Efficient Myerson Value for Union Stable Structures ⋆

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Abstract In this work, an axiomatization of a new value for union stable structures, efficient Myerson value, is shown by average equity, redundant fairness, superfluous component property and other three properties. And the independence of the axioms is illustrated. Besides, the difference of three values, efficient Myerson value, the two-step Shapley value and collective value, is shown.

Keywords: Union stable structure; average equity; redundant fairness.

1. Introduction

A situation in which a finite set of players can obtain payoffs by cooperation can be described by a cooperative game with transferable utility, shortly TU-game, being a pair consisting of a finite set of players and a characteristic function on the set of coalitions of players assigning a worth to each coalition of players. In practice, since the cooperation restrictions exist, only some subgroup of players can form a coalition. One way to describe the structure of partial cooperation in the context of cooperative games is to specify sets of feasible coalitions. Algaba, et al (Algaba, 2000) considered union stable systems as such sets. A union stable system of two intersecting feasible coalitions is also feasible, which can be interpreted as follows: players who are common members of two feasible coalitions are able to act as intermediaries to elicit cooperation among all the players in either of these coalitions, and so their union should be a feasible coalition. And a TU game with a union stable system is called a union stable structure. Besides, the union stable structure is a generalization of games with communication structure and games with permission structure, which are respectively proposed by Myerson (Myerson, 1977) and Gilles (Gilles,1992).

Hamiache (Hamiache, 2012) presented a matrix approach to construct extensions of the Shapley value on the games with coalition structures and communication structures. This paper aims to generalize this matrix approach to union stable structures, a generalized communication structures.

2. Preliminaries

2.1. Matrix Approach To Shapley Value

A cooperative game with transferable utility, or simply a TU-game, being a pair (N, v), where N is the finite set of all players, and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. The collection of all games with player set N is denoted by G. A game (N, v) is called an inessential game if for any two disjoint coalitions

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$S, T \subseteq N$, $v(S \cup T) = v(S) + v(T)$. Here the cardinality of any coalition $S \subseteq N$ is denoted by $|S|$ or the lower case letter s.

A payoff vector for a game is a vector $x \in \mathbb{R}^N$ assigning a payoff x_i to player $i \in N$. In the sequel, for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$. A single-valued solution is a function that assigns to any game $(N, v) \in G$ a unique payoff vector. The most well-known single-valued efficient solution is the Shapley value (1953) given by

$$Sh_i(N, v) = \sum_{i \in N \setminus S} \frac{s!(n-s-1)!}{n!}(v(S \cup i) - v(S)).$$

In fact, the explicit expression of Shapley value can be presented as $Sh_i(N, v) = (M^{Sh} \cdot v)[\{i\}]$, where the matrix $M^{Sh} = [M^{Sh}]_{i \in N, S \subseteq N \setminus \emptyset}$ is defined by

$$[M^{Sh}]_{i,S} = \begin{cases} \frac{(s-1)!(n-s)!}{n!}, & \text{if } i \in S\ , \\ -\frac{s!(n-s-1)!}{n!}, & \text{otherwise.} \end{cases} \tag{1}$$

Next, we recall the axiomatic characterization of Shapley value (Shapley, 1953) illustrated in Hamiache (Hamiache, 2001)and the matrix approach in Xu, et al (Xu, 2008) and Hamiache (Hamiache, 2010) for the analysis of the associated consistency.

For all games $(N, v) \in G$, the associated game (N, v_λ^{Sh}) defined in Hamiache (Hamiache, 2001)for all parameters $\lambda (0 < \lambda < \frac{2}{n})$ as follows,

$$v_\lambda^{Sh}(S) = v(S) + \lambda \sum_{i \in N \setminus S} [v(S \cup i) - v(S) - v(\{i\})] \text{ for all } S \subseteq N$$

Definition 1. A matrix A is called a row (column)-coalition matrix if its rows (column) are indexed by coalitions $S \subseteq N$ in the lexicographic order. A is called square-coalitional if it is both row-coalitional and column-coalitional. And a row-coalition matrix $A = [a]_{S,T}$ is called row-inessential or inessential, if $A = [a]_{S,T} = \sum_{i \in S} a_{i,T}$ for all $S \subseteq N$.

