Stackelberg Oligopoly Games: the Model and the 1-concavity of its Dual Game

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Abstract This paper highlights the role of a significant property for the core of Stackelberg Oligopoly cooperative games arising from the non-cooperative Stackelberg Oligopoly situation with linearly decreasing demand functions. Generally speaking, it is shown that the so-called 1-concavity property for the dual of a cooperative game is a sufficient and necessary condition for the core of the game to coincide with its imputation set. Particularly, the nucleolus of such dual 1-concave TU-games agree with the center of the imputation set. Based on the explicit description of the characteristic function for the Stackelberg Oligopoly game, the aim is to establish, under certain circumstances, the 1-concavity of the dual game of Stackelberg Oligopoly games. These circumstances require the intercept of the inverse demand function to be bounded below by a particular critical number arising from the various cost figures.

Keywords: Stackelberg oligopoly game; imputation set; core; efficiency; 1-concavity

1. Introduction of game theoretic notions

A cooperative savings game (with transferable utility) is given by a pair $\langle N, w\rangle$, where its characteristic function $w : \mathcal{P}(N) \to \mathbb{R}$ is defined on the power set $\mathcal{P}(N) = \{S \mid S \subseteq N\}$ of the finite set N, of which the elements are called players, while the elements of the power set are called coalitions. The so-called real-valued worth $w(S)$ of coalition $S \subseteq N$ in the game $\langle N, w\rangle$ represents the maximal amount of monetary benefits due to the mutual cooperation among the members of the coalition, on the understanding that there are no benefits by absence of players, that is $w(\emptyset) = 0$. In the framework of the division problem of the benefits $w(N)$ of the grand coalition N among the potential players, any allocation scheme of the form $\boldsymbol{x} = (x_i)_{i\in N} \in \mathbb{R}^N$ is supposed to meet, besides the efficiency principle $\sum_{i\in N} x_i = w(N)$, the so-called individual rationality condition in that each player is allocated at least the individual worth, i.e., $x_i \geqslant w(\{i\})$ for all $i \in N$. Concerning the development of the solution part, a (multi- or single-valued) solution concept σ assigns to any cooperative game $\langle N, w\rangle$ a (possibly empty) subset of its imputation set $I(N, w)$, that is $\sigma(N, w) \subseteq I(N, w)$, where

$$I(N,w) = \{(x_i)_{i\in N} \in \mathbb{R}^N \mid \sum_{i\in N} x_i = w(N) \quad \text{and} \quad x_i \geqslant w(\{i\}) \quad \text{for all } i \in N\}.$$

The best known multi-valued solution concept called core requires the group rationality condition in that the aggregate allocation to the members of any coalition is at least its coalitional worth, that is

$$CORE(N,w) = \{\overrightarrow{x} \in I(N,w) \mid \sum_{i\in S} x_i \geqslant w(S) \quad \text{for all } S \subseteq N,\, S \neq N,\, S \neq \emptyset\} \tag{1.1}$$

Of significant importance is the upper core bound composed of the marginal contributions $m_i^w = w(N) - w(N\backslash\{i\})$, $i \in N$, with respect to the formation of the grand coalition N in the game $\langle N,w\rangle$. Obviously, $x_i \leqslant m_i^w$ for all $i \in N$ and all $\overrightarrow{x} \in CORE(N,w)$. In this context, we focus on the following core catcher called CoreCover

$$CC(N,w) = \{(x_i)_{i\in N} \in \mathbb{R}^N \mid \sum_{i\in N} x_i = w(N) \quad \text{and} \quad x_i \leqslant m_i^w \quad \text{for all } i \in N\} \tag{1.2}$$

In the framework of the core, a helpful tool appears to be the so-called gap function $g^w : \mathcal{P}(N) \to \mathbb{R}$ defined by $g^w(S) = \sum_{i\in S} m_i^w - w(S)$ for all $S \subseteq N$, $S \neq \emptyset$, where $g^w(\emptyset) = 0$. So, the gap $g^w(S)$ of any coalition S measures how much the coalitional worth $w(S)$ differs from the aggregate allocation based on the individually marginal contributions. The interrelationship between the gap function and the general inclusion $CORE(N,w) \subseteq CC(N,w)$ is the following equivalence (Driessen, 1988):

$$CORE(N,w) = CC(N,w) \quad \Longleftrightarrow \quad 0 \leqslant g^w(N) \leqslant g^w(S) \quad \text{for all } S \subseteq N,\, S \neq \emptyset \tag{1.3}$$

In words, the core catcher $CC(N,w)$ coincides with the core $CORE(N,w)$ only if the non-negative gap function g^w attains its minimum at the grand coalition N. If the latter property (1.3) holds, the savings game $\langle N,w\rangle$ is said to be 1-convex.

With every cooperative savings game $\langle N,w\rangle$ there is associated its dual game $\langle N,w^*\rangle$ defined by $w^*(S) = w(N) - w(N\backslash S)$ for all $S \subseteq N$. That is, the worth of any coalition in the dual game is given by the coalitionally marginal contribution with respect to the formation of the grand coalition N in the original game. Particularly, $w^*(\emptyset) = 0$, $w^*(N) = w(N)$, and so, $m_i^{(w^*)} = w^*(N) - w^*(N\backslash\{i\}) = w(\{i\})$ for all $i \in N$. We arrive at the first main result.

Proposition 1.1. *Three equivalent statements for any cooperative savings game $\langle N,w\rangle$.*

$$I(N,w) \neq \emptyset \quad \Longleftrightarrow \quad w(N) \geqslant \sum_{i\in N} w(\{i\}) \quad \Longleftrightarrow \quad g^{(w^*)}(N) \leqslant 0 \tag{1.4}$$

In fact, the dual game $\langle N, w^*\rangle$ of any cooperative savings game $\langle N, w\rangle$ is treated as a cost game such that the core equality $CORE(N, w^*) = CORE(N, w)$ holds, on the understanding that the core of any cost game is defined through the reversed inequalities of (1.1). Thus, $\vec{x} \in CORE(N, w^*)$ iff $\vec{x} \in CORE(N, w)$. As the counterpart to 1-convex savings games (with non-negative gap functions), we deal with so-called 1-concave cost games (with non-positive gap functions).

