Contributions to Game Theory and Management, VIII, 300–314

A Fuzzy-Core Extension of Scarf Theorem and Related Topics ⋆

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Abstract The paper deals with a generalization of Scarf (Scarf, 1967) and Bondareva-Shapley (Bondareva, 1962; Shapley, 1967) theorems on the core of cooperative game to the case of fuzzy domination. The approach proposed is based on the concept of balanced collection of fuzzy coalitions, introduced by the author (Vasil'ev, 2012). This extension of classic notion of balanced collection of standard coalitions makes it possible to present a natural analog of balancedness for so-called fuzzy TU cooperative games. Moreover, it turns out that similar to the standard situation the new balancedness-like assumption is a necessary and sufficient condition for the non-emptyness of the core of fuzzy cooperative game with side payments.

Keywords: F-balanced collection, F-balanced fuzzy NTU game, V-balanced fuzzy TU game, core, S^*-representation.

1. Introduction

The goal of the paper is to present a generalization of the famous Scarf theorem on the core of cooperative game (Scarf, 1967) to the case of fuzzy domination. The approach proposed is heavily relies on the concept of balanced collection of fuzzy coalitions, introduced by the author (Vasil'ev, 2012). This generalization of the well-known notion of balanced collection of standard coalitions makes it possible to introduce a natural analog of balancedness for so-called fuzzy NTU cooperative games (fuzzy games without side payments, in terms of (Aubin, 1993)). It turned out that similar to the standard games situation the new balancedness-like condition plays a crucial role in providing the core of the fuzzy game to be nonempty. Moreover, one of the main results of the paper demonstrates that the generalized balancedness assumption is a necessary and sufficient condition for the non-emptyness of the core of fuzzy game with side payments (an extension of the classic Bondareva-Shapley theorem on the core of a standard TU cooperative game (Bondareva, 1962; Shapley, 1967). Strong attention is paid, as well, to special S^*-representation of fuzzy TU cooperative games in order to relax balancedness condition for several types of games under consideration.

Some applications of the results obtained to the fuzzy-core allocation existence problem in the framework of general equilibrium theory and cost-allocation theory are also considered.

⋆ This work was supported by the Russian Foundation for Basic Researches (grant No.13-06-00311) and Russian Foundation for Humanities (grant No.13-02-00226.)

2. Notations and definitions

Let $N = \{1, \ldots, n\}$ be a set of players. Put $I^N = \{\tau \in \mathbf{R}^N \mid 0 \le \tau_i \le 1, i \in N\}$, and denote by σ_F the set $I^N \setminus \{0\}$ of *fuzzy coalitions* over N (Aubin, 1993). As usual, each component τ_i of $\tau = (\tau_1, \ldots, \tau_n) \in \sigma_F$ is treated as the level of participation of player i in the fuzzy coalition τ. Remind (Aubin, 1993), that any standard coalition $S \subseteq N$ is identified with the corresponding indicator function $e_S \in \mathbf{R}^N$, defined by the formula $(e_S)_i = 1$ if $i \in S$, and $(e_S)_i = 0$ if $i \notin S$. Further, for each $\tau = (\tau_1, \ldots, \tau_n) \in \sigma_F$ denote by $N(\tau)$ the support of fuzzy coalition τ : $N(\tau) = \{i \in N \mid \tau_i > 0\}$. Below, we apply the shortening $\mathbf{R}^\tau := \mathbf{R}^{N(\tau)}$ for any $\tau \in \sigma_F$. Finally, in the notations given above, the definition of the games under consideration looks as follows.

Definition 1. A fuzzy NTU cooperative game (fuzzy game without side payments, according to (Aubin, 1993) is a set-valued map $\tau \mapsto G(\tau)$ that associates any fuzzy coalition $\tau \in \sigma_F$ with a subset $G(\tau)$ belonging to $\mathbf{R}^\tau$.

Each vector $(x_i)_{i \in N(\tau)} \in G(\tau)$ is called *an imputation of coalition* τ. Remind (Vasil'ev, 2012), that G is said to be *a regular game*, if the sets $G(e_{\{i\}})$ of imputations of singletons $e_{\{i\}}$, $i \in N$, and the set of imputations $G(e_N)$ of "grand coalition" e_N are nonempty and closed. Following the standard game-theoretic terminology, we say that fuzzy TU cooperative game G is *comprehensive from below*, if for any $\tau \in \sigma_F$ it holds: $x \in G(\tau)$ and $y \le x$ implies $y \in G(\tau)$.

To complete the section, we introduce one more characteristic of the fuzzy TU game, which makes it possible to propose a proper analog of standard balanced game. To this end we extend first the notion of balanced family to the case of fuzzy coalitions.

Definition 2. A finite subset $\{\tau^k\}_{k \in K} \subseteq \sigma_F$ is an F-balanced collection, if for some balancing weights $\lambda_k \ge 0$, $k \in K$, it holds: $\sum_{k \in K} \lambda_k \tau^k = e_N$.

