On the Inverse Problem and the Coalitional Rationality for Binomial Semivalues of Cooperative TU Games

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Abstract In an earlier work (Dragan, 1991), we introduced the Inverse Problem for the Shapley Values and Weighted Shapley Values of cooperative transferable utilities games (TU-games). A more recent work (Dragan, 2004) is solving the Inverse Problem for Semivalues, a more general class of values of TU games. The Binomial Semivalues have been introduced recently (Puerte, 2000), and they are particular Semivalues, including among other values the Banzhaf Value. The Inverse problem for Binomial Semivalues was considered in another paper (Dragan, 2013). As these are, in general, not efficient values, the main tools in evaluating the fairness of such solutions are the Power Game and the coalitional rationality, as introduced in the earlier joint work (Dragan/Martinez-Legaz, 2001). In the present paper, we are looking for the existence of games belonging to the Inverse Set, and for which the a priori given Binomial Semivalue is coalitional rational, that is belongs to the Core of the Power Game. It is shown that there are games in the Inverse Set for which the Binomial Semivalue is coalitional rational, and also games for which it is not coalitional rational. An example is illustrating the procedure of finding games in the Inverse Set belonging to both classes of games just mentioned.

Keywords: Inverse Problem, Inverse Set, Semivalues, Binomial Semivalues, Power Game, Coalitional rationality.

Introduction

In a cooperative transferable utilities game (TU game), (N, v), defined by a finite set of players N, $n = |N|$, and the characteristic function $v : P(N) \to R$, with $v(\emptyset) = 0$, where $P(N)$ is the set of nonempty subsets of N, called coalitions, the main classical problem is to divide fairly the win of the grand coalition $v(N)$. An early solution was the Shapley Value (1953), defined axiomatically, to satisfy some fairness conditions (the axioms), and proved to be given by the formula

$$SH_i(N,v) = \sum_{S:i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}.[v(S) - v(S-\{i\})], \forall i \in N,$$

where $s = |S|, S \subseteq N$. It is easy to prove that SH is always efficient, that is we have the sum of components equal $v(N)$. The Shapley Value may belong to the Core of the game and in this case it is coalitional rational. The Semivalues, introduced by Dubey, Neyman and Weber (1981), who tried to avoid the efficiency axiom, in general are not efficient, so that they do not belong to the Core, and coalitional rationality is a problem in evaluating the fairness. The Binomial Semivalues were introduced by Puerte (2000), as extensions of the most known Semivalue, the Banzhaf Value (1965). To evaluate the fairness of such a solution, an algebraic structure is needed, and let us denote by $G(N)$ the set of all games with the set of players N. Two operations are defined, addition and scalar multiplication by

$$v = v_1 + v_2 \Leftrightarrow v(S) = v_1(S) + v_2(S), \forall S \subseteq N,$$

where (N, v_1) and (N, v_2) are any two TU games in $G(N)$, and

$$v = \gamma v_1 \Leftrightarrow v(S) = \gamma v_1(S), \forall S \subseteq N, \forall \gamma \in R,$$

where (N, v_1) is any TU game. It is easy to check that $G(N)$ is a linear vector space and its dimension is $2^n - 1$. Now, for every coalition $S \subseteq N$, the restriction of (N, v) to S is the game denoted by (S, v). Obviously, this finite set and the operations shown above define on S again a linear vector space, $G(S)$, and the union of all spaces $G(S), \forall S \subseteq N$, is denoted by G^N. A value Φ defined on G^N is any functional defined on each $G(S)$ with values in R^s. The Shapley Value is defined on G^N by a formula similar to the first formula above, where N has been changed into $S \subseteq N$, and n into s.

In the first section we introduce the Semivalues and the Binomial Semivalues; the solution of the Inverse Problem for Semivalues is shown in the second section. The Power Game and the coalitional rationality, together with the main result of the paper about Binomial Semivalues are discussed in the last section.

