On Monotonicity of the SM-nucleolus and the α-nucleolus

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Abstract In this paper two single-valued solution concepts of a TU-game with a finite set of players, the SM-nucleolus and the α-nucleolus, are considered. Based on the procedure of finding lexicographical minimum, there was proposed an algorithm allowing to calculate the SM-nucleolus as well as the prenucleolus. This algorithm is modified to calculate the α-nucleolus for any fixed $\alpha \in [0, 1]$. Using this algorithm the monotonicity properties of the SM-nucleolus and the α-nucleolus are studied by means of counterexamples.

Keywords: cooperative TU-game, solution concept, aggregate and coalitional monotonicity, the SM-nucleolus, the α-nucleolus.

1. Introduction

In this paper we examine two single-valued solution concepts of a transferable utility game (TU-game) with a finite set of players — the SM-nucleolus (Tarashnina, 2011) and the α-nucleolus (Smirnova and Tarashnina, 2011). Both of these solution concepts take into account "the blocking power" of a coalition, the amount which the coalition cannot be prevented from by the complement coalition.

Based on the procedure (Maschler et al., 1979) of finding lexicographical minimum, there was proposed an algorithm (Britvin and Tarashnina, 2013) allowing to calculate the SM-nucleolus as well as the prenucleolus (Schmeidler, 1969). By introducing the special numbering of coalitions the problem of finding the SM-nucleolus of a cooperative n-person game is reduced to solving a single linear program with 2^n rows and $(n+1)$ columns. The initial values of the problem coefficients are $0, 1, -1$. This algorithm is modified to calculate the α-nucleolus for any fixed $\alpha \in [0, 1]$.

In this work we consider two properties of single-valued solution concepts of TU-games: aggregate and coalitional monotonicity. Aggregate monotonicity means that if the worth of the grand coalition icreases while the worths of all other coalitions remain the same, then the players payoffs should not decrease. Coalitional monotonicity applies this rule to any coalition $S \subset N$ in a game. The Shapley value (Shapley, 1953) satisfies aggregate and coalitional monotonicity. N. Megiddo (Megiddo, 1974) presented an example of nine person cooperative TU-game that shows that another well-known single-valued solution, the nucleolus (Schmeidler, 1969), violates aggregate monotonicity. It is known that the nucleolus does not satisfy coalitional monotonicity too (Young, 1985). In this paper we verify that the SM-nucleolus does not satisfy aggregate and coalitional monotonicity.

The paper is organized as follows. In section 2 basic definitions and notations are given. For any fixed α we describe the algorithm of finding the α-nucleolus (including the SM-nucleolus) in section 3. Using this algorithm we study the monotonicity properties of the considered solution concepts by means of counterexamples in section 4.

2. Basic definitions and notations

In this paper we consider cooperative games with transferable utilities (TU-games). A TU-game is a pair (N, v), where $N = \{1, ..., n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. More information about cooperative game theory can be found in (Petrosjan et al., 2012) and (Pecherskiy and Yanovskaya, 2004).

The set of all TU-games with the fixed set of players N is denoted by G^N. Consider a game (N, v) from G^N. Assume that the players have formed the maximal coalition N and consider the distribution of $v(N)$ among all the players. We define the set of feasible payoff vectors as follows:

$$X(N, v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leq v(N)\}.$$

The set $X^0(N, v) \subset X(N, v)$ such that

$$X^0(N, v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)\} \tag{1}$$

is a set of group rational payoff vectors of the game (N, v). It follows from (1) that $x \in X^0(N, v)$ if and only if for all $S \subset N$ it holds

$$x(S) + x(N \backslash S) = v(N),$$

where

$$x(S) = \sum_{i \in S} x_i.$$

Definition 1. A solution of a TU-game on G^N is a mapping f that matches for every game $(N, v) \in G^N$ the subset $f(N, v)$ of $X(N, v)$.

In the paper we study two single-valued solution concepts: the SM-nucleolus and the α-nucleolus. To introduce the definitions of these solution concepts, we should define the excess of a coalition.