Since the associated game is a linear transformation of the original game, the associated game can be expressed as $v_\lambda^{Sh} = M_\lambda \cdot v$, where M_λ is a square-coalitional matrix of order 2^{n-1}, for detailed information, please refer to Xu, et al (Xu, 2008) and Hamiache (Hamiache, 2010).

The sequence of associated games illustrated in Hamiache (2001) can be expressed by matrix approach in Xu, et al (Xu, 2008) and Hamiache(Hamiache, 2010) as follows,

$$v_{k\lambda} = (v_{(k-1)\lambda})_\lambda^{Sh} = M_\lambda \cdot v_{(k-1)\lambda} = ... = (M_\lambda)^k \cdot v, \text{for all } k \geq 2\ .$$

And the sequence of games $\{(N, v_{k\lambda})\}_{k=1}^{\infty}$ converges to an inessential game (N, v_L), denote the corresponding coefficient matrix as M_L, then $\lim_{k \to \infty}(M_\lambda)^k = M_L$, and M_L is inessential.

2.2. Union Stable Structures

Definition 2. A union stable system is a pair $(N, \mathcal{F})$ with $\mathcal{F} \subseteq 2^N$ verifying that $\{i\} \in \mathcal{F}$ for all $i \in N$ and for all $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, $S \cup T \in \mathcal{F}$.

Given a union stable system $(N, \mathcal{F})$, $\mathcal{B}(\mathcal{F})$ is called the basis of $\mathcal{F}$, it is denoted by the set of all feasible coalitions which cannot be expressed as a union of feasible coalitions with nonempty intersection, the elements of the basis $\mathcal{B}(\mathcal{F})$ are called supports of $\mathcal{F}$. Especially, the set of non-singleton supports is denoted by $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$.

A union stable structure is a triple $(N, v, \mathcal{F})$, i.e., a TU game (N, v) with union stable system $(N, \mathcal{F})$. The set of such union stable structure with player set N is denoted by US^N.

Definition 3. Let $\mathcal{E} \subseteq 2^N$ be a set system and $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{E}$-component of S if $T \in \mathcal{E}$ and there exists no $T^{'} \in \mathcal{E}$ such that $T \subsetneq T^{'} \subseteq S$.

Especially, the collection of $\mathcal{F}$-component of N is denoted by $\beta = C_{\mathcal{F}}(N) = \{B_1, B_2, ..., B_r\}$ with $1 \leq r \leq |N|$ and $\cup_{B\in\beta} B = N$, $B_i \cap B_j \neq \emptyset$ for any $B_i, B_j \in \beta$.

Given $(N, v, \mathcal{F}) \in US^N$, define the intermediate game (β, v^β) by $v^\beta(R) = v(\cup_{B\in R} B)$ for all $R \subseteq \beta$ and the quotient game $(N, v^{\mathcal{F}})$ by $v^{\mathcal{F}}(S) = \sum_{T\in C_{\mathcal{F}}(S)} v(T)$ for all $S \subseteq N$.

For coalition $S \subseteq N \setminus \emptyset$, define coalitions $\underline{S}$ and $\overline{S}$ respectively by the following, $\underline{S} = \cup\{K \in \beta | K \subseteq S\}$, i.e., the maximal union of components of N which belongs to coalition S. $\overline{S} = \cup\{K \in \beta | K \cap S \neq \emptyset\}$, i.e., the minimal union of components of N covering coalition S.

3. Efficient Myerson Value For Union Stable Structures

3.1. Definition

In order to give the formal definition of the efficient Myerson value, two matrices closely related to union stable structures are constructed.

Let us define a $\{0, 1\}$-squared matrix P of order $2^n - 1$, which is closely related to union stable structure $(N, v, \mathcal{F})$. So that for all $S, T \subseteq 2^N \setminus \emptyset$,

$$P[S,T] = \begin{cases} 1, & \text{if } T \in C_{\mathcal{F}}(S) \ , \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Note that for all coalitions $S \subseteq N$, $v^{\mathcal{F}}(S) = (P{\cdot}v)[S]$, thus $\varphi_i(N, v, \mathcal{F}) = Sh_i(N, v^{\mathcal{F}}) = (M_L{\cdot}P{\cdot}v)[i]$, where $\varphi(N, v, \mathcal{F})$ is the Myerson value for union stable structure$(N, v, \mathcal{F})$.