1. Semivalues and Binomial Semivalues

To give the definition of a Semivalue, we need a weight vector $p^n \in R^n$, satisfying a normalization condition

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s^n = 1, \tag{1.1}$$

together with the interpretation: p_s^n is the common weight of all coalitions of size s. This would be enough for the games in $G(N)$, but for all the games in G^N we need a sequence of weight vectors defined recursively as follows:

$$p_s^{n-1} = p_s^n + p_{s+1}^n, \quad s = 1, 2, ..., n-1, \tag{1.2}$$

will give the weight vector for the space $G(S)$ with $|S| = n - 1$. Then, the sequence of weight vectors $p^{n-2}, p^{n-3}, ..., p^2, p^1$ is defined by formulas similar to (1.2), going up to $p_1^1 = 1$. From (1.2) it is easy to show that these vectors satisfy a normalization condition like (1.1). It has been said earlier that (1.2) are the inverse Pascal triangle conditions, as any sequence shows triangles similar to those present when the Pascal triangle conditions were defined. Now, we can define the Semivalue associated with any sequence of weight vectors, $p^1, p^2, ..., p^n$ subject to (1.1) and connected by (1.2), as the value defined on G^N by

$$SE_i(T, v, p^t) = \sum_{S: i \in S \subseteq T} p_s^t [v(S) - v(S - \{i\})], \forall i \in T, \forall T \subseteq N, T \neq \emptyset. \tag{1.3}$$

Recall that here (T, v) is the TU game, a subgame of (N, v), obtained as a restriction of the characteristic function to T, so that $(T, v) \in G^T$. Notice that for $t \leqslant n$ the weight vectors

$$p_s^t = \frac{(s-1)!(t-s)!}{t!}, \quad s = 1,2,...,t, \quad p_s^t = 2^{1-s}, s = 1,2,...,t, \tag{1.4}$$

give the Shapley Value, and the Banzhaf Value, respectively.

Example 1. Consider the weight vector $p^3 = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8})$, and $p^2 = (\frac{3}{8}, \frac{5}{8})$, $p^1 = (1)$, derived via (1.2). Consider the game

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(1,2,3\}) = 1, \tag{1.5}$$

a constant sum game. We may compute the Semivalue of this game, by the formula (3), to get for the game (1.5) the outcome $SE(\{1,2,3\}, v, p^3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which is not efficient, because the sum of components makes a number different of 1. Then, the Semivalue does not belong to the Core, as the efficiency is missing. As this is the case in most situations, we have to define the coalitional rationality in some other way, following the ideas from the earlier work (Dragan/Martinez-Legaz, 2001), namely to consider the Power Game of the given game, in which the Semivalue is efficient and may be coalitional rational. Thus we have to define the Power Game. We may compute also the Semivalues of the subgames

$$SE(\{1,2\}, v, p^2) = SE(\{1,3\}, v, p^2) = SE(\{2,3\}, v, p^2) = (\frac{5}{8}, \frac{5}{8}). \tag{1.6}$$

They are looking all similar, due to the symmetry in (1.5) of the worth of the characteristic function of the players. Of course, the Semivalues of singletons are all zero. Then, we got a new game

$$w(\{1\}) = w(\{2\}) = w(\{3\}) = 0, w(\{1,2\}) = w(\{1,3\}) = w(\{2,3\}) = \frac{5}{4},$$
$$w(\{1,2,3\}) = \frac{3}{2}.$$

were we used (6) to satisfy the definition which follows.

Definition 1. For a TU game (N, v), the Power Game, relative to a Semivalue associated with a weight vector p^n, is the game (N, π, p^n) defined by formula

$$\pi(T, v, p^t) = \sum_{i \in T} SE_i(T, v), \forall T \subseteq N, \tag{1.7}$$

where the components of the Semivalue were given by formula (1.3).