Definition 2. The excess $e(x, S, v)$ of a coalition S at $x \in X^0(N, v)$ is calculated as

$$e(x, S, v) = v(S) - x(S). \tag{2}$$

Let (N, v) be a TU-game. The dual game (N, v^*) of (N, v) is defined by

$$v^*(S) = v(N) - v(N \backslash S)$$

for all coalitions S.

Let us clarify the notion of the constructive and the blocking power of S. The constructive power of S is the worth of the coalition, or exactly what S can reach by cooperation. By the blocking power of coalition S we understand the amount $v^*(S)$ that this coalition brings to N if the last will be formed — its contribution to the grand coalition. The difference between $v(N)$ and $v(N \backslash S)$ is a subject which should be taken into account in a solution of a game. In our opinion, the blocking power can be judged as a measure of necessity of S for N — how much S contributes to N. So, each coalition S is estimated by N in this spirit.

In order to introduce the SM-nucleolus we define the sum-excess of the coalition.

Definition 3. The sum-excess $\bar{e}(x,S,v)$ of a coalition S at $x \in X^0(N,v)$ in the game (N,v) is

$$\bar{e}(x,S,v) = \frac{1}{2}e(x,S,v) + \frac{1}{2}e(x,S,v^*).$$

We define for some $\varphi \in \mathbb{R}^n$ the mapping $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi = \theta(\varphi) \in \mathbb{R}^n$ means that ψ is obtained from φ by ordering its components in non-increasing order.

After calculating the sum-excess for each $S \subseteq N$ we obtain the sum-excess vector $\bar{e}(x,v) = \{\bar{e}(x,S,v)\}_{S\subseteq N}$ of dimension 2^n.

Definition 4. The SM-nucleolus of the game (N,v) is the set $X_{SM} \subset X^0(N,v)$ such that for every $x \in X_{SM}$ vector $\theta(\{\bar{e}(x,S,v)\}_{S\subseteq N})$ is lexicographically the smallest:

$$X_{SM}(N,v) = \{x \in X^0 | \theta(\{\bar{e}(x,S,v)\}_{S\subseteq N}) \preceq_{lex} \theta(\{\bar{e}(y,S,v)\}_{S\subseteq N}),\ \forall y \in X^0(N,v)\}.$$

To introduce the α-nucleolus we define the α-excess of a coalition.

Definition 5. The α-excess $e^\alpha(x,S,v)$ of a coalition S at $x \in X^0(N,v)$ is

$$e^\alpha(x,S,v) = \alpha e(x,S,v) + (1-\alpha)e(x,S,v^*),\ \alpha \in [0,1]. \tag{3}$$

After calculating the α-excess for each $S \subseteq N$ we obtain the α-excess vector $e^\alpha(x,v) = \{e^\alpha(x,S,v)\}_{S\subseteq N}$ of dimension 2^n.

Definition 6. The α-nucleolus of the game (N,v) is the set $X_\alpha \subset X^0(N,v)$ such that for every $x \in X_\alpha$ vector $\theta(\{e^\alpha(x,S,v)\}_{S\subseteq N})$ is lexicographically the smallest:

$$X_\alpha(N,v) = \{x \in X^0(N,v) | \theta(\{e^\alpha(x,S,v)\}_{S\subseteq N}) \preceq_{lex} \theta(\{e^\alpha(y,S,v)\}_{S\subseteq N}),\ \forall y \in X^0(N,v)\}.$$

It is important to note that both solution concepts represent a unique point in X^0, so they are single-valued solutions (Smirnova and Tarashnina, 2011).

Obviously, if $\alpha = \frac{1}{2}$, then

$$X_\alpha(N,v) = X_{SM}(N,v).$$

This means that the SM-nucleolus is a special case of the α-nucleolus.

3. Algorithm

In the literature there was presented an algorithm of finding the SM-nucleolus of any TU-game (Britvin and Tarashnina, 2013). Here we modify this algorithm for calculation the α-nucleolus. First, we should replace the excess in the procedure of finding the lexicographical minimum (Maschler et al., 1979) to the α-excess. We obtain the following procedure.

