increasing* if for all $\mathbf{a}, \mathbf{b} \in \prod_{r \in J} Z_r$ one has $\mathbf{a} \geq \mathbf{b} \Rightarrow f(\mathbf{a}) \geq f(\mathbf{b})$.

Proof: by condition IIb we have $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$ or $\Psi_i(\mathbf{b}) \geq \Psi_i(\mathbf{a})$ $(i \in I)$. Without loss of generality we may assume that $\Psi_i(\mathbf{a}) \geq \Psi_i(\mathbf{b})$ $(i \in I)$. Step 3 implies $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$.

Step 4: $\Phi_i(\mathbf{a}_i) = \Phi_i(\mathbf{b}_i)$ $(i \in I)$.
Proof: by Step 3 we have $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$. Now apply Step 1.

2. By contradiction suppose $\#E \geq 2$. Fix $\mathbf{a}, \mathbf{b} \in E$ and $i \in I$ and $j \in M_i$ such that $a_{i;j} \neq b_{i;j}$. We may assume that $a_{i;j} > b_{i;j}$. By part 1, $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b}) =: y_i$ and $\Phi_i(\mathbf{a}_i) = \Phi_i(\mathbf{b}_i) =: w_i$. As $\mathbf{a}$ is an equilibrium and $a_{i;j}$ is not a left boundary point of $X_{i;j}$, it follows that $D_{i;j} f_i(\mathbf{a}) \geq 0$. And, by the same arguments, $D_{i;j} f_i(\mathbf{b}) \leq 0$. By condition IIa

$$\mathcal{T}_{i;j}(a_{i;j}, w_i, y_i) \geq 0 \geq \mathcal{T}_{i;j}(b_{i;j}, w_i, y_i).$$

As $\mathcal{T}_{i;j}$ is strictly decreasing in its first variable, this implies a contradiction. □

Taking, in Theorem 1, $m_i = 1$ $(i \in I)$ and $\Phi_i = \mathrm{Id}$ leads to:

Corollary 1. *For $i \in I$ let $E_i := \{e_i \mid \mathbf{e} \in E\}$. Sufficient for $\#E \leq 1$ is that the following conditions I-III hold.*

I. The partial derivatives $\frac{\partial f_i}{\partial x_i}$ $(i \in I)$ exist at every $\mathbf{e} \in E$ as an element of $\overline{\mathrm{I\!R}}$.
II. There exist functions

$$\varphi_i : E \to \mathrm{I\!R} \ (i \in I),$$

and functions

$$t_i : E_i \times \varphi_i(E) \to \overline{\mathrm{I\!R}} \ (i \in I),$$

such that for all $i \in I$
  a. $\frac{\partial f_i}{\partial x_i}(\mathbf{e}) = t_i(e_i, \varphi_i(\mathbf{e}))$ $(\mathbf{e} \in E)$;
  b. t_i is decreasing in each of its two variables, and strictly decreasing in the first.
III. a. Every φ_i is increasing.
  b. For all $\mathbf{a}, \mathbf{b} \in E$: $\varphi_i(\mathbf{a}) \geq \varphi_i(\mathbf{b})$ $(i \in I)$ or $\varphi_i(\mathbf{b}) \geq \varphi_i(\mathbf{a})$ $(i \in I)$. ⋄

And here is a more practical variant of Theorem 1:

Corollary 2. *For $i \in I$, let*[5] *$t_{i;j} \geq 0$ $(j \in M_i)$, $r_i \geq 0$, $s_i > 0$, $W_i := \sum_{j \in M_i} t_{i;j} X_{i;j}$, $Y_i := s_i \sum_{k \in I} r_k W_k$ and define $\Phi_i : X_i \to \mathrm{I\!R}$ and $\Psi_i : \mathbf{X}_I \to \mathrm{I\!R}$ by*

$$\Phi_i(x_i) := \sum_{j \in M_i} t_{i;j} x_{i;j}, \ \ \Psi_i(\mathbf{x}) := s_i \sum_{k \in I} r_k \Phi_k(x_k).$$

Suppose the following conditions I, IIa and IIb hold.