As seen above in example 1, it is not easy to compute the Power Game by means of (7). However, this may be done by using the following result:

Theorem (Dragan, 2000). *Let a Semivalue $SE(N, v)$ be associated with the weight vector p^n, and the Power Game (N, π, p^n), relative to the Semivalue, given by formula (7). Then, we have*

$$\pi(T, v, p^t) = \sum_{S \subseteq T} [sp_s^t - (t-s)p_{s+1}^t] v(S), \forall T \subseteq N, \tag{1.8}$$

where p_{t+1}^t is an arbitrary number.

Example 2. Return to the game of Example 1 and recall that the computation of the Power Game, relative to the Semivalue, by using the definition (7), led to

$$\pi(\{1,2\},v,p^2) = \pi(\{1,3\},v,p^2) = \pi(\{2,3\},v,p^2) = \frac{5}{4}, \quad \pi(\{1,2,3\},v,p^3) = \frac{3}{2}. \tag{1.9}$$

Now, by using the theorem, as we have the bracket in (8) given by

$$2p_2^2 = \frac{5}{4}, \quad \text{and} \quad 2p_2^3 - p_3^3 = \frac{1}{8}, \quad 3p_3^3 = \frac{9}{8}, \tag{1.10}$$

with the second equality used three times for coalitions of size two, from (1.10) and formula (1.8) we get the same worth for the characteristic function as in (1.9). Beside definition 1, we illustrated the usefulness of (1.8), relative to (1.7).

Now, that we have the Power Game (1.9) for our given game (1.5), and the Semivalue is efficient in this game, which obviously is always true, we may check whether or not, the Semivalue is in the Core of the Power Game, and conclude that this is not true; hence according to the ideas from Dragan/Martinez-Legaz (2001), the Semivalue of (1.5) is not coalitional rational. Of course, it may be possible that the Semivalue does belong to the Core of the Power Game, and in this last case it will be coalitional rational. For example, if we consider the Banzhaf Value, the most popular Semivalue, defined by $p^3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and compute the Power Game, then we find out that the value is coalitional rational. Looking at our Example 2 we think that it is justified to introduce the following:

Definition 2. The Semivalue of a given game is coalitional rational if it belongs to the Core of the Power Game relative to the Semivalue (or, the Power Core of the game).

Now, let us consider the Binomial Semivalues, introduced by Puerte (2000) and discussed also in the work by Puerte/Freixas (2002), where definition 2 applies.

Definition 3. The Semivalue SE associated with the sequence of normalized weight vectors $p^1, p^2, ..., p^n$ connected by the inverse Pascal triangle relationships, is a Binomial Semivalue, if the weight vectors satisfy also for some number $r \in (0,1]$, the equalities

$$\frac{p_2^n}{p_1^n} = \frac{p_3^n}{p_2^n} = ... = \frac{p_n^n}{p_{n-1}^n} = r. \tag{1.11}$$

Now, that the concepts of Power Game and coalitional rationality have been explained and the computation of the Power Game has been given, we notice:

Lemma 1. *In G^N, the weights of a Binomial Semivalue are given by the equalities*

$$p_s^t = \frac{r^{s-1}}{(1+r)^{t-1}}, \quad s = 1,2,...,t, \quad t \leqslant n, \tag{1.12}$$

so that the Binomial Semivalue is given by the formula

$$SE_i(T,v,p^t) = \sum_{S:i\in S\subseteq T} \frac{r^{s-1}}{(1+r)^{t-1}}[v(S) - v(S-\{i\})], \forall i \in T, \forall T \subseteq N, T \neq \emptyset. \tag{1.13}$$

Proof. Follows from (1.11) and the normalization condition (1.1), as well as formula (1.3). □

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Remarks: (a) From the inverse Pascal triangle relationships it follows that all weight vectors of the sequence are given by formulas similar to (1.12), obtained for different values of $t = 1, 2, ..., n$, and while the game is replaced by the Value, a fact which justifies the study of the Binomial Semivalues, that should have properties similar to those of the Banzhaf Value.