Definition 1.2. A cooperative cost game $\langle N, w\rangle$ is said to be 1-concave if its non-positive gap function attains its maximum at the grand coalition N, i.e.,

$$g^w(S) \leqslant g^w(N) \leqslant 0 \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{1.5}$$

Theorem 1.3. *Three equivalent statements for any cooperative savings game $\langle N, w\rangle$.*

(i) The dual game $\langle N, w^*\rangle$ is 1-concave, that is (1.5) applied to $\langle N, w^*\rangle$ holds

(ii)

$$w(N) \geqslant \sum_{i\in N} w(\{i\}) \quad \textit{and} \quad w(S) \leqslant \sum_{i\in S} w(\{i\}) \qquad \textit{for all } S \subseteq N,\ S \neq N,\ S \neq \emptyset \tag{1.6}$$

(iii) $I(N, w) \neq \emptyset$ *and* $CORE(N, w) = I(N, w)$

Proof. In view of Proposition 1.1, together with $CORE(N, w) \subseteq I(N, w)$, it remains to prove the implication $(iii) \Longrightarrow (ii)$. By contra-position, suppose (ii) does not hold in that there exists $S \subseteq N$, $S \neq N$, $S \neq \emptyset$ with $w(S) > \sum_{i\in S} w(\{i\})$. Define the allocation $\vec{x} = (x_i)_{i\in N} \in \mathbb{R}^N$ by $x_i = w(\{i\})$ for all $i \in S$ and $x_i = w(\{i\}) + \frac{1}{n-s} \cdot \left[w(N) - \sum_{j\in N} w(\{j\})\right]$ for all $i \in N\backslash S$. Obviously, $\vec{x} \in I(N, w)\backslash CORE(N, w)$. □

2. The Stackelberg oligopoly game

The normal form game of the non-cooperative Stackelberg oligopoly situation[1] is modeled as a cooperative TU-game as follows.

Throughout the paper we fix the set N of firms with (possibly identical) strategy sets $X_i = [0, w_i)$, $i \in N$, with reference to (possibly unlimited) capacities $w_i \in [0, \infty]$, $i \in N$, (possibly distinct) marginal costs $c_i \geqslant 0$, $i \in N$, and the inverse demand function $p(x) = a - x$ for all $x \leqslant a$ and $p(x) = 0$ for all $x \geqslant a$. In this framework, the corresponding individual profit functions $\pi_i : \Pi_{k\in N} X_k \to \mathbb{R}$, $i \in N$, and coalitional profit functions $\pi_T : \Pi_{k\in N} X_k \to \mathbb{R}$, $T \subseteq N$, $T \neq \emptyset$, are defined by

$$\pi_i((x_k)_{k\in N}) = (a - X(N) - c_i) \cdot x_i \quad \text{and} \quad \pi_T((x_k)_{k\in N}) = \sum_{j\in T} \pi_j((x_k)_{k\in N}) \tag{2.7}$$

where $X(N) = \sum_{k\in N} x_k \in \mathbb{R}$ represents the aggregate production and $a \geqslant 2 \cdot n \cdot \max_{i\in N} c_i$.

The Stackelberg oligopoly model is based on a two-stage procedure. Given that the

[1] For a description of the oligopoly situation, we refer to the PhD thesis of Aymeric Lardon (Lardon, 2011).

members of the coalition S are supposed to perform their leadership in the first stage maximizing its coalitional profit by taking into account the best responses of individual followers $j \in N\backslash S$ (i.e., the non-members of S) during the second stage. So, the second stage is devoted to the maximization problems $\max_{x_j \in X_j} \pi_j(x_j, (x_k)_{k \in N\backslash\{j\}})$ for all $j \in N\backslash S$.

Theorem 2.1. *Let $S \subseteq N$, $S \neq N$. Write $c_T = \sum\limits_{k \in T} c_k$ for all $T \subseteq N$, $T \neq \emptyset$.*

(i) The best response of any individual $i \in N\backslash S$ during the second stage is given by

$$y_i = \frac{c_{N\backslash S} + X(N\backslash S)}{n-s} - c_i = \frac{a - X(S) + c_{N\backslash S}}{n+1-s} - c_i$$

(ii) The worth $v(S)$ of coalition S is determined by

$$v(S) = \frac{(x^*_{i_S})^2}{n+1-s} \qquad \text{where} \qquad x^*_{i_S} = \tfrac{1}{2} \cdot \Big[a + c_{N\backslash S} - (n+1-s) \cdot c_{i_S}\Big]$$

is the maximizer of the profit function of player $i_S \in S$ with the smallest marginal contribution among members of S, supposing other members of S produce nothing. Note that $x^*_{i,S} \geqslant 0$ because of $a \geqslant n \cdot \max_{i \in N} c_i$.

Proof. Fix coalition $S \neq N$. For all $i \in N\backslash S$ the maximization problem of the player's profit function $\pi_i((x_k)_{k \in N}) = (a - X(N) - c_i) \cdot x_i = -(x_i)^2 + x_i \cdot (a - c_i - X(N\backslash\{i\}))$ is solved through its first order condition $\frac{\partial \pi_i}{\partial x_i} = 0$, yielding

$$x_i = \tfrac{1}{2} \cdot \Big[a - c_i - X(N\backslash\{i\})\Big] \qquad \text{or equivalently,} \qquad x_i = a - c_i - X(N)$$

Summing up the latter equations over all $i \in N\backslash S$ yields

$$X(N\backslash S) = (n-s) \cdot a - c_{N\backslash S} - (n-s) \cdot X(N)$$
$$\text{and so,} \quad a - X(N) = \frac{c_{N\backslash S} + X(N\backslash S)}{n-s}$$

Hence, by substitution, it holds for all $i \in N\backslash S$

$$y_i = (a - X(N)) - c_i = \frac{c_{N\backslash S} + X(N\backslash S)}{n-s} - c_i = \frac{a - X(S) + c_{N\backslash S}}{n+1-s} - c_i$$

This proves part (i). Given these best responses by players in $N\backslash S$, the maximization problem of the coalitional profit function π_S is, due to unlimited capacities, equivalent to the maximization problem of the profit function of the firm $i_S \in S$ with smallest marginal cost among members of S, (i.e., $c_{i_S} \leqslant c_i$ for all $i \in S$), supposing that the other members of S produce nothing. In this framework,

$$y_i = \frac{a - x_{i_S} + c_{N\backslash S}}{n+1-s} - c_i \qquad \text{for all } i \in N\backslash S \text{ and thus,}$$

$$\begin{aligned} y(N\backslash S) &= \frac{n-s}{n+1-s} \cdot \Big[a - x_{i_S} + c_{N\backslash S}\Big] - c_{N\backslash S} \\ &= a - x_{i_s} - \frac{1}{n+1-s} \cdot \Big[a - x_{i_S} + c_{N\backslash S}\Big] \end{aligned}$$

Hence, we focus on the player's profit function of the form

$$
\begin{aligned}
&\pi_{i_S}((y_i)_{i\in N\setminus S}, x_{i_S}, (\overrightarrow{0})_{i\in S\setminus\{i_S\}}) \\
&= \Big[a - X(N) - c_{i_S}\Big]\cdot x_{i_S} = \Big[a - x_{i_S} - y(N\setminus S) - c_{i_S}\Big]\cdot x_{i_S} \\
&= \Big[\frac{1}{n+1-s}\cdot\Big[a - x_{i_S} + c_{N\setminus S}\Big] - c_{i_S}\Big]\cdot x_{i_S} \\
&= \frac{1}{n+1-s}\cdot\Big[a + c_{N\setminus S} - (n+1-s)\cdot c_{i_S} - x_{i_S}\Big]\cdot x_{i_S}
\end{aligned}
$$