I. Each player i's payoff function f_i is partially differentiable with respect to each variable $x_{i;j}$.
II. There exist functions

$$\mathcal{T}_{i;j} : X_{i;j} \times W_i \times Y_i \to \mathrm{I\!R} \ (i \in I, \ j \in M_i),$$

such that for all $i \in I$ and $j \in M_i$

[5] The sum in W_i and Y_i is a Minkowski-sum.

a. $\frac{\partial f_i}{\partial x_{i;j}}(\mathbf{x}) = \mathcal{T}_{i;j}(x_{i;j}, \Phi_i(x_i), \Psi_i(\mathbf{x}))$ $(\mathbf{x} \in \mathbf{X}_I)$*;*
b. $\mathcal{T}_{i;j}$ *is decreasing in each of its three variables, and strictly decreasing in the first or second.*

1. For all $\mathbf{a}, \mathbf{b} \in E$*:* $\Psi_i(\mathbf{a}) = \Psi_i(\mathbf{b})$ $(i \in I)$ *and even* $\Phi_i(a_i) = \Phi_i(b_i)$ $(i \in I)$*.*
2. If every $\mathcal{T}_{i;j}$ *is strictly decreasing in the first variable, then* $\#E \leq 1$*.* $\diamond$

5. Uniqueness of coalitional equilibria for Cournot-like games

In the following definition a class of games in strategic form is introduced for which we provide sufficient conditions for uniqueness of coalitional equilibria.

Definition 2. A *Cournot-like game* is a game in strategic form

$$\Gamma = (N, (K_i)_{i \in N}, (\pi_i)_{i \in N})$$

where every K_i is a proper interval of $\mathbb{R}$ with $0 \in K_i \subseteq \mathbb{R}_+$ and

$$\pi_i(\mathbf{x}) = a_i(x_i) - x_i^{\beta_i} b_i(\sum_{l \in N} \gamma_l x_l)$$

where, with $Y := \sum_{l \in N} \gamma_l K_l$,

- $a_i : K_i \to \mathbb{R}$;
- $\beta_i \in \{0, 1\}$;
- $\gamma_i > 0$;
- $b_i : Y \to \mathbb{R}$. $\diamond$

In case K_i is bounded, i.e. where $K_i = [0, m_i]$ or $K_i = [0, m_i[$ we say that player i has a *capacity constraint.* Note that some players may have a capacity constraint while others may not have. The class of Cournot-like games contains various heterogeneous Cournot oligopoly games: take every $\beta_i = 1$. It contains[6] all homogeneous Cournot oligopoly games: take in addition all b_i equal and each $\gamma = 1$. It also contains various public good games: take every $\beta_i = 0$. We call β_l the *type* of player l.

In the next theorem and proposition we consider a Cournot-like game Γ and fix a coalition structure $\mathcal{C}$ of N. We suppose that all players belonging to a same coalition $C \in \mathcal{C}$ are of the same type β_C. Also we suppose for every $C \in \mathcal{C}$ that $\gamma_l = \gamma_{l'}$ $(l, l' \in C)$ and in case $\beta_C = 1$ that $b_l, b_{l'}$ $(l, l' \in C)$.

Theorem 2. *Suppose that each function* a_i *and* b_i *is differentiable. Consider the with the coalition structure* $\mathcal{C}$ *associated game*

$$\Gamma_{\mathcal{C}} = (\mathcal{C}, (\mathbf{K}_C)_{C \in \mathcal{C}}, (\pi_C)_{C \in \mathcal{C}}).$$

For $C \in \mathcal{C}$*, let* $W_C := \sum_{l \in C} \gamma_l K_l$ *and define the functions* $T_{C;j} : K_j \times W_C \times Y \to \mathbb{R}$ $(j \in C)$ *by*

$$T_{C;j}(x_j, w, y) := Da_j(x_j) - \frac{w^{\beta_C} \gamma_j}{\#C \cdot \beta_C + (1 - \beta_C)} \sum_{i \in C} Db_i(y) - \beta_C b_j(y).$$

Suppose every $T_{C;j}$ *is decreasing in each of its three variables and strictly decreasing in its first or second variable.*

[6] Disregarding Cournot oligopoly games with finite action sets.

1. *For all Nash equilibria* $\boldsymbol{\eta}, \boldsymbol{\mu}$ *of* $\Gamma_{\mathcal{C}}$ *one has*

$$\sum_{C \in \mathcal{C}} \sum_{i \in C} \eta_{C;i} = \sum_{C \in \mathcal{C}} \sum_{i \in C} \mu_{C;i}$$

and even $\sum_{i \in C} \eta_{C;i} = \sum_{i \in C} \mu_{C;i}$ $(C \in \mathcal{C})$.