Now, the problem to be considered in this paper is: **for a given vector** L, **and a given TU game** (N, v), **such that the Binomial Semivalue corresponding to a parameter** r **is not coalitional rational, find out in the Inverse Set of** L **a TU game** (N, w), **for which the Binomial Semivalue with parameter** r **is the same, but belongs to the Core of the Power Game.** Clearly, we have to explain the procedure in two steps:

- How do we find the Inverse Set of a Binomial Semivalue associated with a parameter r, and an a priori given value L? From our previous work on general Semivalues we know that this should be determined by an explicit formula, hence the Inverse Set will be available. This will be discussed in the second section.
- In the Inverse Set, how do we get a TU game for which the Binomial Semivalue of the original game is in the Core of its Power Game? This will be discussed in the last section. What about games for which the Binomial Semivalue of the original game does not belong to the Power Core? like the one from example 1.

2. The Null Space and the Inverse Set

In a recent work (Dragan, 2004), it has been shown that the Semivalue, associated with a sequence of weight vectors derived from p^n by means of formulas of type (1.2), has a potential function and for a game (N, v), it is given by the formula

$$P(N, v, p^n) = \sum_{S \subseteq N} p^n_s v(S). \tag{2.1}$$

Thus, for a Binomial Semivalue (2.1) becomes

$$P(N, v, p^n) = \frac{1}{(1+r)^{n-1}} \sum_{S \subseteq N} r^{s-1} v(S). \tag{2.2}$$

Obviously, to make (2.2) computationally better, the sum may be written as

$$P(N, v, p^n) = \frac{1}{(1+r)^{n-1}} \sum_{s=1}^{n} r^{s-1} d_s(N, v), \tag{2.3}$$

where $d_s(N, v)$ is the sum of worth of the characteristic function for all subcoalitions of size s in the set of players N.

Example 3. Returning to the game (1.5) of Example 1, and the weight vector $p^3 = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8})$, we see that $d_1(N, v) = 0, d_2(N, v) = 3, d_3(N, v) = 1$, so that from (2.3) we get $P(N, v, p^3) = \frac{1}{(1+r)^2}(3r + r^2)$, where the expressions of weights in terms of the ratio r were used. Now, for any coalition of size two, again from (2.3), we get the potential $P(N-\{i\},v,p^2)=\frac{r}{1+r}, i=1,2,3$, Hence, the Binomial Semivalue is

$$SE_i(N,v,p^3)=P(N,v,p^3)-P(N-\{i\},v,p^2)=\frac{3r+r^2}{(1+r)^2}-\frac{r}{1+r}=\frac{2r}{(1+r)^2},\quad i=1,2,3, \quad (2.4)$$

which for the Banzhaf Value ($r=1$) becomes as above $B(N,v)=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

In the more recent work (Dragan, 2013), we considered a basis for the space G(N), consisting of the linearly independent vectors in the set

$$W=\{w_T\in R^n : T\subseteq N, T\neq\emptyset\}, \quad (2.5)$$

defined by the formulas

$$w_T(T)=\frac{1}{p_t^t},\quad w_T(S)=\sum_{l=0}^{s-t}\frac{(-1)^l\binom{s-t}{l}}{p_{t+l}^{t+l}}, \forall S\supset T,\quad w_T(S)=0,\quad \text{otherwise.} \quad (2.6)$$

For all these vectors were computed the Binomial Semivalues associated with the weight vectors in the sequence generated by the vector p^n, via (1.2), by using the formulas for the Binomial Semivalues (1.3) obtained in Lemma 1. Here, we try to compute the same Binomial Semivalues by means of the potential, using

$$SE_i(N,w_T,p^n)=P(N,w_T,p^n)-P(N-\{i\},w_T,p^{n-1}), \forall i\in N, \quad (2.7)$$

like in example 3 above. To get the experience needed in this computation let us consider first an example.

