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Strategically Supported Cooperation in Differential Games with Coalition Structures*

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Abstract The problem of strategic stability of long-range cooperative agreements in differential games with coalition structures is investigated. We build a general theoretical framework of the cooperative differential game with a coalition structure basing on imputation distribution procedure. The notion of imputation distribution procedure is the basic ingredient in our theory. This notion may be interpreted as an instantaneous payoff of an individual at some moment which prescribes distribution of the total gain among the members of a group and yields the existence of a Nash equilibrium. Moreover, a few assumptions about deviation instant for a coalition are made concerning behavior of a group of many individuals in certain dynamic environments; thus, the time-consistent cooperative agreement can be strategically supported by an ε-Nash equilibrium or a strong ε-Nash equilibrium.

Keywords: differential game, coalition structure, strategic stability, imputation distribution procedure, deviation instant, ε-Nash equilibrium, strong ε-Nash equilibrium.

1. Introduction

Human behavior is dynamic, and cooperation runs through human behavior. It happens often that players agree to cooperate over a certain period. It also happens often that some cooperative agreements are abandoned before reaching the maturity. It is important that cooperation remains stable on a time interval. When we analyze the problem of stability of long-range cooperative agreements there are three important aspects which must be taken into account, including time consistency, strategic stability and the irrational-behavior-proof condition.

Time consistency involves the property that as the cooperation develops, partners are guided by the same optimal principle at each instant of time and hence do not possess incentives to deviate from the previous cooperative behavior.

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The concept of time consistency and its implementation was initially proposed in (Petrosyan, 1977), (Petrosyan and Danilov, 1979), (Petrosyan and Danilov, 1982), (Petrosyan and Danilov, 1986) and was developed in (Petrosyan, 1993), (Petrosyan and Zenkevich, 1996), and (Petrosyan, 1997). Some new results about time consistency can be found in (Petrosyan and Zaccour, 2003), (Yeung and Petrosyan, 2005), and (Gao et al., 2014).

Strategic stability means that the outcome of the cooperative agreement must be attained in some Nash equilibrium, which will guarantee the strategic support of the cooperation. The agreement will be developed in such a manner that at least individual deviations from the cooperation will not give any advantage to the deviator. Some results about strategic stability can be found in (Petrosyan and Grauer, 2002), (Gao and Petrosyan, 2009), and (Petrosyan and Zenkevich, 2009).

The irrational-behavior-proof condition means that the partners involved in the cooperation must be sure that even in the worst scenario they will not lose compared with non-cooperative behavior. Since one cannot be sure that the partners will behave rational on a long time interval, this aspect must be also taken into account. The concept of the irrational-behavior-proof condition was initially proposed in (Yeung, 2006). A further investigation can be found in (Gao et al., 2013).

Some results about dynamic games with coalition structures are given in (Petrosjan and Mamkina, 2003), (Kozlovskaya et al., 2010). In this paper we focus on *the problem of strategic stability* in cooperative differential games with coalition structures. We build a general theoretical framework of the cooperative differential game with a coalition structure basing on *imputation distribution procedure (IDP).* The notion of imputation distribution procedure (IDP) is the basic ingredient in our theory. This notion may be interpreted as a instantaneous payoff of an individual at some moment which prescribes distribution of the total gain among the members of a group. This notion yields the existence of a Nash equilibrium. Moreover to construct an ε-Nash equilibrium or a strong ε-Nash equilibrium in such a game a few assumptions about *deviation instant of a coalition* concerning the behavior of a group of many individuals in certain dynamic environments are made. It turns out that ε-Nash equilibrium or strong ε-Nash equilibrium exist in such a differential game with a coalition structure which guarantee the strategic support of cooperation.

The paper is organized as follows. In Section 2 we define the basic concepts and set up standard terminology and notation about a cooperative differential game with a coalition structure. In Section 3 we prove the existence of ε-Nash equilibrium in a regularized differential game with a coalition structure and the existence of strong ε-Nash equilibrium in a strictly regularized differential game with a coalition structure.

2. Formal Definitions and Terminology

In this section we define the basic concepts of a cooperative differential game with a coalition structure and set up standard terminology and notation.

Differential Game $\Gamma(x_0, T - t_0)$ Let $N = \{1, 2, \ldots, n\}$ be the set of players. We consider an n-person differential game $\Gamma(x_0, T - t_0)$ with independent motions on the time interval $[t_0, T]$ (see (Dockner et al., 2000)). Motion equations have the form:

$$\dot{x_i} = f_i(x_i, u_i), \quad u_i \in U_i \subset R^l, \quad x_i \in R^m, \quad i = 1, \dots, n. \tag{1}$$

It is assumed that the system of differential equations (1) satisfies all conditions necessary for the existence, sustainability and uniqueness of the solution for any n-tuple of measurable controls $u_1(t), \dots, u_n(t)$. The payoff of player i is given by:

$$H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = \int_{t_0}^{T} h_i(x(\tau))d\tau, \tag{2}$$

where $h_i(x)$ is a continuous function, $x(\tau) = (x_1(\tau), \dots, x_n(\tau))$ is a solution (a trajectory) of (1) when open-loop controls $u_1(\tau), \dots, u_n(\tau)$ are used, and $x(t_0) = (x_1(t_0), \dots, x_n(t_0)) = x_0$.