The first order condition yields that the maximizer of this quadratic profit function is given by

$$
x^*_{i_S} = \tfrac{1}{2}\cdot\Big[a + c_{N\setminus S} - (n+1-s)\cdot c_{i_S}\Big] \qquad \text{and finally,}
$$

$$
\begin{aligned}
v(S) &= \pi_{i_S}((y_i)_{i\in N\setminus S}, x^*_{i_S}, (\overrightarrow{0})_{i\in S\setminus\{i_S\}}) \\
&= \frac{1}{n+1-s}\cdot\Big[a + c_{N\setminus S} - (n+1-s)\cdot c_{i_S} - x^*_{i_S}\Big]\cdot x^*_{i_S} \\
&= \frac{1}{n+1-s}\cdot\Big[2\cdot x^*_{i_S} - x^*_{i_S}\Big]\cdot x^*_{i_S} = \frac{(x^*_{i_S})^2}{n+1-s}
\end{aligned}
$$

It remains to check the non-negativity constraint for the maximizer $x^*_{i_S}$ (since production levels are supposed to be non-negative). Of course, the non-negativity constraint also applies to any player $j\in N\setminus S$. For that purpose, choose a sufficiently large in that $a \geqslant 2\cdot n\cdot \max_{i\in N} c_i$ (or to be exact, $a \geqslant 2\cdot n\cdot \max_{i\in N} c_i$). Recall that production levels of all firms are supposed to be unlimited. This proves part (ii). □

In the context of the resulting cooperative TU game, the following significant notions appear. For any non-trivial coalition $T\subseteq N$, $T\neq\emptyset$, let c_T, $\bar{c}_T$, and $\underline{c}_T$ respectively, denote the aggregate, average, and minimal cost of coalition T, that is

$$
c_T = \sum_{k\in T} c_k \qquad \bar{c}_T = \frac{c_T}{|T|} \qquad \underline{c}_T = \min\{c_k \mid \quad k\in T\}. \qquad \text{Moreover,} \tag{2.8}
$$

$$
\bar{c}\bar{c}_T = \frac{1}{|T|}\cdot\sum_{k\in T}(c_k)^2 \quad \text{Note that} \quad \sum_{k\in T}\Big[c_k - \bar{c}_T\Big]^2 = \sum_{k\in T}(c_k)^2 - |T|\cdot(\bar{c}_T)^2 \tag{2.9}
$$

Generally speaking, $\bar{c}_T \geqslant \underline{c}_T$ and moreover, the equality is met only by identical marginal costs, that is, for any coalition, the average cost equals the minimum cost if and only if all the marginal costs of its members do not differ. By (2.9), $\bar{c}\bar{c}_T \geqslant (\bar{c}_T)^2$, that is the average of the squares of marginal costs covers the square of the average cost.

Theorem 2.2. *Given the normal form game $\langle N, (c_k)_{k\in N}, (w_k)_{k\in N}, a\rangle$ of the non-cooperative Stackelberg oligopoly situation with unlimited capacities ($w_i = +\infty$ for all $i \in N$) and possibly distinct marginal costs, then the corresponding cooperative n-person Stackelberg oligopoly game $\langle N, v\rangle$ is determined by $v(\emptyset) = 0$ and for all $S \subseteq N$, $S \neq \emptyset$,*

$$v(S) = \frac{\left[a + c_{N\setminus S} - (n+1-s)\cdot \underline{c}_S\right]^2}{4\cdot(n+1-s)} = \frac{n+1-s}{4}\cdot\left[\frac{a+c_{N\setminus S}}{n+1-s} - \underline{c}_S\right]^2 \tag{2.10}$$

Here $c_\emptyset = 0$. In case all marginal costs are identical, say $c_i = c$ for all $i \in N$, then (2.10) reduces to $v(S) = \frac{(a-c)^2}{4}\cdot\frac{1}{n+1-s}$ for all $S \subseteq N$, $S \neq \emptyset$, and so, the Stackelberg oligopoly game is a multiple of the symmetric n-person cooperative game $v(s) = \frac{1}{n+1-s}$, the imputation set of which degenerates into the single core allocation $\frac{1}{n}\cdot(1,1,\ldots,1) \in \mathbb{R}^n$.

3. 1-Concavity of the dual game of Stackelberg oligopoly games

Assuming non-emptiness of its core, our goal is to study whether or not the 1-convexity property applies to the Stackelberg oligopoly game. For that purpose, we are interested in the structure of the corresponding non-negative gap function. Generally speaking, the validity of the inequality $v(S) \leqslant \sum_{k\in S} v(\{k\})$ is equivalent to the reversed inequality $g^v(S) \geqslant \sum_{k\in S} g^v(\{k\})$ for all $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. Together with the non-negativity of the gap function g^v, it follows that $g^v(S) \geqslant g^v(\{i\})$ whenever $i \in S$. In words, the gap function g^v of the Stackelberg oligopoly game attains among non-trivial coalitions containing a given player its minimum either at the one-person coalition or the grand coalition N. According to the next proposition, the gap of the grand coalition is not minimal and hence, the 1-convexity property fails to hold for the Stackelberg oligopoly games. However, the solution concept called τ-value (cf. Tijs, 1981) agrees, concerning its allocation to any player i, with the efficient compromise between the marginal contribution $m_i^v = v(N) - v(N\setminus\{i\})$ and the stand-alone worth $v(\{i\})$, $i \in N$ (treated as upper and lower core bounds respectively).