Example 4. From (2.6), the general three person game shows the basis W, that taking into account (1.2) becomes

$$\begin{aligned}
w_{\{1\}}&=(1,0,0,1-\frac{1}{p_2^2},1-\frac{1}{p_2^2},0,1-\frac{2}{p_2^2}+\frac{1}{p_3^3})=(1,0,0,-\frac{1}{r},-\frac{1}{r},0,\frac{1}{r^2}),\\
w_{\{2\}}&=(0,1,0,1-\frac{1}{p_2^2},0,1-\frac{1}{p_2^2},1-\frac{2}{p_2^2}+\frac{1}{p_3^3})=(0,1,0,-\frac{1}{r},0,-\frac{1}{r},\frac{1}{r^2}),\\
w_{\{3\}}&=(0,0,1,0,1-\frac{1}{p_2^2},1-\frac{1}{p_2^2},1-\frac{2}{p_2^2}+\frac{1}{p_3^3})=(0,0,1,0,-\frac{1}{r},-\frac{1}{r},\frac{1}{r^2}),\\
w_{\{1,2\}}&=(0,0,0,\frac{1}{p_2^2},0,0,\frac{1}{p_2^2}-\frac{1}{p_3^3})=(0,0,0,\frac{1+r}{r},0,0,-\frac{1+r}{r^2}), \qquad (2.8)\\
w_{\{1,3\}}&=(0,0,0,0,\frac{1}{p_2^2},0,\frac{1}{p_2^2}-\frac{1}{p_3^3})=(0,0,0,0,\frac{1+r}{r},0,-\frac{1+r}{r^2}),\\
w_{\{2,3\}}&=(0,0,0,0,0,\frac{1}{p_2^2},\frac{1}{p_2^2}-\frac{1}{p_3^3})=(0,0,0,0,0,\frac{1+r}{r},-\frac{1+r}{r^2}),\\
w_{\{1,2,3\}}&=(0,0,0,0,0,0,\frac{1}{p_3^3})=(0,0,0,0,0,0,\frac{(1+r)^2}{r^2}).
\end{aligned}$$

Obviously, these are linearly independent vectors and their number equals the dimension of the space, hence they form a basis. Let us compute the potentials of all basic vectors, by using formula (2.3). For $i, j, k = 1, 2, 3$, we have the sums

$$\begin{aligned} &d_1(N, w_{\{i\}}) = 1, d_2(N, w_{\{i\}}) = -\frac{2}{r}, d_3(N, w_{\{i\}}) = \frac{1}{r^2}, \\ &d_1(N, w_{\{i,j\}}) = 0, d_2(N, w_{\{i,j\}}) = \frac{1+r}{r}, d_3(N, w_{\{i,j\}}) = -\frac{1+r}{r^2}, \\ &d_1(N, w_{\{i,j,k\}}) = d_2(N, w_{\{i,j,k\}}) = 0, d_3(N, w_{\{i,j,k\}}) = \frac{(1+r)^2}{r^2}, \end{aligned} \tag{2.9}$$

and from (2.3) we obtain

$$p_s^3 = \frac{r^{s-1}}{(1+r)^2}, \quad s = 1, 2, 3. \tag{2.10}$$

Now, by (2.9) and (2.10), for every basic vector (N, w) shown above in (2.8), formula (2.1) written as

$$P(N, w, p^3) = \frac{1}{(1+r)^2} \sum_{s=1}^{3} r^{s-1} d_s(N, w), \tag{2.11}$$

will give

$$P(N, w_s, p^3) = \frac{1}{(1+r)^{s-1}} [1 + (-1)]^{s-1} = 0, \forall S \subset N. \tag{2.12}$$

while the potential of the last game equals 1. Now, the potentials of the subgames would be computed, by using a formula similar to (2.11), namely

$$P(N - \{i\}, w_S, p^2) = \frac{1}{1+r} \sum_{s=0}^{2} r^{s-1} d_S(N, w), \forall S \subset N - \{i\}, i \in N. \tag{2.13}$$

In the same way, we get zero, and the potential of the last game equals 1.
Then, the Semivalues computed by the formula (2.7) are

$$SE(N, w_{\{i\}}, p^1) = (0, 0, 0), \forall i \in N, \tag{2.14}$$

$$SE_j(N, w_{N-\{i\}}, p^2) = -\delta_j^i, \quad i = 1, 2, 3; \quad SE(N, w_N, p^3) = (1, 1, 1).$$