1. Consider a pair (X^0, J^0), where J^0 consists of all possible coalitions except the empty one.
2. Recursively find

$$u^t = \min_{x \in X^{t-1}} \max_{S \subseteq J^{t-1}} e^\alpha(x, S, v), \tag{4}$$

$$X^t = \{x \in X^{t-1} \mid e^\alpha(x, S, v) \le u^t, \forall S \subseteq J^{t-1}\},$$

$$J_t = \{S \subseteq J^{t-1} \mid e^\alpha(x, S, v) = u^t, \forall x \in X^t\},$$

$$J^t = J^{t-1} \backslash J_t.$$

3. If $J^t = \emptyset$, then we stop, otherwise we go to step 2 with $t = t + 1$.

In the game there may be formed 2^n coalitions (including the empty one). We will not consider the empty coalition. Let us enumerate all the other coalitions in the following way.

Suppose that n-person game has been built dynamically by adding one player at each step. In a game with one player the single coalition $\{1\}$ is number 1. When the second player enters the game he brings there two additional coalitions. The sequence of coalitions in ascending order for two-person game is as follows: $\{1\}, \{2\}, \{1, 2\}$. Further, for a three-person game we have: {1}, {2}, {1,2}, {3}, {1,3}, {2,3}, {1,2,3}. And so on.

Assume that the sequence of coalitions has been formed in ascending order for a k-person game. Adding to this game the $(k + 1)$-th player entails forming 2^k coalitions. We determine the order of the added coalitions. Let coalition $\{k + 1\}$ be the first of the added coalitions. Among the remaining coalitions we do not pay attention to the $(k + 1)$-th player, then we obtain a set of coalitions for a k-person game, which is already built in ascending order. Finally, we extend this numbering to the additional coalitions. As a result, each coalition in a $(k+1)$-person game will be numbered.

For example, the coalitions in 4-person game in ascending order look like

$$\{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\},$$

$$\{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$

Problem (4) for $t = 1$ is equivalent to the following task

$$\begin{cases} \min u^1, \\ u^1 \ge e^\alpha(x, S, v), \\ S \subseteq J^0, \\ x \in X^0. \end{cases}$$

By using formulas (1), (2) and (3) we transform it to the following form

$$\begin{cases} \min u^1, \\ u^1 + \sum_{i \in S} x_i \ge \alpha v(S) + (1 - \alpha)(v(N) - v(N \backslash S)),\ S \subseteq J^0, \\ \sum_{i \in N} x_i = v(N). \end{cases}$$

The resulting problem is a linear programming. Given the suggested order of the coalitions in J^0 we obtain the matrix form

$$\begin{cases} \min c^T z, \\ Az \geq b, \\ A_{eq} z = b_{eq}, \end{cases}$$

with

$$z = \begin{pmatrix} u^1 \\ x \end{pmatrix}. \quad (5)$$

The parameters of this linear programming are the following:

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ A = \begin{pmatrix} I \ A^* \end{pmatrix}, \ I = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \ A^* = \begin{pmatrix} 1\,0\,0\,...\,0\,0 \\ 0\,1\,0\,...\,0\,0 \\ 1\,1\,0\,...\,0\,0 \\ \vdots \quad \ddots \quad \vdots \\ 0\,1\,1\,...\,1\,1 \end{pmatrix},$$

$$b = \alpha \begin{pmatrix} v(1) \\ v(2) \\ v(1,2) \\ \vdots \\ v(2,3,...,n) \end{pmatrix} + (1-\alpha) \begin{pmatrix} v(N) - v(2,3,...,n) \\ v(N) - v(1,3,...,n) \\ v(N) - v(3,4,...,n) \\ \vdots \\ v(N) - v(1) \end{pmatrix},$$

$$A_{eq} = \begin{pmatrix} 1\ 1\ 1\ ...\ 1 \end{pmatrix}, \quad b_{eq} = v(N).$$

Based on the theorem from (Britvin and Tarashnina, 2013), it is easy to prove that there exists a unique solution z^* of this linear programming. So, the calculation procedure is stopped and we obtain the α-nucleolus in the form

$$X_\alpha = \begin{pmatrix} z_2^*(\alpha) \\ z_3^*(\alpha) \\ \vdots \\ z_{n+1}^*(\alpha) \end{pmatrix}, \ \alpha \in [0,1].$$

4. The monotonicity of the SM-nucleolus and the α-nucleolus

In this paper we investigate the monotonicity properties of single-valued solution concepts of TU-games: aggregate monotonicity and coalitional monotonicity. First, let us define these properties.