Optimal Cooperative Trajectory $\bar{x}(t)$ Suppose that there exist an n-tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \dots, \bar{u}_n(t))$ and a trajectory $\bar{x}(t), t \in [t_0, T]$, such that

$$\max_{u_1(t),\dots,u_n(t)} \sum_{i=1}^{n} H_i(x_0, T - t_0; u_1(t), \dots, u_n(t)) =$$
$$\sum_{i=1}^{n} H_i(x_0, T - t_0; \bar{u}_1(t), \dots, \bar{u}_n(t)) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}(\tau))d\tau.$$

The trajectory $\bar{x}(t) = (\bar{x}_1(t), \dots, \bar{x}_n(t))$ satisfying (2) we call the *optimal cooperative trajectory*.

Characteristic Function The characteristic function in $\Gamma(x_0, T - t_0)$ is defined in a classical way:

$$V(x_0, T - t_0; N) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}(\tau))d\tau,$$
$$V(x_0, T - t_0; \emptyset) = 0,$$
$$V(x_0, T - t_0; S) = Val\Gamma_{S,N\setminus S}(x_0, T - t_0),$$

where $Val\Gamma_{S,N\setminus S}(x_0, T - t_0)$ is a value of a zero-sum game between coalition S acting as player 1 and coalition $N\setminus S$ acting as player 2, with the payoff of S: $\sum_{i\in S} H_i(x_0, T - t_0; u_1(t), \dots, u_n(t))$.

Imputation Set $L(x_0, T - t_0)$ Define $L(x_0, T - t_0)$ as the imputation set of the game $\Gamma(x_0, T - t_0)$ (see (von Neumann and Morgenstern, 1994)):

$$L(x_0, T - t_0) = \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \geq V(x_0, T-t_0; \{i\}), \sum_{i\in N} \alpha_i = V(x_0, T-t_0; N)\}.$$

Core $C(x_0, T - t_0)$ Define $C(x_0, T - t_0)$ as the core of $\Gamma(x_0, T - t_0)$:

$$C(x_0, T - t_0) = \{\alpha = (\alpha_1, \dots, \alpha_n) \in L(x_0, T - t_0) : \sum_{i\in S} \alpha_i \geq V(x_0, T-t_0; S), S \subset N\}.$$

Imputation Distribution Procedure $\beta(\tau)$ Let $\alpha \in L(x_0, T - t_0)$. Define imputation distribution procedure (IDP) (see (Petrosyan, 1993)) as a function $\beta(\tau) = (\beta_1(\tau), \dots, \beta_n(\tau))$, $\tau \in [t_0, T]$, such that

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau)d\tau. \tag{3}$$

Regularized Game $\Gamma_\alpha(x_0, T - t_0)$ For every $\alpha \in L(x_0, T - t_0)$, we define a non-cooperative game $\Gamma_\alpha(x_0, T - t_0)$ which differs from game $\Gamma(x_0, T - t_0)$ only by payoffs defined along the optimal cooperative trajectory $\bar{x}(\tau), \tau \in [t_0, T]$.

Denote a payoff function in $\Gamma_\alpha(x_0, T - t_0)$ by $H_i^\alpha(x_0, T - t_0; u_1(t), \ldots, u_n(t))$ and the corresponding trajectory by $x(\tau)$. Then $H_i^\alpha(x_0, T - t_0; u_1(t), \ldots, u_n(t)) = H_i(x_0, T - t_0; u_1(t), \ldots, u_n(t))$, if there does not exist $\tau \in (t_0, T]$ such that $x(\tau) = \bar{x}(\tau)$. Let $t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in (t_0, t']\}$. Then

$$\begin{aligned} H_i^\alpha(x_0, T - t_0; u_1(t), \ldots, u_n(t)) &= \int_{t_0}^{t} \beta_i(\tau) d\tau + H_i(\bar{x}(t), T - t; u_1(t), \ldots, u_n(t)) \\ &= \int_{t_0}^{t} \beta_i(\tau) d\tau + \int_{t}^{T} h_i(x(\tau)) d\tau. \end{aligned}$$

In a special case, when $x(\tau) = \bar{x}(\tau), \tau \in (t_0, T]$, we have

$$H_i^\alpha(x_0, T - t_0; \bar{u}_1(t), \ldots, \bar{u}_n(t)) = \int_{t_0}^{T} \beta_i(\tau) d\tau = \alpha_i.$$

Consider subgames $\Gamma(\bar{x}(t), T - t)$, imputation sets $L(\bar{x}(t), T - t)$ and cores $C(\bar{x}(t), T - t)$. Let $\alpha(t) \in L(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as a differentiable function of $t, t \in [t_0, T]$. Game $\Gamma_\alpha(x_0, T - t_0)$ is called a *regularized game* of $\Gamma(x_0, T - t_0)$ (*α-regularization*) if IDP β is defined in such a way that

$$\alpha_i(t) = \int_{t}^{T} \beta_i(\tau) d\tau,$$

or

$$\beta_i(t) = -\alpha_i'(t). \tag{4}$$

In particular, if $\alpha(t) \in C(\bar{x}(t), T - t)$, $\Gamma_\alpha(x_0, T - t_0)$ is called a *strictly regularized game* of $\Gamma(x_0, T - t_0)$.