Next, we shall make a modification of the matrix, and define the matrix Q as follows,

$$Q[S,T] = \begin{cases} 1, & \text{if } T = \underline{S} \ , \\ 1, & \text{if } T \in C_{\mathcal{F}}(S \setminus \underline{S}), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Lemma 1. *Given $(N, v, \mathcal{F}) \in US^N$ and the intermediate game (β, v^β) defined before, the vector of weights are $w = (b_1, b_2, ..., b_r), b_l = |B_l|$ for all $l \in \{1, 2, ..., r\}$. Then for all $B_l \in \beta$ and all players $i \in B_l$,*

$$\begin{aligned} (M_L \cdot (Q - P) \cdot v)[\{i\}] &= Sh_i(N, (Q - P) \cdot v) \\ &= \frac{1}{b_l}(Sh^w_{B_l}(\beta, v^\beta) - v(B_l)). \end{aligned}$$

The proof is similar to the computation in Hamiache (Hamiache, 2012),we omit here. Next we give the definition of the efficient Myerson value and some axioms that will be used to axiomatize the value.

Definition 4. For all union stable structures $(N, v, \mathcal{F})$, define the the efficient Myerson value η as

$$\eta_i(N, v, \mathcal{F}) = (M_L \cdot Q \cdot v)[\{i\}] = Sh_i(N, Q \cdot v) \text{ for all } i \in N.$$

From Lemma 1, it is obvious that for solution η and all $i \in N$,

$$\eta_i(N, v, \mathcal{F}) = \varphi_i(N, v, \mathcal{F}) + \frac{1}{b_l}(Sh^w_{B_l}(\beta, v^\beta) - v(B_l)).$$

We can interpret efficient Myerson value in the following sense. In the first step, every player obtains the payoff of Myerson value. In the second step, since the allocation rule satisfies efficiency, define a quotient game, and every component B_l obtains the payoff of weighted Shapley value, following the principle of fairness among the members of component B_l , the surplus $Sh^w_{B_l}(\beta, v^\beta) - v(B_l)$ is split equally.

It can be seen that the efficient Myerson value and the collective value, which was proposed by Kamijo(Kamijo, 2011), is similar, and the difference lies in the allocation rule of first step. Compared with a prior coalition structure, given a component of union stable system, some subset of the component may be not feasible. And the formation of the component of union stable system lies on the contribution of common players, while collective value cannot illustrate the contribution of the intermediate members make during the cooperation, so the collective value is not suitable for union stable structures. So they are irreplaceable for each other.

Let $\mathcal{C}_i(\mathcal{F})$ denote the collection given by $\{C \in \mathcal{C}(\mathcal{F}) : i \in C\}$, to provide axiomatic characterizations of the efficient Myerson value, the following definitions and properties are introduced.

3.2. Axiomatization

Definition 5. A union stable structure $(N, v, \mathcal{F})$ is called point anonymous if there exists a function $f : \{0, 1, ..., |D|\} \to \mathbb{R}$ with $f(0) = 0$ such that $v^{\mathcal{F}}(S) = f(|S \cap D|)$ for all $S \subseteq N$, where $D = \{i \in N : C_i(\mathcal{F}) \neq \emptyset\}$.

Definition 6. For any $(N, v, \mathcal{F}) \in US^N$, a player $i \in N$ is called superfluous for $(N, v, \mathcal{F})$ if $v^{\mathcal{F}}(S \cup i) = v^{\mathcal{F}}(S)$ for all $S \subseteq N \setminus \{i\}$.

Let $\psi : US^N \to \mathbb{R}^n$ be a solution, then we call it satisfies the above properties, if
Efficiency (EFF) For all $(N, v, \mathcal{F}) \in US^N$, $\sum_{i \in N} \psi_i(N, v, \mathcal{F}) = v(N)$.
Additivity (ADD) For any $(N, u, \mathcal{F}), (N, v, \mathcal{F}) \in US^N$, $\psi(u + v) = \psi(u) + \psi(v)$, where $(u + v)(S) = u(S) + v(S)$ for all $S \subseteq N$.
Average equity (AE) For all unanimity games u_T with $T \subseteq N \setminus \emptyset$, if there exists two components $B_l, B_k \in \beta$ with $B_l \cap T \neq \emptyset, B_k \cap T \neq \emptyset$, then

$$|B_l|^{-1} \sum_{i \in B_l} \psi_i(N, u_T, \mathcal{F}) = |B_k|^{-1} \sum_{j \in B_k} \psi_j(N, u_T, \mathcal{F}).$$