Definition 7. A single-valued solution concept f satisfies aggregate monotonicity if for every pair of games (N, v) and (N, w) such that

$$v(N) < w(N), \quad (6)$$

$$v(S) = w(S) \text{ for all } S \subset N, \quad (7)$$

it follows that

$$f_i(N, v) \leq f_i(N, w) \text{ for all } i \in N. \quad (8)$$

Definition 8. A single-valued solution concept f satisfies coalitional monotonicity if for every pair of games (N, v) and (N, w) such that

$$v(T) < w(T) \text{ for any } T \subset N, \tag{9}$$

$$v(S) = w(S) \text{ for all } S \subset N, S \neq T, \tag{10}$$

it follows that

$$f_i(N, v) \leq f_i(N, w) \text{ for all } i \in T. \tag{11}$$

Let us give the following example (Megiddo, 1974) that contains of two cooperative games and illustrates the absence of aggregate monotonicity of the SM-nucleolus.

Example 1. Let $N = \{1, 2, ..., 9\}$ and $\varphi = (1, 1, 1, 2, 2, 2, 1, 1, 1)$. Consider two groups of coalitions:

$$A = \{(1, 2, 3), (1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (7, 8, 9)\},$$

$$B = \{(1, 2, 3, 6, 7), (1, 2, 3, 6, 8), (1, 2, 3, 6, 9), (4, 5, 6)\}.$$

Define the characteristic function v as

$$v = \begin{cases} 6, \text{ if } S \in A, \\ 9, \text{ if } S \in B, \\ 12, \text{ if } S = N, \\ \sum_{i \in S} \varphi_i - 1, \text{ otherwise.} \end{cases}$$

The characteristic function w has the following form

$$w = \begin{cases} 6, \text{ if } S \in A, \\ 9, \text{ if } S \in B, \\ 13, \text{ if } S = N, \\ \sum_{i \in S} \varphi_i - 1, \text{ otherwise.} \end{cases}$$

It is obvious that conditions (6) and (7) hold for characteristic functions v and w. For some fixed $\alpha \in [0, 1]$ we calculate the α-nucleolus for games (N, v) and (N, w) using the algorithm, presented in section 3. Assuming that the parameter α is moving along the interval $[0, 1]$ with the step of 0.1, we have the following payoff vectors presented in Table 1. The special case $\alpha = \frac{1}{2}$ with $X_{SM} = X_\alpha$ is shown in bold.

Consider the payoffs that player 6 gets according to the α-nucleolus for $\alpha \geq 0.1$. We can see that $X^6_\alpha(N, v) > X^6_\alpha(N, w)$. Therefore, inequality (8) is not satisfied. So, by means of the counterexample we can verify that aggregate monotonicity does not hold for the α-nucleolus with $0.1 \leq \alpha \leq 1$.

By using the dichotomy method for this pair of games we can approximately calculate the maximum α^* such that $0 < \alpha^* < 0.1$ for which aggregate monotonicity is satisfied and for some $\alpha > \alpha^*$ aggregate monotonicity is not satisfied. In the current example $\alpha^* \approx 0.075$.

Table 1: The α-nucleolus for (N, v) and (N, w).

α	$X_\alpha(N, v)$	$X_\alpha(N, w)$
0	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.11, 2.11, 2.11, 1.11, 1.11, 1.11)$
0.1	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.92, 1.11, 1.11, 1.11)$
0.2	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.3	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.4	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.5	**(1,1,1,2,2,2,1,1,1)**	**(1.11,1.11,1.11,2.22,2.22,1.89,1.11,1.11,1.11)**
0.6	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.7	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.8	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
0.9	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$
1	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.11, 1.11, 1.11, 2.22, 2.22, 1.89, 1.11, 1.11, 1.11)$

Let us give one more example.