Proposition 3.1. *Given the non-emptiness of the core of the Stackelberg oligopoly game $\langle N, v\rangle$, the corresponding gap function g^v satisfies $g^v(N) \geqslant g^v(\{i\})$ for all $i \in N$.*

Proof. Fix $i \in N$. Recall that $g^v(N\setminus\{k\}) = g^v(N)$ for all $k \in N$. Fix $j \in N$, $j \neq i$. Due to the non-emptiness of the core, $m_k^v \geqslant v(\{k\})$ for all $k \in N$. We conclude that

$$\begin{aligned}
g^v(N) - g^v(\{i\}) &= g^v(N\setminus\{j\}) - m_i^v + v(\{i\}) \\
&= \sum_{k\in N\setminus\{i,j\}} m_k^v - v(N\setminus\{j\}) + v(\{i\}) \\
&\geqslant \sum_{k\in N\setminus\{i,j\}} m_k^v - \sum_{k\in N\setminus\{j\}} v(\{k\}) + v(\{i\}) \\
&= \sum_{k\in N\setminus\{i,j\}} (m_k^v - v(\{k\})) \geqslant 0
\end{aligned}$$

Here we applied the inequality $v(S) \leqslant \sum_{k\in S} v(\{k\})$ to $S = N\backslash\{j\})$. □

Now we arrive at the main result stating that the core of any Stackelberg oligopoly game coincides with its imputation set, provided its non-emptiness. By Theorem 1.3, the class of dual games of Stackelberg oligopoly savings games is a significant class of 1-concave (cost) games. The proof proceeds by checking the validity of (1.6). Note that the worth of any single player $i \in N$ and the grand coalition N respectively, are given as follows:

$$v(\{i\}) = \frac{\Big[a + c_N - (n+1)\cdot c_i\Big]^2}{4\cdot n} \quad \text{for all } i \in N, \text{ and} \quad v(N) = \frac{(a - \underline{c}_N)^2}{4} \tag{3.11}$$

Theorem 3.2. *The dual game $\langle N, v^*\rangle$ of the cooperative n-person Stackelberg oligopoly game $\langle N, v\rangle$ of the form (2.10) with distinct marginal costs is 1-concave only if the intercept $a > 0$ of the inverse demand function is large enough. On the one hand,*

$$g^{(v^*)}(N) \leqslant 0 \qquad \textit{if and only if} \qquad a \geqslant \frac{L_1}{2} - c_N \tag{3.12}$$

where the critical number L_1 represents the lower bound given by

$$L_1 = \Big[\bar{c}_N - \underline{c}_N\Big]^{-1} \cdot \Big[(n+1)^2 \cdot \bar{c}\bar{c}_N - [c_N + \underline{c}_N]^2\Big] \qquad \textit{On the other,} \tag{3.13}$$

$$g^{(v^*)}(S) \leqslant g^{(v^*)}(N) \qquad \textit{for all } S \subseteq N,\ S \neq \emptyset \tag{3.14}$$

Proof of Theorem 3.2. (The full proof consists of two parts.) **Part 1.**

Firstly, we check $g^{(v^*)}(N) \leqslant 0$ or equivalently, by Proposition (1.1) 1.1, $v(N) \geqslant \sum_{i\in N} v(\{i\})$. Put the substitution $x = a + c_N$. By using (3.11), it holds that $4\cdot n\cdot v(\{i\}) = [x - (n+1)\cdot c_i]^2$ for all $i \in N$ as well as $4\cdot n\cdot v(N) = n\cdot[a - \underline{c}_N]^2 = n\cdot[x - (c_N + \underline{c}_N)]^2$. Thus, we obtain the following chain of equalities:

$$\begin{aligned}
&4\cdot n\cdot\Big[v(N) - \sum_{i\in N} v(\{i\})\Big] \\
&= n\cdot\Big[x - (c_N + \underline{c}_N)\Big]^2 - \sum_{i\in N}\Big[x - (n+1)\cdot c_i\Big]^2 \\
&= n\cdot\Big[x^2 - 2\cdot x\cdot(c_N + \underline{c}_N) + \Big[c_N + \underline{c}_N\Big]^2\Big] \\
&- \sum_{i\in N}\Big[x^2 - 2\cdot x\cdot(n+1)\cdot c_i + (n+1)^2\cdot(c_i)^2\Big] \\
&= 2\cdot x\cdot\Big[(n+1)\cdot c_N - n\cdot(c_N + \underline{c}_N)\Big]
\end{aligned}$$

$$+ n \cdot \left[c_N + \underline{c}_N \right]^2 - (n+1)^2 \cdot \sum_{i \in N} (c_i)^2$$

$$= 2 \cdot x \cdot \left[c_N - n \cdot \underline{c}_N \right] + n \cdot \left[c_N + \underline{c}_N \right]^2 - (n+1)^2 \cdot n \cdot \bar{c}\bar{c}_N$$

$$= n \cdot \left[2 \cdot x \cdot \left[\bar{c}_N - \underline{c}_N \right] + \left[c_N + \underline{c}_N \right]^2 - (n+1)^2 \cdot \bar{c}\bar{c}_N \right]$$

So far, we conclude that $g^{(v^*)}(N) \leqslant 0$ if and only if

$$2 \cdot x \cdot \left[\bar{c}_N - \underline{c}_N \right] \geqslant (n+1)^2 \cdot \bar{c}\bar{c}_N - \left[c_N + \underline{c}_N \right]^2 \qquad \text{where} \qquad x = a + c_N \quad (3.15)$$

or equivalently, $a \geqslant \frac{L_1}{2} - c_N$ where the critical lower bound L_1 is given by (3.13). Notice that the quadratic term x^2 vanishes in the inequality (3.15). □

Remark 3.3. For future convenience, we treat an alternative proof of part 1 of Theorem 3.2 in the appendix. They differ in that this second proof is based on the variable a itself instead of the variable x. In the new setting, the description of the critcal lower bound (3.15) has to be replaced by a similar inequality:

$$2 \cdot a \cdot \left[\bar{c}_N - \underline{c}_N \right] \geqslant (n^2 + 2 \cdot n) \cdot \left[\bar{c}\bar{c}_N - (\bar{c}_N)^2 \right] + \bar{c}\bar{c}_N - (\underline{c}_N)^2 \qquad (3.16)$$

This second approach yields an alternative description of the same lower bound of the form

$$L_2 = \left[\bar{c}_N - \underline{c}_N \right]^{-1} \cdot \left[(n^2 + 2 \cdot n) \cdot \left[\bar{c}\bar{c}_N - (\bar{c}_N)^2 \right] + \bar{c}\bar{c}_N - (\underline{c}_N)^2 \right] \qquad (3.17)$$

It is left to the reader to verify the validity of the equality $L_2 = L_1 - 2 \cdot c_N$. □

Proof of Theorem 3.2. (The full proof consists of two parts.) **Part 2.**

Secondly, we check $g^{(v^*)}(S) \leqslant g^{(v^*)}(N)$ for all $S \subseteq N$, $S \neq \emptyset$, or equivalently, by Theorem 1.3, $v(S) \leqslant \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. Put the fundamental substitutions

$$x := a + c_N \qquad \text{as well as} \qquad A_S := c_S + (n + 1 - s) \cdot \underline{c}_S$$

From (2.10), we derive the following shortened notation for the worth of any multi-person coalition S as well as the one-person coalitions respectively, in the Stackelberg oligopoly game.