Time-consistency From (4) we get

$$\alpha_i = \int_{t_0}^{t} \beta_i(\tau) d\tau + \alpha_i(t). \tag{5}$$

Now suppose that $M(x_0, T - t_0) \subset L(x_0, T - t_0)$ is some optimality principle in the cooperative version of game $\Gamma(x_0, T - t_0)$, and $M(\bar{x}(t), T - t) \subset L(\bar{x}(t), T - t)$ is the same optimality principle defined in the subgame $\Gamma(\bar{x}(t), T - t)$ with an initial condition on the optimal trajectory. M can be the core, the stable set, the Shapley value, nucleolus, ect. If $\alpha \in M(x_0, T - t_0)$ and $\alpha(t) \in M(\bar{x}(t), T - t)$, condition (5) gives us *time consistency* of the chosen imputation α or the chosen optimality principle in game $\Gamma_\alpha(x_0, T - t_0)$.

Differential Game with Coalition Structure $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ Let $\mathcal{P} = (S_1, \ldots, S_m)$ be a partition of player set N such that $S_i \cap S_j = \emptyset, i \neq j, \bigcup_{i=1}^m S_i = N, |S_i| = n_i, \sum_{i=1}^m n_i = n$. Suppose that each player i from N is playing in the interests of coalition $S_k \in \mathcal{P}$ to which he belongs trying to maximize the sum of payoffs of its members, i. e.

$$\max_{u_i, i \in S_k} \sum_{i \in S_k} H_i(x_0, T - t_0; u_1(t), \ldots, u_n(t)). \tag{6}$$

Define $u_{S_k} = \{u_i, i \in S_k\}$ as the strategy of coalition S_k and $x_{S_k} = \{x_i, i \in S_k\}$ as a trajectory of coalition S_k. Write

$$H_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t)) = \sum_{i \in S_k} H_i(x_0, T - t_0; u_1(t), \ldots, u_n(t))$$

as the payoff of coalition S_k. Suppose that coalitions in $\mathcal{P}$ are playing cooperatively with objective (2) and state dynamics (1). We call the above game as a *cooperative differential game with a coalition structure* denoted by $\Gamma^{\mathcal{P}}(x_0, T - t_0)$. Suppose that there exist an n-tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \ldots, \bar{u}_n(t))$ and a trajectory $\bar{x}(t) = (\bar{x}_1(t), \ldots, \bar{x}_n(t)), t \in [t_0, T]$ satisfying (2). Then trajectory $\bar{x}(t)$ is an optimal cooperative trajectory of $\Gamma(x_0, T - t_0)$. We define $\bar{x}(t)$ as the *optimal cooperative trajectory* of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ at the same time.

The characteristic function in $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ is defined by:

$$\begin{aligned} V(x_0, T - t_0; \mathcal{P}) &= \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}(\tau)) d\tau, \\ V(x_0, T - t_0; \emptyset) &= 0, \\ V(x_0, T - t_0; \mathcal{S}) &= Val\Gamma_{\mathcal{S}, \mathcal{P} \setminus \mathcal{S}}(x_0, T - t_0), \end{aligned}$$

where $Val\Gamma_{\mathcal{S}, \mathcal{P} \setminus \mathcal{S}}(x_0, T - t_0)$ is a value of zero-sum game played between coalition $\mathcal{S}$ acting as player 1 and coalition $\mathcal{P} \setminus \mathcal{S}$ acting as player 2 in which the payoff of coalition $\mathcal{S}$ equals $\sum_{S_k \in \mathcal{S}} H_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t))$.

Define $L^{\mathcal{P}}(x_0, T - t_0)$ as the imputation set in $\Gamma^{\mathcal{P}}(x_0, T - t_0)$:

$$\begin{aligned} L^{\mathcal{P}}(x_0, T - t_0) = \{\alpha = (\alpha_{S_1}, \ldots, \alpha_{S_m}) : \\ \alpha_{S_k} \geq V(x_0, T - t_0; \{S_k\}), \sum_{S_k \in \mathcal{P}} \alpha_{S_k} = V(x_0, T - t_0; \mathcal{P})\}. \end{aligned}$$