Point anonymity (PA) For all point anonymous union stable structures $(N, v, \mathcal{F})$,

there exists $b \in \mathbb{R}$ such that $\psi_i(N, v, \mathcal{F}) = b$ for all $i \in D$, $\psi_i(N, v, \mathcal{F}) = 0$ otherwise.
Redundant fairness (RF) If there exists two superfluous players $i, j \in B_k$ with $B_k \in \beta$, then $\psi_i = \psi_j$.
Superfluous component property (SCP) Given component $B_k \in \beta$, if $v(R \cup B_k) = v(R)$ for all $R \subseteq \beta$, then $\sum_{i \in B_k} \psi_i(N, v, \mathcal{F}) = 0$.

Theorem 1. *The efficient Myerson value is the unique value on US^N that satisfies efficiency, additivity, average fairness, point anonymity, redundant fairness and superfluous component property.*

Proof. It is straightforward to verify that the efficient Myerson value satisfies EFF, ADD, AE, RF and SCP. In the following, we will only verify the property of point anonymity.

Let $(N, v, \mathcal{F}) \in US^N$ be point anonymous. If $D = \emptyset$, then the restricted game $v^{\mathcal{F}}(S) = f(|S \cap \emptyset|) = f(0) = 0$ for all $S \subseteq N$. Hence, the efficient Myerson value $\eta_i(N, v, \mathcal{F}) = 0$ for all $i \in N$. Let $D \neq \emptyset$, we will show that there exists a unique component $B_k \in \beta$ such that $D \subseteq B_k$. Otherwise, assume there are two components $B_i, B_j \in \beta$ such that $\overline{D} = B_i \cup B_j$, let $S = \overline{D}$, we have $v^{\mathcal{F}}(S) = v(B_i) + v(B_j) = f(|B_i \cap D|) + f(|B_j \cap D|)$, which contradicts with $v^{\mathcal{F}}(S) = f(|S \cap D|) = f(|D|)$. Hence, let us suppose $B_k \in \beta$ is the unique component such that $D \subseteq B_k$, then for any $R \subseteq \beta$, $v(|(B_l \cup R) \cap D|) = f(|R \cap D|) = v(R)$ for all $B_l \neq B_k$, $Sh^w_{B_k}(\beta, v^\beta) = v(B_k)$. Consequently, for any $B_l \in \beta$, $Sh^w_{B_k}(\beta, v^\beta) - v(B_k) = 0$, the efficient Myerson value is equal to the Myerson value, i.e., $\eta_i = \varphi_i = f(|D|)/|D|$ for all $i \in D$, otherwise, $\eta_i = 0$. Thus the efficient Myerson value verifies point anonymity.

Next, we will show the converse part. Let $\psi \in \mathbb{R}^n$ be a solution on US^N satisfying the above six properties. Given $T \subseteq N \setminus \emptyset$, let $(N, u_T, \mathcal{F})$ be a unanimity game with union stable system. Given $c \in \mathbb{R}$, let cu_T be a unanimity game u_T multiplied by a scalar c, Then by additivity, it suffices to show that $\psi(N, v, \mathcal{F})$ is uniquely determined by the above six properties. For all $T \subseteq N \setminus \emptyset$, , let us consider the following two cases: $T \notin \mathcal{F}$ and $T \in \mathcal{F}$.
Case 1 $T \notin \mathcal{F}$, define $\overline{T} \subseteq \beta$ by $\{B \in \beta, B \cap T \neq \emptyset\}$. Then the unanimity game $(\beta, (cu_T)^\beta)$ is a $\overline{T}$-unanimity game multiplied by c, i.e., $(\beta, cu_{\overline{T}})$. It is obvious that any component $B_l \in \beta \setminus \overline{T}$ is superfluous. From superfluous component property, we have $\sum_{i \in B_l} \psi_i(N, cu_T, \mathcal{F}) = 0$ for all $B_l \in \beta \setminus \overline{T}$. Together with average equity together and efficiency, we have that $\sum_{i \in B_l} \psi_i(N, cu_T, \mathcal{F}) = c(\sum_{B_l \in \overline{T}} |B_l|)^{-1} |B_l|$ for all $B_l \in \overline{T}$, $\sum_{i \in B_l} \psi_i(N, cu_T, \mathcal{F}) = 0$ otherwise.