$$4 \cdot n \cdot v(S) = \frac{n \cdot \Big[x - A_S\Big]^2}{n+1-s} = \frac{n \cdot \Big[x^2 - 2 \cdot x \cdot A_S + (A_S)^2\Big]}{n+1-s} \quad \text{for all } S \subseteq N,$$

$$4 \cdot n \cdot v(\{i\}) = \Big[x - (n+1) \cdot c_i\Big]^2 \qquad \text{for all } i \in N, \text{ and next}$$

$$4 \cdot n \cdot \sum_{i \in S} v(\{i\}) = \sum_{i \in S} \Big[x^2 - 2 \cdot x \cdot (n+1) \cdot c_i + (n+1)^2 \cdot (c_i)^2\Big]$$

$$= s \cdot x^2 - 2 \cdot x \cdot (n+1) \cdot c_S + (n+1)^2 \cdot \sum_{i \in S} (c_i)^2$$

$$4 \cdot n \cdot \Big[\sum_{i \in S} v(\{i\}) - v(S)\Big] = \alpha_2 \cdot x^2 + \alpha_1 \cdot x + \alpha_0 \tag{3.18}$$

Our main goal is to describe the coalitional notion of surplus in terms of a quadratic function of the variable x, say $f(x) = \alpha_2 \cdot x^2 + \alpha_1 \cdot x + \alpha_0$ where $\alpha_2 > 0$.

Definition 3.4. The three real numbers α_k, $k = 0, 1, 2$, are given as follows:

$$\alpha_2 = s - \frac{n}{n+1-s} = \frac{(n-s) \cdot (s-1)}{n+1-s} \tag{3.19}$$

$$\alpha_1 = \frac{1}{n+1-s} \cdot \Big[-2 \cdot (n+1-s) \cdot (n+1) \cdot c_S + 2 \cdot n \cdot A_S\Big] \tag{3.20}$$

$$= \frac{1}{n+1-s} \cdot \Big[2 \cdot \Big[n - (n+1) \cdot (n+1-s)\Big] \cdot c_S + 2 \cdot n \cdot (n+1-s) \cdot \underline{c}_S\Big] \tag{3.21}$$

$$\alpha_0 = \frac{1}{n+1-s} \cdot \Big[(n+1-s) \cdot (n+1)^2 \cdot \sum_{i \in S} (c_i)^2 - n \cdot (A_S)^2\Big] \tag{3.22}$$

Clearly, $\alpha_2 > 0$ since $s \neq n$, $s \neq 1$. Further, it holds that $\alpha_1 < 0$ due to $\bar{c}_S \geqslant \underline{c}_S$ as well as

$$n \cdot (n+1-s) < s \cdot \Big[(n+1) \cdot (n+1-s) - n\Big] \quad \text{or equivalently,} \quad n < s \cdot (n+1-s)$$

So far, we conclude that the quadratic function $f(x) = \alpha_2 \cdot x^2 + \alpha_1 \cdot x + \alpha_0$ attains its minimum at $x = \frac{-\alpha_1}{2 \cdot \alpha_2}$ and the corresponding minimal function values $f(\frac{-\alpha_1}{2 \cdot \alpha_2}) = \frac{-(\alpha_1)^2}{4 \cdot \alpha_2} + \alpha_0$. This minimal function value is non-negative if and only if $4 \cdot \alpha_0 \cdot \alpha_2 \geqslant (\alpha_1)^2$. For the sake of the forthcoming computational matters, recall that $\bar{c}_T = \frac{c_T}{t}$ for all $T \subseteq N$, $T \neq \emptyset$, as well as (2.9). In order to apply shortened notation, put the substitution $\delta_s = n - (n+1) \cdot (n+1-s)$. As one out of two options for a possible representation of α_1, we choose (3.21) to evaluate the square of α_1, as well as the product $4 \cdot \alpha_2 \cdot \alpha_0$. Finally, we arrive at a reasonable description of their difference as stated in the next lemma.

Lemma 3.5. *Consider the setting of Definition 3.4. Then the following equality holds:*

(i)

$$\alpha_2 \cdot \alpha_0 - \frac{(\alpha_1)^2}{4} = s \cdot (s-1) \cdot (n-s) \cdot (n+1)^2 \cdot \Big[\overline{cc}_S - (\bar{c}_S)^2 \Big] - s \cdot n \cdot (n+1-s)^2 \cdot \Big[\underline{c}_S - \bar{c}_S \Big]^2 \tag{3.23}$$

(ii) Moreover, a sufficient condition for $4 \cdot \alpha_2 \cdot \alpha_0 - (\alpha_1)^2 \geqslant 0$ *is given by the following inequality:*

$$(s-1) \cdot (n-s) \cdot (n+1)^2 \cdot \Big[\overline{cc}_S - (\bar{c}_S)^2 \Big] \geqslant n \cdot (n+1-s)^2 \cdot \Big[\underline{c}_S - \bar{c}_S \Big]^2 \tag{3.24}$$

(iii) The sufficient condition (3.24) holds.

Proof of Lemma 3.5. The current approach proceeds as follows. Firstly, we evaluate the square $(\alpha_1)^2$ and secondly, we study the two contributions within the product $\alpha_2 \cdot \alpha_0$, particularly the main contribution $-\frac{(n-s)\cdot(s-1)}{n+1-s} \cdot n \cdot (A_S)^2$, while its second contribution will not be changed at all and kept till the end in the form

$$(n+1)^2 \cdot \frac{(n-s)\cdot(s-1)}{n+1-s} \cdot \sum_{i \in S} (c_i)^2 \quad \text{that is} \quad (n+1)^2 \cdot \frac{(n-s)\cdot(s-1)}{n+1-s} \cdot s \cdot \overline{cc}_S$$

In order to apply shortened notation, put $\rho_s = \frac{1}{n+1-s}$. Firstly, straightforward calculations involving the relevant square $(\alpha_1)^2$ and secondly, straightforward calculations involving the remaining part of the product $\alpha_2 \cdot \alpha_0$, yield the following:

$$\begin{aligned}
\frac{(\alpha_1)^2}{4} &= (\rho_s)^2 \cdot \Big[\delta_s \cdot c_S + n \cdot (n+1-s) \cdot \underline{c}_S \Big]^2 \\
&= (\rho_s)^2 \cdot \Big[(\delta_s)^2 \cdot (c_S)^2 + n^2 \cdot (n+1-s)^2 \cdot (\underline{c}_S)^2 + 2 \cdot \delta_s \cdot c_S \cdot n \cdot (n+1-s) \cdot \underline{c}_S \Big] \\
&= (\rho_s)^2 \cdot \Big[(\delta_s)^2 \cdot s^2 \cdot (\bar{c}_S)^2 + n^2 \cdot (n+1-s)^2 \cdot (\underline{c}_S)^2 + 2 \cdot \delta_s \cdot s \cdot n \cdot (n+1-s) \cdot \bar{c}_S \cdot \underline{c}_S \Big]
\end{aligned} \tag{3.25}$$