Define $C^{\mathcal{P}}(x_0, T - t_0)$ as the core in $\Gamma^{\mathcal{P}}(x_0, T - t_0)$:

$$\begin{aligned} C^{\mathcal{P}}(x_0, T - t_0) = \{\alpha = (\alpha_{S_1}, \ldots, \alpha_{S_m}) \in L^{\mathcal{P}}(x_0, T - t_0) : \\ \sum_{S_k \in \mathcal{S}} \alpha_{S_k} \geq V(x_0, T - t_0; \mathcal{S}), \mathcal{S} \subset \mathcal{P}\}. \end{aligned}$$

Let $\alpha \in L^{\mathcal{P}}(x_0, T - t_0)$. Define imputation distribution procedure (IDP) of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ as a function $\beta(\tau) = (\beta_{S_1}(\tau), \ldots, \beta_{S_m}(\tau)), \tau \in [t_0, T]$, such that

$$\alpha_{S_k} = \int_{t_0}^{T} \beta_{S_k}(\tau) d\tau, S_k \in \mathcal{P}. \tag{7}$$

Regularized Game with Coalition Structure $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T - t_0)$ For every $\alpha \in L^{\mathcal{P}}(x_0, T - t_0)$, we define a non-cooperative game $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T - t_0)$ which differs from game $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ only by payoffs defined along the optimal cooperative trajectory $\bar{x}(\tau), \tau \in [t_0, T]$. Denote the payoff function in game $\Gamma_{\alpha}^{\mathcal{P}}(x_0, T - t_0)$ by $H_{S_k}^{\alpha}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t))$ and the corresponding trajectory by $x(\tau)$. Then $H_{S_k}^{\alpha}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t)) = H_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t))$, if there does not exist $\tau \in (t_0, T]$ such that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, T]$. Let $t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in (t_0, t']\}$. Then

$$H^{\alpha}_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t)) = \int_{t_0}^{t} \beta_{S_k}(\tau)d\tau + H_{S_k}(\bar{x}(t), T - t; u_{S_1}(t), \ldots, u_{S_m}(t)) = \int_{t_0}^{t} \beta_{S_k}(\tau)d\tau + \int_{t}^{T} h_{S_k}(x(\tau))d\tau,$$

where $h_{S_k}(x(\tau)) = \sum_{i \in S_k} h_i(x(\tau))$. In a special case, when $x(\tau) = \bar{x}(\tau), \tau \in (t_0, T]$, we have

$$H^{\alpha}_{S_k}(x_0, T - t_0; u_{S_1}(t), \ldots, u_{S_m}(t)) = \int_{t_0}^{T} \beta_{S_k}(\tau)d\tau = \alpha_{S_k}.$$

Consider subgames $\Gamma^{\mathcal{P}}(\bar{x}(t), T - t)$, imputation sets $L^{\mathcal{P}}(\bar{x}(t), T - t)$ and cores $C^{\mathcal{P}}(\bar{x}(t), T - t)$. Let $\alpha(t) \in L^{\mathcal{P}}(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as a differentiable function of $t, t \in [t_0, T]$. Game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ is called a *regularized game* of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$ (α*-regularization*) if IDP β is defined in such a way that

$$\alpha_{S_k}(t) = \int_{t}^{T} \beta_{S_k}(\tau)d\tau,$$

or

$$\beta_{S_k}(t) = -\alpha'_{S_k}(t). \tag{8}$$

In particular, if $\alpha(t) \in C^{\mathcal{P}}(\bar{x}(t), T - t)$, $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ is called a *strictly regularized game* of $\Gamma^{\mathcal{P}}(x_0, T - t_0)$.

From (8) we get

$$\alpha_{S_k} = \int_{t_0}^{t} \beta_{S_k}(\tau)d\tau + \alpha_{S_k}(t), \quad S_k \in \mathcal{P}. \tag{9}$$

Now suppose that $M^{\mathcal{P}}(x_0, T - t_0) \subset L^{\mathcal{P}}(x_0, T - t_0)$ is some optimality principle in the cooperative version of game $\Gamma^{\mathcal{P}}(x_0, T - t_0)$, and $M^{\mathcal{P}}(\bar{x}(t), T - t) \subset L^{\mathcal{P}}(\bar{x}(t), T - t)$ is the same optimality principle defined in the subgame $\Gamma^{\mathcal{P}}(\bar{x}(t), T - t)$ with an initial condition on the optimal trajectory. If $\alpha \in M^{\mathcal{P}}(x_0, T - t_0)$ and $\alpha(t) \in M^{\mathcal{P}}(\bar{x}(t), T - t)$, condition (9) gives us *time consistency* of the chosen imputation α or the chosen optimality principle in game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$.

ε-Nash Equilibrium and strong ε-Nash Equilibrium of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ In a differential game with a coalition structure, different members of a coalition may deviate from their strategies at different time moments. And the trajectory realized by the deviations possibly has no changing, which cannot be regarded as the actual deviation. To define ε-Nash Equilibrium and strong ε-Nash Equilibrium of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$, we shall define deviation instant for a coalition.