2. *If every* $T_{C;j}$ *is strictly decreasing in its first variable, then* $\Gamma_{\mathcal{C}}$ *has at most one Nash equilibrium.* $\diamond$

Proof. Let $\gamma_l =: \gamma_C$ $(l \in C)$ and in case $\beta_C = 1$, let $b_l =: b_C$ $(l \in C)$.

Consider the game $\Gamma_{\mathcal{C}}$. The payoff function of player $C \in \mathcal{C}$ is

$$\begin{aligned}\pi_C(\boldsymbol{\xi}) &= \sum_{i \in C} (\pi_i \circ J^{\mathcal{C}})(\boldsymbol{\xi}) = \sum_{i \in C} \Big(a_i(\xi_{C;i}) - (\xi_{C;i})^{\beta_C} b_i(\sum_{m \in N} \gamma_m \xi_{C_m;m}) \Big) \\ &= \sum_{i \in C} \Big(a_i(\xi_{C;i}) - (\xi_{C;i})^{\beta_C} b_i(\sum_{A \in \mathcal{C}} \sum_{m \in A} \gamma_m \xi_{A;m}) \Big) \\ &= \sum_{i \in C} \Big(a_i(\xi_{C;i}) - (\xi_{C;i})^{\beta_C} b_i(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m}) \Big).\end{aligned}$$

If $\beta_C = 0$, then $\pi_C(\boldsymbol{\xi}) = \sum_{i \in C} a_i(\xi_{C;i}) - \sum_{i \in C} b_i(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m})$ and therefore for $j \in \mathcal{C}$

$$\frac{\partial \pi_C}{\partial \xi_{C;j}}(\boldsymbol{\xi}) = Da_j(\xi_{C;j}) - \gamma_j \sum_{i \in \mathcal{C}} Db_i(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m}).$$

If $\beta_C = 1$, then

$$\pi_C(\boldsymbol{\xi}) = \sum_{i \in C} a_i(\xi_{C;i}) - (\sum_{i \in C} \xi_{C;i}) b_C(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m})$$

and therefore for $j \in \mathcal{C}$

$$\frac{\partial \pi_C}{\partial \xi_{C;j}}(\boldsymbol{\xi}) = Da_j(\xi_{C;j}) - b_C(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m}) - \gamma_j (\sum_{i \in C} \xi_{C;i}) Db_C(\sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m}).$$

Noting that for the above functions $T_{C;j}$ one has

$$T_{C;j}(x_j, w, y) := Da_j(x_j) - \gamma_j p_C(w, y) - \beta_C b_j(y)$$

where $p_C : W_C \times Y \to \mathbb{R}$ is defined by

$$p_C(w, y) = \begin{cases} \sum_{i \in \mathcal{C}} Db_i(y) \text{ if } \beta_C = 0, \\ wDb_C(y) \text{ if } \beta_C = 1, \end{cases}$$

we obtain

$$\frac{\partial \pi_C}{\partial \xi_{C;j}}(\boldsymbol{\xi}) = T_{C;j}(\xi_{C;j}, \sum_{i \in C} \xi_{C;i}, \sum_{A \in \mathcal{C}} \gamma_A \sum_{m \in A} \xi_{A;m}).$$

Having the above, we can apply[7] Corollary 2 which implies the desired results. $\square$

[7] Taking $I = \mathcal{C}$, $M_C = C$ $(C \in \mathcal{C})$, $X_{C;j} = K_j$ $(C \in \mathcal{C},\ j \in C)$, $X_C = \prod_{j \in M_C} X_{C;j} = \mathbf{K}_C$ $(C \in \mathcal{C})$, $t_{C;j} = 1$ $(C \in \mathcal{C},\ j \in M_C)$, $r_C = \gamma_C$ $(C \in \mathcal{C})$, $s_C = 1$ $(C \in \mathcal{C})$, $f_C = \sum_{i \in C} \pi_i \circ J^{\mathcal{C}} = \pi_C$ $(C \in \mathcal{C})$, $\mathcal{T}_{C;j} = T_{C;j}$ $(C \in \mathcal{C},\ j \in M_C)$, $\Phi_C(\boldsymbol{\xi}_C) = \sum_{j \in M_C} \xi_{C;j}$ $(C \in \mathcal{C})$ and $\Psi_C(\boldsymbol{\xi}) = \sum_{D \in \mathcal{C}} \gamma_D \Phi_D(\boldsymbol{\xi}_D)$ $(C \in \mathcal{C})$.