In addition,

$$+\frac{(n-s)\cdot(s-1)}{n+1-s}\cdot\frac{n\cdot(A_S)^2}{n+1-s}=(\rho_s)^2\cdot(n-s)\cdot(s-1)\cdot n\cdot\Big[c_S+(n+1-s)\cdot\underline{c}_S\Big]^2$$
$$=(\rho_s)^2\cdot(n-s)\cdot(s-1)\cdot n\cdot\Big[s^2\cdot(\bar{c}_S)^2+(n+1-s)^2\cdot(\underline{c}_S)^2+2\cdot s\cdot(n+1-s)\cdot\bar{c}_S\cdot\underline{c}_S\Big] \tag{3.26}$$

Summing up the two negative expressions (3.25)–(3.26) to be multiplied by the square $(\rho_s)^2$ yields

$$s^2\cdot\Big[(\delta_s)^2+(n-s)\cdot(s-1)\cdot n\Big]\cdot(\bar{c}_S)^2 \tag{3.27}$$

$$+\,(n+1-s)^2\cdot\Big[n^2+(n-s)\cdot(s-1)\cdot n\Big]\cdot(\underline{c}_S)^2 \tag{3.28}$$

$$+\,2\cdot s\cdot n\cdot(n+1-s)\cdot\Big[\delta_s+(n-s)\cdot(s-1)\Big]\cdot\bar{c}_s\cdot\underline{c}_S \tag{3.29}$$

In order to simplify these calculations, we use the following simple equalities:

$$(n-s)\cdot(s-1)+n=s\cdot(n+1-s)$$
$$(n-s)\cdot(s-1)\cdot n+n^2=s\cdot n\cdot(n+1-s)$$
$$\delta_s+(n-s)\cdot(s-1)=-(n+1-s)^2$$
$$-s^2\Big[(\delta_s)^2+(n-s)(s-1)n\Big]+n\cdot s\cdot(n+1-s)^3=-(n+1-s)(n+1)^2(n-s)(s-1) \tag{3.30}$$

The final computations are as follows:

$$\begin{aligned}(\rho_s)^{-2}\cdot\Big[\alpha_2\cdot\alpha_0-\frac{(\alpha_1)^2}{4}\Big]&=-s\cdot n\cdot(n+1-s)^3\cdot(\underline{c}_S)^2\\&\quad+2\cdot s\cdot n\cdot(n+1-s)^3\cdot\bar{c}_S\cdot\underline{c}_S\\&\quad-s^2\cdot\Big[(\delta_s)^2+(n-s)\cdot(s-1)\cdot n\Big]\cdot(\bar{c}_S)^2\\&\quad+(n+1-s)\cdot(n+1)^2\cdot(n-s)\cdot(s-1)\cdot s\cdot\bar{c}\bar{c}_S\\&=-n\cdot s\cdot(n+1-s)^3\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2\end{aligned}$$

$$-s^2\cdot\Big[(\delta_s)^2+(n-s)\cdot(s-1)\cdot n\Big]\cdot(\bar{c}_S)^2+\Big[n\cdot s\cdot(n+1-s)^3\Big]\Big]\cdot(\bar{c}_S)^2$$

$$+\,(n+1-s)\cdot(n+1)^2\cdot(n-s)\cdot(s-1)\cdot s\cdot\overline{cc}_S$$

$$=-n\cdot s\cdot(n+1-s)^3\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2$$

$$+\,(n+1-s)\cdot(n+1)^2\cdot(n-s)\cdot(s-1)\cdot s\cdot(\overline{cc}_S-(\bar{c}_S)^2)$$

The last equality is due to (3.30). □

4. APPENDIX: Alternative Proofs.

Alternative proof of Theorem 3.2. The full proof consists of two parts. **Part 1.**

Firstly, we check $g^{(v^*)}(N)\leqslant 0$ or equivalently, by Proposition 1.1, $v(N)\geqslant\sum_{i\in N}v(\{i\})$. By using (3.11), we obtain the following chain of equalities:

$$4\cdot n\cdot\Big[v(N)-\sum_{i\in N}v(\{i\})\Big]$$

$$=n\cdot\Big[a-\underline{c}_N\Big]^2-\sum_{i\in N}\Big[a+c_N-(n+1)\cdot c_i\Big]^2$$

$$=n\cdot\Big[a-\underline{c}_N\Big]^2-\sum_{i\in N}\Big[a^2+2\cdot a\cdot\Big[c_N-(n+1)\cdot c_i\Big]+\Big[c_N-(n+1)\cdot c_i\Big]^2\Big]$$

$$=n\cdot\Big[a^2-2\cdot a\cdot\underline{c}_N+(\underline{c}_N)^2\Big]-n\cdot a^2-2\cdot a\cdot\Big[n\cdot c_N-(n+1)\cdot c_N\Big]$$

$$-\sum_{i\in N}\Big[(c_N)^2-2\cdot(n+1)\cdot c_N\cdot c_i+(n+1)^2\cdot(c_i)^2\Big]$$

$$=2\cdot a\cdot\Big[c_N-n\cdot\underline{c}_N\Big]+n\cdot(\underline{c}_N)^2-\Big[n-2\cdot(n+1)\Big]\cdot(c_N)^2-(n+1)^2\cdot\sum_{i\in N}(c_i)^2$$

$$=2\cdot a\cdot n\cdot\Big[\bar{c}_N-\underline{c}_N\Big]+n\cdot(\underline{c}_N)^2+(n+2)\cdot(c_N)^2-(n+1)^2\cdot n\cdot\overline{cc}_N$$

$$=n\cdot\Big[2\cdot a\cdot\Big[\bar{c}_N-\underline{c}_N\Big]+(\underline{c}_N)^2+n\cdot(n+2)\cdot(\bar{c}_N)^2-(n+1)^2\cdot\overline{cc}_N\Big]$$

So far, we conclude that $g^{(v^*)}(N)\leqslant 0$ if and only if

$$2\cdot a\cdot\Big[\bar{c}_N-\underline{c}_N\Big]\geqslant(n+1)^2\cdot\overline{cc}_N-(\underline{c}_N)^2-n\cdot(n+2)\cdot(\bar{c}_N)^2 \tag{4.31}$$

or equivalently,

$$a \geqslant \frac{1}{2} \cdot \left[\bar{c}_N - \underline{c}_N\right]^{-1} \cdot \left[(n+1)^2 \cdot \overline{cc}_N - (\underline{c}_N)^2 - n \cdot (n+2) \cdot (\bar{c}_N)^2\right] = \frac{L_2}{2}$$