Furthermore, we assert that any player $i \in N$ is superfluous for $(N, cu_T, \mathcal{F}) \in US^N$, i.e., given any $i \in N$, $u^{\mathcal{F}}_T(S) = u^{\mathcal{F}}_T(S \cup i)$ for all $S \subseteq N$. Consequently, given any $B_k \in \beta$, due to the redundancy fairness of $\psi(N, cu_T, \mathcal{F})$, then $\psi_i = \psi_j$ for all $i, j \in B_k$. From the above arguments, we have that $\psi_i(N, cu_T, \mathcal{F}) = c(\sum_{B_l \in \overline{T}} |B_l|)^{-1}$ for all $i \in B_k$ and $B_k \in \overline{T}$, $\psi_i(N, cu_T, \mathcal{F}) = 0$ otherwise. Hence, for any unanimity game with union stable structure $(N, cu_T, \mathcal{F}) \in US^N$ with $T \notin \mathcal{F}$, $\psi(N, cu_T, \mathcal{F})$ is uniquely determined. The remaining task is to show all players are superfluous.

In the following, we show that any player $i \in N$ is superfluous for $(N, cu_T, \mathcal{F}) \in US^N$. If there exists a unique component $B_k \in \beta$ such that $T \subseteq B_k$, then $u^{\mathcal{F}}_T(S) = u^{\mathcal{F}}_T(S \cup i) = 1$, $u^{\mathcal{F}}_T(S) = 0$ for all $S \subseteq N \setminus B_k$. Otherwise, there exists no such component, then $u^{\mathcal{F}}_T(S) = 0$ for all $S \subseteq N$. This completes the proof for case 1.
Case 2 $T \in \mathcal{F}$, we show that $(N, cu_T, \mathcal{F}) \in US^N$ is point anonymous. First we show

that for $T \in \mathcal{F}$, $(cu_T)^{\mathcal{F}}(S) = c$ if and only if $T \subseteq S$. Due to whether the coalition S is feasible or not, we distinguish the following two cases:
(1)If $S \in \mathcal{F}$, then $(cu_T)^{\mathcal{F}}(S) = cu_T(S) = c$ if and only if $T \subseteq S$, i.e., $T \cap S = T$.
(2)If $S \notin \mathcal{F}$ and $T \subseteq S$, we will show that there exists a unique feasible coalition $K \in \mathcal{F}$ and $K \subseteq S$ such that $T \subseteq K$. If $T \in C_{\mathcal{F}}(S)$, let $K = T$, $(cu_T)^{\mathcal{F}}(S) = cu_T(T) = c$. Otherwise, there exists a series of feasible coalitions $A_1, A_2, ..., A_l \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$ for any $i, j = 1, 2, ..., l (l \geq 2)$ and $i \neq j$ such that $S = \cup_{k=1}^{l} A_k$, since $S \notin \mathcal{F}$ and $|S| \geq 2$. Hence there exists a unique feasible coalition $A_j (1 \leq j \leq l)$ such that $T \subseteq A_j$, let $K = A_j$, consequently, $(cu_T)^{\mathcal{F}}(S) = cu_T(A_j) = c$. If $S \notin \mathcal{F}$ and $T \nsubseteq S$, it is easy to verify that $(cu_T)^{\mathcal{F}}(S) = 0$.

From the arguments above, we have that if $T \in \mathcal{F}$, $(cu_T)^{\mathcal{F}}(S) = c$ if and only if $T \subseteq S$. Therefore, there exists a function $f : \{0, 1, 2, ..., |T|\} \to \mathbb{R}$ such that $cu_T^{\mathcal{F}}(S) = f(|S \cap T|)$ for all $S \subseteq N$ where $f(0) = f(1) = ... = f(|T| - 1) = 0$ and $f(|T|) = c$. Hence,$(N, cu_T, \mathcal{F})$ is point anonymous, applying the point anonymity to the solution ψ, there exists $b \in \mathbb{R}$ such that $\psi_i = b$ if $i \in T$ and $\psi_i = 0$ otherwise. By efficiency, we have that $cu_T(N) = \sum_{i \in T} \psi_i = b|T| = c$, let $b = c/|T|$, thus the solution $\psi(N, v, \mathcal{F})$ is uniquely determined by $\psi_i = b$ for all $i \in T$, $\psi_i = 0$ otherwise.