In $\Gamma(x_0, T - t_0)$, we say that for player $i \in N$ strategy $u_i(\cdot)$ is *essentially different* from strategy $\bar{u}_i(\cdot)$ under n-tuple $\bar{u}(\cdot)$ if trajectory $x_i(\cdot)$ under n-tuple $\bar{u}(\cdot)\|u_i(\cdot)$ is different from trajectory $\bar{x}_i(\cdot)$ under $\bar{u}(\cdot)$, i. e. there is $t \in (t_0, T]$ such that $x_i(t) \neq$

$\bar{x}_i(t)$. If strategies $u_i(\cdot)$ and $\bar{u}_i(\cdot)$ are essentially different, we define $\bar{t}_i(\bar{u}(\cdot)\|u_i(\cdot)) = \sup\{t : x_i(t) = \bar{x}_i(t), t \in (t_0, T]\}$ as the *deviation instant* between strategies $u_i(\cdot)$ and $\bar{u}_i(\cdot)$.

We say that coalition $S_k \in \mathcal{P}$ has *the same deviation instant* under n-tuple $\bar{u}(\cdot)$ if $\bar{t}_i(\bar{u}(\cdot)\|u_i(\cdot))$ is the same for every $i \in S_k$. We shall write $\bar{t}(\bar{u}(\cdot)\|u_{S_k}(\cdot))$ to denote $\bar{t}_i(\bar{u}(\cdot)\|u_i(\cdot))$ if S_k has the same deviation instant. We say that $\mathcal{S} \subset \mathcal{P}$ has *the same deviation instant* if $\bar{t}(\bar{u}(\cdot)\|u_{S_k}(\cdot))$ is the same for every $S_k \in \mathcal{S}$.

Suppose that every $S_k \in \mathcal{P}$ has the same deviation instant. An m-tuple $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{s_m}(\cdot))$ is an *ε-Nash equilibrium* of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ if and only if

$$H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot)) \geq H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot) \parallel u_{S_k}(\cdot)) - \varepsilon, \tag{10}$$

for all $S_k \in \mathcal{P}$ and all u_{S_k}.

Suppose that every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant. An m-tuple $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{S_m}(\cdot))$ is a *strong ε-Nash equilibrium* of $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ if and only if

$$\sum_{S_k \in \mathcal{S}} H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot)) \geq \sum_{S_k \in \mathcal{S}} H^{\alpha}_{S_k}(x_0, T - t_0; u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)) - \varepsilon, \tag{11}$$

for all $\mathcal{S} \subset \mathcal{P}$ and all $u_{\mathcal{S}} = \{u_{S_k}, S_k \in \mathcal{S}\}$.

3. Existence of ε-Nash Equilibrium and Strong ε-Nash Equilibrium in Differential Games with Coalition Structures

Theorem 1. *Suppose that every $S_k \in \mathcal{P}$ has the same deviation instant. For every $\varepsilon > 0$, the regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ has an ε-Nash equilibrium with payoff α.*

Proof. The proof is based on the construction of ε-Nash equilibrium in piecewise open-loop (POL) strategies with memory. Remind the definition of POL strategies with memory in a differential game. Denote any admissible trajectory of the system (1) on the time interval $[t_0, t], t \in [t_0, T]$ by $\hat{x}(t)$. The strategy $u_{S_i}(\cdot)$ of player S_i is called POL if it consists of the pair (a, σ), where σ is a partition of time interval $[t_0, T], t_0 < t_1 < \ldots < t_l = T, t_{k+1} - t_k = \delta > 0, k = 0, 1, 2, \ldots, l - 1$, and a is a map which corresponds an open-loop control $u_{S_i}(t), t \in [t_k, t_{k+1})$ for each point $(\hat{x}(t_k), t_k), t_k \in \sigma$.

Consider POL strategies $\bar{u}(\cdot) = (\bar{a}, \sigma)$, where $\bar{a}$ maps each point $(\bar{x}(t_k), t_k)$ on the optimal trajectories to an open-loop control $\bar{u}_{S_i}(t), t \in [t_k, t_{k+1})$ satisfying (2) and $\bar{a}$ is arbitrary at other points.

Consider a family of zero-sum games $\Gamma^{\mathcal{P}}_{\{S_i\}, \mathcal{P}\backslash\{S_i\}}(x, T - t)$ from the initial position x and duration $T - t$ between coalition $\mathcal{S}$ consisting from a single player S_i and coalition $\mathcal{P}\backslash\{S_i\}$. The payoff of player S_i is equal to $H_{S_i}(x, T - t; u_{S_1}(t), \ldots, u_{S_m}(t))$ and the payoff of player $\mathcal{P}\backslash\{S_i\}$ is equal to $(-H_{S_i})$. Let $\hat{u}(x, t; \cdot)$ be an $\frac{\varepsilon}{2}$-optimal POL strategy of player $\mathcal{P}\backslash\{S_i\}$ in $\Gamma^{\mathcal{P}}_{\{S_i\}, \mathcal{P}\backslash\{S_i\}}(x, T - t)$. Note that $\hat{u}(x, t; \cdot) = \{u_{S_j}, S_j \in \mathcal{P}\backslash\{S_i\}\}$.