A similar approach is hepful in proving, by computing the potentials, the result:

Theorem (Thm.3, Dragan, 2013). *Let a Binomial Semivalue be defined by a parameter r, and let W be the basis of the space provided by formulas (2.5), (2.6). Then, we have*

$$SE(N, w_T, p^t) = 0, \forall T \subset N, |T| \leqslant n - 2, T \neq \emptyset,$$

$$SE_i(N, w_{N-\{i\}}p^n) = -1, \forall i \in N,\ SE_j(N, w_{N-\{i\}}, p^n) = 0, j \neq i,\ \forall i \in N, \quad (2.15)$$

$$SE_i(N, w_N, p^n) = 1, \forall i \in N.$$

As a Corollary, by the linearity of the Semivalue, we get the Inverse Set, where r enters only the basic vectors:

Theorem (Thm.6, Dragan, 2013). *Let a Binomial Semivalue for a game* (N, w)*, defined by a parameter* r*, be* $SE(N, v, p^n) = L$*. Let* W *given by (2.5), (2.6) be a basis for the space* $G(N)$*. Then, the solution of the Inverse Problem is expressed by the formula*

$$w = \sum_{S \subset N, |S| \leqslant n-2} a_S w_S + a_N (w_N + \sum_{i \in N} w_{N-\{i\}}) - \sum_{i \in N} L_i w_{N-\{i\}}, \quad (2.16)$$

where the constants multiplying the basic games are arbitrary.

3. The Power Game and the coalitional rational inverse

Consider in the Inverse Set the family of games, to be called the "almost null games", obtained for $a_S = 0, \quad \forall S \subset N, \quad |S| \leqslant n-2, \quad S \neq \emptyset$. This family, as seen in the formula (2.16), is given by

$$w = a_N (w_N + \sum_{i \in N} w_{N-\{i\}}) - \sum_{i \in N} L_i w_{N-\{i\}}, \quad (3.1)$$

where a_N is the parameter of the family; of course, the parameter r of the Binomial Semivalue occurs in the basic vectors. Now, by using the weight vectors (2.6), written in terms of r, as shown in (Dragan, 2013), we have

$$w_T(T) = \frac{(1+r)^{t-1}}{r^{t-1}}, \forall T \subseteq N, \quad w_T(S) = \frac{(-1)^{s-t}(1+r)^{t-1}}{r^{s-1}}, \forall S \supset T, \quad (3.2)$$

and $w_T(S) = 0$, otherwise, so that from (3.2) we obtain

$$w_{N-\{i\}}(N - \{i\}) = \frac{(1+r)^{n-2}}{r^{n-2}}, \forall i \in N, \quad w_{N-\{i\}}(N) = -\frac{(1+r)^{n-2}}{r^{n-1}}, \quad (3.3)$$

$$w_{N-\{i\}}(N - \{j\}) = 0, \forall j \neq i, \quad w_N(N) = \frac{(1+r)^{n-1}}{r^{n-1}}. \quad (3.4)$$

In this way, from (3.3), (3.4), the components different of zero in (3.1) are:

$$w(N - (i\}) = (a_N - L_i)\frac{(1+r)^{n-2}}{r^{n-2}}, \forall i \in N, \quad (3.5)$$

$$w(N) = \frac{(1+r)^{n-2}}{r^{n-1}}[a_N(r - n + 1) + \sum_{i \in N} L_i]. \quad (3.6)$$