Here the critical number L_2 represents the lower bound given by

$$L_2 = \left[\bar{c}_N - \underline{c}_N\right]^{-1} \cdot \left[(n+1)^2 \cdot \overline{cc}_N - (\underline{c}_N)^2 - n \cdot (n+2) \cdot (\bar{c}_N)^2\right] \quad \text{Notice that}$$

$$L_2 = \left[\bar{c}_N - \underline{c}_N\right]^{-1} \cdot \left[(n^2 + 2 \cdot n) \cdot \left[\overline{cc}_N - (\bar{c}_N)^2\right] + \overline{cc}_N - (\underline{c}_N)^2\right] \quad \text{while} \tag{4.32}$$

$$L_1 = \left[\bar{c}_N - \underline{c}_N\right]^{-1} \cdot \left[(n+1)^2 \cdot \overline{cc}_N - \left[c_N + \underline{c}_N\right]^2\right] \tag{4.33}$$

Recall that $\overline{cc}_N \geqslant (\bar{c}_N)^2 \geqslant (\underline{c}_N)^2$. Thus, $L_2 > 0$. In fact, it is left to the reader to verify the validity of the equality $\frac{L_2}{2} = \frac{L_1}{2} - c_N$, that is $L_2 = L_1 - 2 \cdot c_N$. □

Alternative proof of Theorem 3.2. The full proof consists of two parts. **Part 1.**

Secondly, we check $g^{(v^*)}(S) \leqslant g^{(v^*)}(N)$ for all $S \subseteq N$, $S \neq \emptyset$, or equivalently, by Theorem 1.3, $v(S) \leqslant \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. This second proof differs from the first one in that it uses different fundamental substitutions

$$y_S := a + c_{N \setminus S} \quad \text{as well as} \quad c(S,i) := c_S - (n+1) \cdot c_i \quad \text{for all} \quad i \in S,$$
$$\text{instead of} \quad x = a + c_N$$

Fix $S \subseteq N$, $S \neq N$, $S \neq \emptyset$. Note that $\sum\limits_{i \in S} c(S,i) = -(n+1-s) \cdot c_S$. By using (2.10) it holds that

$$4 \cdot n \cdot v(S) = \frac{n}{n+1-s} \cdot \left[a + c_{N \setminus S} - (n+1-s) \cdot \underline{c}_S\right]^2$$

$$= \frac{n}{n+1-s} \cdot \left[y_S - (n+1-s) \cdot \underline{c}_S\right]^2 \quad \text{Further,}$$

$$4 \cdot n \cdot v(\{i\}) = \left[a + c_N - (n+1) \cdot c_i\right]^2 = \left[y_S + c(S,i)\right]^2 \quad \text{for all } i \in N$$

Recall that $\sum\limits_{i \in S} c(S,i) = -(n+1-s) \cdot c_S$ for all $S \subseteq N$. We obtain the following chain of equalities:

$$4 \cdot n \cdot \left[\sum_{i \in S} v(\{i\}) - v(S)\right] = \sum_{i \in S} \left[y_S + c(S,i)\right]^2$$
$$- \frac{n}{n+1-s} \cdot \left[y_S - (n+1-s) \cdot \underline{c}_S\right]^2$$
$$= \sum_{i \in S} \left[(y_S)^2 + 2 \cdot y_S \cdot c(S,i) + (c(S,i))^2\right]$$
$$- \frac{n}{n+1-s} \cdot \left[(y_S)^2 - 2 \cdot y_S \cdot (n+1-s) \cdot \underline{c}_S + (n+1-s)^2 \cdot (\underline{c}_S)^2\right]$$
$$= s \cdot (y_S)^2 + 2 \cdot y_S \cdot \sum_{i \in S} c(S,i) + \sum_{i \in S} (c(S,i))^2$$
$$- \frac{n}{n+1-s} \cdot (y_S)^2 + 2 \cdot y_S \cdot n \cdot \underline{c}_S - n \cdot (n+1-s) \cdot (\underline{c}_S)^2$$

Our main goal is to describe the coalitional notion of surplus in terms of a quadratic function of the variable y, say $g(y) = \beta_2 \cdot y^2 + \beta_1 \cdot y + \beta_0$ where $\beta_2 > 0$. □

Definition 4.1. The three real numbers β_k, $k = 0, 1, 2$, are given as follows:

$$\beta_2 = s - \frac{n}{n+1-s} = \frac{(n-s) \cdot (s-1)}{n+1-s} \tag{4.34}$$

$$\beta_1 = 2 \cdot \left[n \cdot \underline{c}_S - s \cdot (n+1-s) \cdot \bar{c}_S\right] \tag{4.35}$$

$$\beta_0 = \sum_{i \in S} (c(S,i))^2 - n \cdot (n+1-s) \cdot (\underline{c}_S)^2 \tag{4.36}$$

Clearly, $\beta_2 > 0$ since $s \neq n$, $s \neq 1$. Further, it holds that $\beta_1 < 0$ due to $\bar{c}_S \geqslant \underline{c}_S$ as well as $n < s \cdot (n+1-s)$ since $s \cdot (s-1) < n \cdot (s-1)$. So far, we conclude that the quadratic function $g(y) = \beta_2 \cdot y^2 + \beta_1 \cdot y + \beta_0$ attains its minimum at $y = \frac{-\beta_1}{2 \cdot \beta_2}$ and the corresponding minimal function values $g(\frac{-\beta_1}{2 \cdot \beta_2}) = \frac{-(\beta_1)^2}{4 \cdot \beta_2} + \beta_0$. This minimal function value is non-negative if and only if $4 \cdot \beta_0 \cdot \beta_2 \geqslant (\beta_1)^2$. For the sake of the forthcoming computational matters, recall (3.11) as well as $\bar{c}_T = \frac{c_T}{|T|}$ for all $T \subseteq N$, $T \neq \emptyset$. Recall the fundamental substitution $c(S,i) := c_S - (n+1) \cdot c_i$ for all $i \in S$. Based upon (4.34)–(4.36), we evaluate the square of β_1, as well as the product $4 \cdot \beta_2 \cdot \beta_0$. Finally, we arrive at a reasonable description of their difference as stated in the next lemma.