Remark: sufficient for every $T_{C;j}$ to be decreasing in each of its three variables and strictly decreasing in its first variable is that the following practical condition holds:

every a_i is strictly concave and every b_i is increasing and convex.

Proposition 1. *Consider the with the coalition structure $\mathcal{C}$ associated game*

$$\Gamma_{\mathcal{C}} = (\mathcal{C}, (\mathbf{K}_C)_{C \in \mathcal{C}}, (\pi_C)_{C \in \mathcal{C}}).$$

Given $C \in \mathcal{C}$, the following condition guarantees strict concavity of all conditional payoff functions of player C: the function $a_C : \mathbf{K}_C \to \mathbb{R}$ given by

$$a_C(\boldsymbol{\xi}_C) := \sum_{i \in C} a_i(\xi_{C;i})$$

is concave and

a. if $\beta_C = 0$, then the function $\sum_{i \in \mathcal{C}} b_i$ is strictly convex or a_C is strictly concave;
b. if $\beta_C = 1$, then the function $y \mapsto y b_C(y)$ is convex, and this function is strictly convex or a_C is strictly concave. $\diamond$

Proof. With $\boldsymbol{\xi}_{\hat{C}} \in \mathbf{K}_{\hat{C}}$, the conditional payoff function $\pi_C^{(\boldsymbol{\xi}_{\hat{C}})} : \mathbf{K}_C \to \mathbb{R}$ reads

$$\pi_C^{(\boldsymbol{\xi}_{\hat{C}})}(\boldsymbol{\xi}_C) = \sum_{i \in C} a_i(\xi_{C;i}) + \sum_{i \in C} -(\xi_{C;i})^{\beta_C} b_i(\gamma_C \sum_{m \in C} \xi_{C;m} + z),$$

where $z = \sum_{A \in \mathcal{C} \setminus C} \gamma_A \sum_{m \in A} \xi_{A;m}$ The first sum in this expression is by assumption a concave function.

Case $\beta_C = 0$: the second equals

$$\sum_{i \in C} -b_i(\gamma_C \sum_{m \in C} \xi_{C;m} + z)$$

and also is concave. As the first or second sum is strictly concave, $\pi_C^{(\boldsymbol{\xi}_{\hat{C}})}$ is strictly concave. Case $\beta_C = 1$: the second sum equals

$$-b_C(\gamma_C \sum_{m \in C} \xi_{C;m} + z) \sum_{m \in C} \xi_{C;m}$$

and also is concave. As the first or second sum is strictly concave, $\pi_C^{(\boldsymbol{\xi}_{\hat{C}})}$ is strictly concave. $\square$

The last paragraph in Section 2, the remark after Theorem 2 and Proposition 1 imply:

Corollary 3. *Let Γ be a Cournot-like game with compact action sets, $\beta_i = \beta$ ($i \in N$), $\gamma_i = \gamma_{i'}$ ($i, i' \in N$) and $\beta = 1 \Rightarrow b_i = b_{i'}$ ($i, i' \in N$). Suppose each function a_i is differentiable and strictly concave and each function b_i is differentiable, increasing and convex. Then for every coalition structure $\mathcal{C}$ the game Γ has a unique $\mathcal{C}$-equilibrium.* $\diamond$

References

Bartl, D. (2012). *Application of cooperative game solution concepts to a collusive oligopoly game*. In School of Business Administration in Karviná, editor, Proceedings of the 30th International Conference Mathematical Methods in Economics, pages 14–19.

Bloch, F. (2003). *Non–cooperative models of coalition formation in games with spillovers.* In C. Carraro, editor, Endogenous Formation of Economic Coalitions, chapter 2, pages 35–79. Edward Elgar, Cheltenham.

Finus, M. and B. Rundshagen (2009). *Membership rules and stability of coalition structures in positive externality games.* Social Choice and Welfare, **32(0)**, 389–406.

Folmer, H. and P. H. M. von Mouche (2004). *On a less known Nash equilibrium uniqueness result.* Journal of Mathematical Sociology, **28(0)**, 67–80.

Tan, K. J. Yu, and X. Yuan (1995). *Existence theorems of Nash equilibria for non-cooperative n-person games.* International Journal of Game Theory, **24(0)**, 217–222.

Yi, S. (1997). *Stable coalition structures with externalities.* Games and Economic Behavior, **20(0)**, 201–237.