Let $\hat{x}(t) = \{\hat{x}_{S_1}(t), \ldots, \hat{x}_{S_m}(t)\}$ be the segment of an admissible trajectory satisfying (1) on time interval $[t_0, t], t \in [t_0, T]$. For each $S_i \in \mathcal{P}$ define $\bar{t}(S_i) = \sup\{t : \hat{x}_{S_i}(t) = \bar{x}_{S_i}(t), t \in (t_0, T]\}$ and $\bar{\bar{t}}(S_j) = \min_{S_i} \bar{t}(S_i) = \bar{t}(S_j)$. $\bar{\bar{t}}(S_j)$ lies in one of the intervals $[t_k, t_{k+1}), k = 0, 1, 2, \ldots, l - 1$. And $\bar{\bar{t}}(S_j) - t_0$ is the length of the time interval starting from t_0 on which $\hat{x}(t)$ coincides with cooperative trajectory $\bar{x}(t)$.

Define the following strategies of player $S_i \in \mathcal{P}$.

$$u^*_{S_i}(\cdot) = \begin{cases} \bar{u}_{S_i}(t), & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative trajectory;} \\ \hat{u}_{S_i}(\hat{x}(t_{k+1}), t_{k+1}; \cdot), & S_i\text{-th component of the } \frac{\varepsilon}{2}\text{-optimal POL strategy of player } \mathcal{P}\backslash\{S_j\} \text{ in game } \Gamma^{\mathcal{P}}_{\{S_j\},\mathcal{P}\backslash\{S_j\}}(\hat{x}(t_{k+1}), T - t_{k+1}), \\ & \text{if } t_k \leq \bar{\bar{t}}(S_j)) < t_{k+1}; \\ \text{arbitrary}, & \text{for all other positions.} \end{cases}$$

To show that $u^*(\cdot) = (u^*_{S_1}(\cdot), \ldots, u^*_{S_m}(\cdot))$ is an ε-Nash equilibrium in $\Gamma^{\mathcal{P}}_\alpha(x_0, T - t_0)$, we have to show that

$$H^\alpha_{S_i}(x_0, T - t_0; u^*(\cdot)) \geq H^\alpha_{S_i}(x_0, T - t_0; u^*(\cdot) \parallel u_{S_i}(\cdot)) - \varepsilon, \tag{12}$$

for all $S_i \in \mathcal{P}$ and all u_{S_i}. It is easy to see that when m-tuple $u^*(\cdot)$ is played, the game develops along the optimal trajectory $\bar{x}(t)$. If under $u^*(\cdot) \parallel u_{S_i}(\cdot))$ trajectory $\bar{x}(t)$ is also realized then (12) will be true.

Now suppose that under $u^*(\cdot) \parallel u_{S_i}(\cdot))$ trajectory $x(t)$ is different from $\bar{x}(t)$. Suppose $\bar{t}(S_i) \in [t_k, t_{k+1})$. Since the motion of players are independent, we get $x_{S_j}(t_{k+1}) = \bar{x}_{S_j}(t_{k+1})$ for $S_j \in \mathcal{P}\backslash\{S_i\}$. From the definition of $u^*(\cdot)$ it follows that the players in $\mathcal{P}\backslash\{S_i\}$ will use their strategies $\hat{u}_{S_j}(x(t_{k+1}), t_{k+1}; \cdot)$ and player S_i starting from position $(x(t_{k+1}), t_{k+1})$ will get no more than $V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) + \frac{\varepsilon}{2}$, where $V(x(t_{k+1}), T - t_{k+1}; \{S_i\})$ is the value of game $\Gamma^{\mathcal{P}}_{\{S_i\},\mathcal{P}\backslash\{S_i\}}(x(t_{k+1}), T - t_{k+1})$. By choosing $\delta = t_{k+1} - t_k$ sufficiently small one can achieve that integral $\int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau$ will be small (less than $\frac{\varepsilon}{4}$). Then the total payoff $H^\alpha_{S_i}(x_0, T - t_0; u^*(\cdot) \parallel u_{S_i}(\cdot))$ of player S_i in game $\Gamma^{\mathcal{P}}_\alpha(x_0, T - t_0)$ when m-tuple of strategies $u^*(\cdot) \parallel u_{S_i}(\cdot))$ is played cannot exceed the amount

$$\int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau + V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) + \frac{\varepsilon}{2} \leq$$
$$\int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) + \frac{3\varepsilon}{4}. \tag{13}$$