Now, we compute the Power Game of an arbitrary game in the almost null family set, given in (3.5), (3.6), where the null values of the characteristic function are omitted, and we get:

$$\pi(N-\{i\}, v, p^{n-1}) = (n-1)(a_N - L_i), \; \forall i \in N, \tag{3.7}$$

$$\pi(N, v, p^n) = \sum_{i \in N} L_i. \tag{3.8}$$

As stated in definition 2 before, a Semivalue of a given game is coalitional rational if it belongs to the Core of the Power Game, or Power Core. If the Binomial Semivalue is $L \geqslant 0$, and, as seen in (3.8), this is efficient in the Power Game, then the only Core conditions are those obtained from the coalitions of size $n-1$, that have the worth shown in (3.7), namely

$$\sum_{j \in N-\{i\}} L_j \geqslant (n-1)(a_N - L_i), \; \forall i \in N, \tag{3.9}$$

or

$$a_N \leqslant \frac{1}{n-1}\Big[\sum_{j \in N-\{i\}} L_j + (n-1)L_i\Big], \; \forall i \in N. \tag{3.10}$$

We proved

Theorem 2. *A Binomial Semivalue associated with the parameter r, and given by a nonnegative vector $L \in R^n$, is coalitional rational, in the Power Game of the game*

$$w = a_N(w_N + \sum_{i \in N} w_{N-\{i\}}) - \sum_{i \in N} L_i w_{N-\{i\}}, \tag{3.11}$$

If and only if a_N satisfies the inequality

$$a_N \leqslant \frac{1}{n-1} Min\{\sum_{j \in N-\{i\}} L_j + (n-1)L_i\}, \tag{3.12}$$

Notice that there is also an infinite set of games in the almost null Inverse Set for which the Binomial Semivalue is not coalitional rational.

Example 5. Return to the game considered in Example 1, for which the Banzhaf Value is $B(N, v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so that the inequality (3.12) is $a_N \leqslant 1$. We compute the almost null game, relative to the Banzhaf Value by (3.5) and (3.6) , and we obtain

$$w(\{1\}) = w(\{2\}) = w(\{3\}) = 1, w(\{1,2\}) = w(\{1,3\}) = w(\{2,3\}) = w(\{1,2,3\}) = 1, \tag{3.13}$$

that incidentally coincides with the given game (1.5). Obviously, the Banzhaf Value is the same, that is $B(N, w) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then, we compute the Power Game, of (3.13) by (3.7) and (3.8) and we get

$$\pi(\{1,2\}, w, p^2) = \pi(\{1,3\}, w, p^2) = \pi(\{2,3\}, w, p^2) = 1, \pi(\{1,2,3\}, w, p^3) = \frac{3}{2}, \tag{3.14}$$

while we have null values for the singletons. Now, in the new game (3.14) the old Banzhaf Value is efficient and we may see that it is also in the Power Core. This happened because for our game we have $a_N = 1$, which satisfies (3.12) and we have to modify only $w(\{1,2,3\}) = 1$, into $\pi(\{1,2,3\}, w, p^3) = \frac{3}{2}$, to get the Banzhaf Value in the Power Core. Notice that the Power Game does not have the same Banzhaf Value as the original one, or the almost null game in the Inverse Set; for example, in our case the Banzhaf Value of (3.14) is $B(N, \pi) = (\frac{5}{8}, \frac{5}{8}, \frac{5}{8})$. Obviously, this is not efficient again and the coalitional rationality conditions do not hold.

Consider the same game, but take $a_N = \frac{3}{2}$, and compute again the almost null game, relative to the Banzhaf Value by (3.5) and (3.6), and we obtain

$$w(\{1\}) = w(\{2\}) = w(\{3\}) = 0, w(\{1,2\}) = w(\{1,3\}) = w(\{2,3\}) = 2, \\ w(\{1,2,3\}) = 0, \quad (3.15)$$

which gives the same old Banzhaf Value. Further, we compute the Power Game and we get

$$\pi(\{1,2\}, w, p^2) = \pi(\{1,3\}, w, p^2) = \pi(\{2,3\}, w, p^2) = 2, \pi(\{1,2,3\}, w, p^3) = \frac{3}{2}, \quad (3.16)$$

in which the old Banzhaf Value is efficient, but it is not coalitional rational, because (3.12) does not hold. These two examples illustrate theorem 2 and the technique to build the game in the almost null inverse family, for which the given Banzhaf Value is coalitional rational.

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