So, $\psi(N, v, \mathcal{F})$ is unique determined in both cases. Since the efficient Myerson value verifies the six properties, $\psi(N, v, \mathcal{F}) = \eta(N, v, \mathcal{F})$.

Also the axioms of theorem 1 are logically independent as shown by the following alternative solutions.

Example 1. The zero solution given by $\psi_i(N, v, \mathcal{F}) = 0$ for all $i \in N$ satisfies ADD, AE, PA, RF and SCP. It does not satisfy efficiency.

Example 2. The equal division given by

$$\psi_i(N, v, \mathcal{F}) = \begin{cases} \frac{Sh^w_{B_k}(\beta, v^\beta) - v(B_k)}{|B_k \setminus SU|} + \psi_i(N, v, \mathcal{F}), & \text{if } i \in B_k \setminus SU, \\ 0, & \text{if } i \in B_k \cap SU. \end{cases} \tag{4}$$

for all $i \in B_k$, $B_k \in \beta$, where SU denotes the set of all superfluous players in $(N, v, \mathcal{F})$ and the weight system is the same with the definition of Lemma 1. This solution satisfies all properties except additivity.

Example 3. The solution given by $\psi_i(N, v, \mathcal{F}) = \varphi_i(N, v, \mathcal{F}) + \frac{Sh_{B_k}(\beta, v^\beta) - v(B_k)}{|B_k|}$ for all $i \in B_k$, $B_k \in \beta$, satisfies EFF, ADD, PA, RF and SCP. It does not satisfy average equity.

Example 4. The solution given by $\psi_i(N, v, \mathcal{F}) = \frac{Sh^w_{B_k}(\beta, v^\beta)}{|B_k|}$ for all $i \in B_k$ satisfies all properties except point anonymity.

Example 5. Define the solution $\psi(N, v, \mathcal{F})$ by $\psi_i(N, v, \mathcal{F}) = \varphi_i(N, v, \mathcal{F}) + \frac{[Sh_{B_k}(\beta, v^\beta) - v(B_k)]w_i}{\sum_{j \in B_k} w_j}$ for all $i \in B_k, B_k \in \beta$, for some exogenous weight system $w \in \mathbb{R}^n$ with $w_i \neq w_j$ for any two players $i \neq j$ in the same component, and there exists a constant number $a \in \mathbb{R}$ such that for any component $B_k \in \beta$, $w(B_k) = \sum_{i \in B_k} w_i = |B_k| \cdot a$. It is straightforward to verify that this solution satisfies EFF, ADD, AE, PA and SCP, except redundant fairness.

Example 6. The solution given by

$$\psi_i(N,v,\mathcal{F}) = \begin{cases} \frac{v(N)-\alpha(v)}{|D|}, & \text{if } i \in D, \\ 0, & \text{if } i \in B_k \setminus D, \\ \frac{\alpha(v)}{|N\setminus B_k|}, & \text{if } i \in N \setminus B_k. \end{cases} \tag{5}$$

for all $i \in B_k$, $B_k \in \beta$, where $\alpha : v \to \mathbb{R}$ is a linear operator, i.e., satisfying $\alpha(v+w) = \alpha(v) + \alpha(w)$, and $\alpha(v) = 0$ when the union stable structure$(N, v, \mathcal{F})$ is point anonymous, otherwise $0 < \alpha(v) < v(N)/|D|$. Since there exists only one component B_k such that $D \subseteq B_k$. It is straightforward to verify that this solution satisfies EFF, ADD, AE, PA and RF. It does not verify superfluous component property.

4. Conclusion

This paper mainly focus on the axiomatization of efficient Myerson value for union stable structures. And three new axioms:average equity, redundant fairness, superfluous component property and other three properties. And the independence of the axioms is illustrated. Besides, the difference between the value and collective value is remarked.

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