When m-tuple $u^*(\cdot)$ is played, payoff $H^\alpha_{S_i}(x_0, T - t_0; u^*(\cdot))$ of player S_i is equal to $\alpha_{S_i} = \int_{t_0}^T \beta_{S_i}(\tau)d\tau = \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \alpha_{S_i}(t_k)$. But $\alpha_{S_i}(t_k) \in L^{\mathcal{P}}(\bar{x}(t_k), T - t_k)$, then we get $\alpha_{S_i}(t_k) \geq V(\bar{x}(t_k), T - t_k; \{S_i\})$. From the continuity of function V and continuity of trajectory $x(t)$ by appropriate choice of $\delta = t_{k+1} - t_k$ the following inequality can be guaranteed: $V(\bar{x}(t_k), T - t_k; \{S_i\}) \geq V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) - \frac{\varepsilon}{4}$. So $H^\alpha_{S_i}(x_0, T - t_0; u^*(\cdot))$ will be no less than

$$\int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T - t_{k+1}; \{S_i\}) - \frac{\varepsilon}{4}. \tag{14}$$

Combining (13) and (14) we finish the proof of Theorem 1. This means that the cooperative solution (any imputation) can be strategically supported in a regularized game $\Gamma^{\mathcal{P}}_\alpha(x_0, T - t_0)$ by a specially constructed ε-Nash equilibrium. □

Theorem 2. *Suppose that every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant. For every $\varepsilon > 0$, the strictly regularized game $\Gamma^{\mathcal{P}}_\alpha(x_0, T - t_0)$ has a strong ε-Nash equilibrium with payoff α.*

Proof. The proof is based on the construction of strong ε-Nash equilibrium in piecewise open-loop (POL) strategies with memory. Consider POL strategies $\bar{u}(\cdot) = (\bar{a}, \sigma)$, where $\bar{a}$ maps each point $(\bar{x}(t_k), t_k)$ on the optimal trajectories to an open-loop control $\bar{u}_{S_i}(t), t \in [t_k, t_{k+1}), S_i \in \mathcal{P}$, satisfying (2) and $\bar{a}$ is arbitrary at other points.

Consider a family of zero-sum games $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x, T-t)$ from the initial position x and duration $T-t$ between coalition $\mathcal{S}$ and coalition $\mathcal{P}\backslash\mathcal{S}$ in which the payoff of coalition $\mathcal{S}$ equals $\sum_{S_i \in \mathcal{S}} H_{S_i}(x, T-t; u_{S_1}(t), \dots, u_{S_m}(t))$. Let $\hat{u}_{\mathcal{P}\backslash\mathcal{S}}(x, t; \cdot)$ be an $\frac{\varepsilon}{2}$-optimal POL strategy of player $\mathcal{P}\backslash\mathcal{S}$ in $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x, T-t)$. Note that $\hat{u}_{\mathcal{P}\backslash\mathcal{S}}(x, t; \cdot) = \{u_{S_j}, S_j \in \mathcal{P}\backslash\mathcal{S}\}$.

Let $\hat{x}(t) = \{\hat{x}_{S_1}(t), \dots, \hat{x}_{S_m}(t)\}$ be the segment of an admissible trajectory satisfying (1) on time interval $[t_0, t], t \in [t_0, T]$. Since every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant, for every $\mathcal{S} \subseteq \mathcal{P}$ we can define $\bar{t}(\mathcal{S}) = \bar{t}(S_i) = \sup\{t : \hat{x}_{S_i}(t) = \bar{x}_{S_i}(t), t \in (t_0, T]\}, S_i \in \mathcal{S}$ and $\bar{\bar{t}}(\mathcal{S}) = \min_{\mathcal{S}} \bar{t}(\mathcal{S})$. $\bar{\bar{t}}(\mathcal{S})$ belongs to one of the intervals $[t_k, t_{k+1}), k = 0, 1, 2, \dots, l-1$. And $\bar{\bar{t}}(\mathcal{S}) - t_0$ is the length of the time interval starting from t_0 on which $\hat{x}(t)$ coincides with cooperative trajectory $\bar{x}(t)$.

Define the following strategies of player $S_i \in \mathcal{P}$:

$$u^*_{S_i}(\cdot) = \begin{cases} \bar{u}_{S_i}(t), & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative} \\ & \text{trajectory;} \\ \hat{u}_{S_i}(\hat{x}(t_{k+1}), t_{k+1}; \cdot), & S_i\text{-th component of the } \frac{\varepsilon}{2}\text{-optimal POL strategy} \\ & \text{of player } \mathcal{P}\backslash\mathcal{S} \text{ in game } \Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(\hat{x}(t_{k+1}), T-t_{k+1}), \\ & \text{if } t_k \le \bar{\bar{t}}(\mathcal{S}) < t_{k+1}; \\ \text{arbitrary}, & \text{for all other positions.} \end{cases}$$

We shall show that $u^*(\cdot) = (u^*_{S_1}(\cdot), \dots, u^*_{S_m}(\cdot))$ is a strong ε-Nash equilibrium in $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T-t_0)$. We have to show that

$$\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot)) \ge \sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)) - \varepsilon, \tag{15}$$

for all $\mathcal{S} \subset \mathcal{P}$ and all $u_{\mathcal{S}} = \{u_{S_i}, S_i \in \mathcal{S}\}$. It is easy to see that when m-tuple $u^*(\cdot)$ is played, the game develops along the optimal trajectory $\bar{x}(t)$. If under $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ trajectory $\bar{x}(t)$ is also realized then (15) will be true.