Lemma 4.2. Consider the setting of Definition 4.1. Firstly, we evaluate β_0 in the following form:

$$\begin{aligned}
\beta_0 &= \sum_{i\in S}(c(S,i))^2 - n\cdot(n+1-s)\cdot(\underline{c}_S)^2\\
&= \sum_{i\in S}\Big[c_S-(n+1)\cdot c_i\Big]^2 - n\cdot(n+1-s)\cdot(\underline{c}_S)^2\\
&= s\cdot(c_S)^2 - 2\cdot(n+1)\cdot(c_S)^2 + (n+1)^2\cdot s\cdot\overline{cc}_S - n\cdot(n+1-s)\cdot(\underline{c}_S)^2\\
&= (-2\cdot n-2+s)\cdot(c_S)^2 + (n+1)^2\cdot s\cdot\overline{cc}_S - n\cdot(n+1-s)\cdot(\underline{c}_S)^2
\end{aligned}$$

Secondly, we add the following chain of computations:

$$\begin{aligned}
&\beta_2\cdot\beta_0 - \frac{(\beta_1)^2}{4} = \beta_2\cdot\beta_0 - \Big[n\cdot\underline{c}_S - s\cdot(n+1-s)\cdot\bar{c}_S\Big]\\
&= \frac{(n-s)\cdot(s-1)}{n+1-s}\cdot\Big[(-2\cdot n-2+s)\cdot s^2\cdot(\bar{c}_S)^2 + (n+1)^2\cdot s\cdot\overline{cc}_S\\
&- n\cdot(n+1-s)\cdot(\underline{c}_S)^2\Big] - \Big[n^2\cdot(\underline{c}_S)^2 - 2\cdot n\cdot s\cdot(n+1-s)\cdot\underline{c}_S\cdot\bar{c}_S\\
&+ s^2\cdot(n+1-s)^2\cdot(\bar{c}_S)^2\Big]\\
&= \Big[-n\cdot(n-s)\cdot(s-1)-n^2\Big]\cdot(\underline{c}_S)^2\\
&+ \Big[\frac{(n-s)\cdot(s-1)}{n+1-s}\cdot(-2\cdot n-2+s)\cdot s^2 - s^2\cdot(n+1-s)^2\Big]\cdot(\bar{c}_S)^2\\
&+ \frac{(n-s)\cdot(s-1)}{n+1-s}\cdot(n+1)^2\cdot s\cdot\overline{cc}_S + 2\cdot n\cdot s\cdot(n+1-s)\cdot\underline{c}_S\cdot\bar{c}_S\\
&= -n\cdot s\cdot(n+1-s)\cdot(\underline{c}_S)^2 + 2\cdot n\cdot s\cdot(n+1-s)\cdot\underline{c}_S\cdot\bar{c}_S\\
&+ \frac{(n-s)\cdot(s-1)}{n+1-s}\cdot(n+1)^2\cdot s\cdot\overline{cc}_S\\
&- \frac{s^2}{n+1-s}\cdot\Big[(n-s)\cdot(s-1)\cdot(2\cdot n+2-s) + (n+1-s)^3\Big]\cdot(\bar{c}_S)^2\\
&= -n\cdot s\cdot(n+1-s)\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2 + \frac{(n-s)\cdot(s-1)}{n+1-s}\cdot(n+1)^2\cdot s\cdot\overline{cc}_S\\
&+ \Big[n\cdot s\cdot(n+1-s) - \frac{s^2}{n+1-s}\Big[(n-s)\cdot(s-1)\cdot(2\cdot n+2-s)\\
&+ (n+1-s)^3\Big]\Big]\cdot(\bar{c}_S)^2
\end{aligned}$$

Thus,

$$
\begin{aligned}
&(n+1-s)\cdot\Big[\beta_0\cdot\beta_2-\frac{(\beta_1)^2}{4}\Big]\\
&=-s\cdot n\cdot(n+1-s)^2\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2+(s-1)\cdot(n-s)\cdot(n+1)^2\cdot s\cdot\overline{c}\overline{c}_S\\
&+\Big[n\cdot s\cdot(n+1-s)^2-s^2\cdot(n+1-s)^3\\
&-s^2\cdot(n-s)\cdot(s-1)\cdot(2\cdot n+2-s)\Big]\cdot(\bar{c}_S)^2\\
&=-s\cdot n\cdot(n+1-s)^2\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2+s\cdot(s-1)\cdot(n-s)\cdot(n+1)^2\cdot\overline{c}\overline{c}_S\\
&-s\cdot(s-1)\cdot(n-s)\cdot(n+1)^2\cdot(\bar{c}_S)^2\\
&=-s\cdot n\cdot(n+1-s)^2\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2\\
&+s\cdot(s-1)\cdot(n-s)\cdot(n+1)^2\cdot\Big[\overline{c}\overline{c}_S-(\bar{c}_S)^2\Big]
\end{aligned}
$$

It suffices to prove the next inequality:

$$
(s-1)\cdot(n-s)\cdot(n+1)^2\cdot\Big[\overline{c}\overline{c}_S-(\bar{c}_S)^2\Big]\geqslant n\cdot(n+1-s)^2\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2 \tag{4.37}
$$

or equivalently, by (2.9)

$$
(s-1)\cdot(n-s)\cdot(n+1)^2\cdot\sum_{i\in S}\Big[c_i-\bar{c}_S\Big]^2\geqslant s\cdot n\cdot(n+1-s)^2\cdot\Big[\underline{c}_S-\bar{c}_S\Big]^2
$$

By (4.37), we observe the same inequality as in Lemma 3.5 and hence, we may state the same sufficiency condition. □

References

Chander, P., and H. Tulkens, (1997). *The Core of an Economy with Multilateral Environmental Externalities.* International Journal of Game Theory, **26**, 379–401.

Driessen, T. S. H. (1988). *Cooperative Games, Solutions, and Applications.* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Driessen, T. S. H., and H. I. Meinhardt (2005). Convexity of oligopoly games without transferable technologies, Mathematical Social Sciences, **50**, 102–126.

Driessen, T. S. H., Hou, Dongshuang, and Aymeric Lardon (2011a). *Stackelberg Oligopoly TU-games: Characterization of the core and* 1*-concavity of the dual game.* Working Paper, Department of Applied Mathematics, University of Twente, Enschede, The Netherlands.

Driessen, T. S. H., Hou, Dongshuang, and Aymeric Lardon (2011b). *A necessary and sufficient condition for the non-emptiness of the core in Stackelberg Oligopoly TU games.* Working paper, University of Saint-Etienne, France.

Lardon, A. (2009). *The γ-core of Cournot oligopoly TU games with capacity constraints.* Working paper, University of Saint-Etienne, France.

Lardon, A. (2011). *PhD Thesis: Five Essays on Cooperative Oligopoly Games*, University of Saint-Etienne, Universite Jean Monnet de Saint-Etienne, France, 13 October 2011.

Norde, H., Do, K. H. P., and S. H. Tijs (2002). *OLigopoly Games with and without transferable technologies.* Mathematical Social Sciences, **43**, 187–207.

Sherali, H. D., Soyster, A. L., and F. H. Murphy (1983). *Stackelberg–Nash–Cournot Equilibria, Characterizations and Computations.* Operations Research, **31(2)**, 253–276.

Tijs, S. H. (1981). *Bounds for the core and the τ-value.* In: Game Theory and Mathematical Economics, North-Holland Publishing Company, Amsterdam, 123–132.

Zhao, J. (1999). *A necessary and sufficient condition for the convexity in oligopoly games.* Mathematical Social Sciences, **37**, 187–207.

Zhao, J. (1999). *A β-core existence result and its application to oligopoly markets.* Games and Economics Behavior, **27**, 153–168.