Example 2. Consider the characteristic function v' that coincides with v from Example 1:

$$v'(S) = v(S) \text{ for all } S \subseteq N.$$

The characteristic function w' is constructed in the following way

$$w'(S) = \begin{cases} 7, \text{ if } S = (4, 7), \\ v(S), \text{ otherwise.} \end{cases}$$

It is obvious that conditions (9) and (10) hold for characteristic functions v' and w'. For some fixed $\alpha \in [0, 1]$ we calculate the α-nucleolus for games (N, v') and (N, w') using the algorithm, presented in section 3. Assuming that the parameter α is moving along the interval $[0, 1]$ with the step of 0.1, we have the payoff vectors presented in Table 2. The special case $\alpha = \frac{1}{2}$ with $X_{SM} = X_\alpha$ is shown in bold.

Table 2: The α-nucleolus for (N, v') and (N, w').

α	$X_\alpha(N, v')$	$X_\alpha(N, w')$
0	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$
0.1	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.01, 1.01, 1.01, 1.98, 1.98, 2.03, 1.11, 0.94, 0.94)$
0.2	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.01, 1.01, 1.01, 1.96, 1.96, 2.05, 1.21, 0.88, 0.88)$
0.3	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.02, 1.02, 1.02, 1.94, 1.94, 2.08, 1.32, 0.82, 0.82)$
0.4	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.03, 1.03, 1.03, 1.93, 1.93, 2.10, 1.43, 0.76, 0.76)$
0.5	**(1,1,1,2,2,2,1,1,1)**	**(1.04,1.04,1.04,1.91,1.91,2.13,1.54,0.70,0.70)**
0.6	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.04, 1.04, 1.04, 1.89, 1.89, 2.16, 1.64, 0.64, 0.64)$
0.7	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.05, 1.05, 1.05, 1.87, 1.87, 2.18, 1.75, 0.59, 0.59)$
0.8	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.06, 1.06, 1.06, 1.85, 1.85, 2.21, 1.86, 0.53, 0.53)$
0.9	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.07, 1.07, 1.07, 1.83, 1.83, 2.23, 1.97, 0.47, 0.47)$
1	$(1, 1, 1, 2, 2, 2, 1, 1, 1)$	$(1.07, 1.07, 1.07, 1.81, 1.81, 2.26, 2.07, 0.41, 0.41)$

Consider the payoffs that player 4 gets according to the α-nucleolus for $\alpha \geq 0.1$. We can see that $X^4_\alpha(N, v) > X^4_\alpha(N, w)$. Therefore, inequality (11) is not satisfied.

So, by means of counterexample we can verify that coalitional monotonicity does not hold for the α-nucleolus with $0.1 \leq \alpha \leq 1$.

By using the dichotomy method for this pair of games we can approximately calculate the maximum α^* such that $0 < \alpha^* < 0.1$ for which coalitional monotonicity is satisfied and for some $\alpha > \alpha^*$ coalitional monotonicity is not satisfied. In the current example $\alpha^* \approx 0$.

5. Conclusion

The result of this work is not surprising. Although aggregate and coalitional monotonicity are considered to be desirable and natural properties of a solution in a TU-game (Maschler, 1992). There are very few solution concepts satisfying even aggregate monotonicity, the weakest form of it. In the paper we have investigated the monotonicity of the SM-nucleolus and come across to some negative conclusion. In a general game it violates the both aggregate and coalitional monotonicity. At the same time, the α-nucleolus due to the arbitrary choice of a real parameter α demonstrates for some α better properties than the SM-nucleolus. The intervals for which the α-nucleolus satisfies aggregate and coalitional monotonicity are approximately calculated for the given examples. The investigation may be extended and deepened in the direction of getting analytical formulas for this interval.

In (Tauman and Zapechelnyuk, 2010), the authors argue that monotonicity may not be a proper requirement for some economic context from which a cooperative game arises. They provide an example of a simple 4-person game that marks out a class of economic problems where the monotonicity property of a solution concept is not as attractive as it may seem at the beginning. So, sometimes there is a competition between monotonicity and other attractive properties of a solution in a TU-game.

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