Now suppose that under $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ trajectory $x(t)$ is different from $\bar{x}(t)$. Suppose $\bar{t}(\mathcal{S}) \in [t_k, t_{k+1})$. Since the motion of players are independent we get $x_{S_j}(t_{k+1}) = \bar{x}_{S_j}(t_{k+1})$ for $S_j \in \mathcal{P}\backslash\mathcal{S}$. From the definition of $u^*(\cdot)$ it follows that players in $\mathcal{P}\backslash\mathcal{S}$ will use their strategies $\hat{u}_{S_j}(x(t_{k+1}), t_{k+1}; \cdot)$ and coalition $\mathcal{S}$ starting from position $(x(t_{k+1}), t_{k+1})$ will get no more than $V(x(t_{k+1}), T-t_{k+1}; \mathcal{S}) + \frac{\varepsilon}{2}$, where $V(x(t_{k+1}), T-t_{k+1}; \mathcal{S})$ is the value of game $\Gamma^{\mathcal{P}}_{\mathcal{S},\mathcal{P}\backslash\mathcal{S}}(x(t_{k+1}), T-t_{k+1})$. By choosing $\delta = t_{k+1} - t_k$ sufficiently small one can achieve that integral $\int_{t_k}^{t_{k+1}} \sum_{S_i \in \mathcal{S}} h_{S_i}(x(\tau))d\tau$ will be small (less than $\frac{\varepsilon}{4}$). Then the total payoff $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T-t_0; u^*(\cdot) \parallel u_{S_i}(\cdot))$ of coalition $\mathcal{S}$ in game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T-t_0)$ when m-tuple of strategies $u^*(\cdot) \parallel u_{\mathcal{S}}(\cdot)$ is played cannot exceed the amount

$$\sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + \sum_{S_i \in \mathcal{S}} \int_{t_k}^{t_{k+1}} h_{S_i}(x(\tau))d\tau + V(x(t_{k+1}), T-t_{k+1}; \mathcal{S}) + \frac{\varepsilon}{2} \le$$

$$\sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau)d\tau + V(x(t_{k+1}), T-t_{k+1}; \mathcal{S}) + \frac{3\varepsilon}{4}. \tag{16}$$

When m-tuple $u^*(\cdot)$ is played, payoff $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ of coalition $\mathcal{S}$ is equal to

$$\sum_{S_i \in \mathcal{S}} \alpha_{S_i} = \sum_{S_i \in \mathcal{S}} \int_{t_0}^{T} \beta_{S_i}(\tau) d\tau = \sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau) d\tau + \sum_{S_i \in \mathcal{S}} \alpha_{S_i}(t_k).$$

But $\alpha_{S_i}(t_k) \in C^{\mathcal{P}}(\bar{x}(t_k), T - t_k)$, then we get $\sum_{S_i \in \mathcal{S}} \alpha_{S_i}(t_k) \geq V(\bar{x}(t_k), T - t_k; \mathcal{S})$. From the continuity of function V and continuity of trajectory $x(t)$ by appropriate choice of $\delta = t_{k+1} - t_k$ the following inequality can be guaranteed:

$$V(\bar{x}(t_k), T - t_k; \mathcal{S}) \geq V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) - \frac{\varepsilon}{4}.$$

So $\sum_{S_i \in \mathcal{S}} H^{\alpha}_{S_i}(x_0, T - t_0; u^*(\cdot))$ will be no less than

$$\sum_{S_i \in \mathcal{S}} \int_{t_0}^{t_k} \beta_{S_i}(\tau) d\tau + V(x(t_{k+1}), T - t_{k+1}; \mathcal{S}) - \frac{\varepsilon}{4}. \tag{17}$$

Combining (16) and (17) we finish the proof of Theorem 2. □

It should be noticed that if every $\mathcal{S} \subset \mathcal{P}$ has the same deviation instant, a strong ε-Nash equilibrium is also an ε-Nash equilibrium in the strictly regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$. So the existence of strong ε-Nash equilibrium implies the existence of ε-Nash equilibrium in $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$. And if every $S \subset N$ has the same deviation instant, we can easily construct a strong ε-Nash equilibrium in the strictly regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$ from a strong ε-Nash equilibrium in the strictly regularized game $\Gamma_{\alpha}(x_0, T - t_0)$ (see Petrosyan and Zenkevich (2009)). So the existence of strong ε-Nash equilibrium in the strictly regularized game $\Gamma_{\alpha}(x_0, T - t_0)$ implies the existence of strong ε-Nash equilibrium in the strictly regularized game $\Gamma^{\mathcal{P}}_{\alpha}(x_0, T - t_0)$.

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