Definition 3. A fuzzy NTU cooperative game G is said to be *F-balanced game*, if any vector $x = (x_1, \ldots, x_n)$ belongs to $G(e_N)$ whenever its restrictions $x_{N(\tau^k)}$ belong to the corresponding sets $G(\tau^k)$ for some F-balanced collection $\{\tau^k\}_{k \in K}$ (as usual, restriction of $x = (x_1, \ldots, x_n)$ to $N(\tau^k)$ is the vector $x_{N(\tau^k)} \in \mathbf{R}^{N(\tau^k)}$ with the same components just as x for each $i \in N(\tau^k)$).

3. Main result

We start with the definition of improvement in case of fuzzy NTU games.

Definition 4. We say that a fuzzy coalition $\tau \in \sigma_F$ can improve upon an imputation $x = (x_1, \ldots, x_n) \in G(e_N)$, if there exists an imputation $y \in G(\tau)$ such that $y_i > x_i$ for each $i \in N(\tau)$.

An analog of the standard core for the fuzzy NTU cooperative game G, given below, seems to be very close to the original (for more information, see (Aubin, 1993)).

Definition 5. The core $C(G)$ of a fuzzy NTU cooperative game G is the set of imputations $x \in G(e_N)$ such that no coalition $\tau \in \sigma_F$ can improve upon.

For any fuzzy NTU cooperative game G put $x^G = (x_1^G, \dots, x_n^G)$ with

$$x_i^G = \sup \{x_i \in \mathbf{R} \mid \ x_i \in G(e_{\{i\}})\}, \quad i \in N,$$

and denote by $\widehat{G}(e_N)$ the set of individually rational imputations of "grand coalition" of the game G

$$\widehat{G}(e_N) := \{x \in G(e_N) \ \mid \ x \geq x^G\}.$$

The main result of the paper is the following extension of the famous Scarf theorem to the case of fuzzy NTU cooperative games.

Theorem 1. *For any regular, comprehensive from below, and F-balanced fuzzy NTU cooperative game G with $\widehat{G}(e_N)$ to be bounded from above, the core $C(G)$ is nonempty.*

To prove Theorem 1 we apply an approach based on approximation of fuzzy NTU games, similar to that employed by Scarf for the standard NTU games. Namely, a key role in our approach belongs to fuzzy NTU finitely generated games. Results obtained for these games are interesting in itself. Moreover, we apply them to get more general facts, like Theorem 1. To describe finitely generated games remind (Vasil'ev, 2012), that a fuzzy NTU game G is said to be F-finite, if G is a regular game, and its efficiency set $e(G) := \{\tau \in \sigma_F \bigm| G(\tau) \neq \emptyset\}$ is finite.

Definition 6. A fuzzy NTU game is finitely generated, if G is F-finite, and for any $\tau \in e(G)$ there exists a finite family of vectors $u^{\tau,k} \in \mathbf{R}^\tau, k \in K(\tau), \tau \in e(G)$, such that $u^{\tau,k} \geq x^G_{N(\tau)}$, $\tau \in e(G)$, $k \in K(\tau)$, and

$$G(\tau) = \bigcup_{k \in K(\tau)} \{\omega \in \mathbf{R}^\tau \bigm| \omega \leq u^{\tau,k}\}, \quad \tau \in e(G).$$

Certainly, in comparison with general case, finitely generated fuzzy NTU games admit more simple conditions providing non-emptyness of the core.

Theorem 2. *For any fuzzy NTU F-balanced and finitely generated game G, the core $C(G)$ is nonempty.*

More details on the core of a fuzzy NTU games can be found in (Vasil'ev, 2012).

4. A fuzzy-core extension of Bondareva-Shapley theorem

As a corollary of Theorem 1 we propose an extension of the well-known Bondareva-Shapley criterion for nonemptyness of the core (Bondareva, 1962; Shapley, 1967) to the case of fuzzy TU cooperative games (fuzzy games with side payments in terms of (Aubin, 1993)). As usual, for any $x = (x_1, \dots, x_n) \in \mathbf{R}^N$ and $S \subseteq N$ denote by $x_S \in \mathbf{R}^S$ the restriction of x to $S : (x_S)_i = x_i$, $i \in S$. Further, for any $S, T \subseteq N$ and $x \in \mathbf{R}^S$, $y \in \mathbf{R}^T$ we use the notation $x \cdot y := \sum_{i \in S \cap T} x_i y_i$ (thus generalizing the standard notation for inner product to the case $S \neq T$). In the notations given above we isolate a class of games under consideration.

Definition 7. A fuzzy TU cooperative game with coalition function $v : \sigma_F \to \mathbf{R}$ is a set-valued map $\tau \mapsto G_v(\tau)$ that associates any fuzzy coalition $\tau \in \sigma_F$ with a subset of its imputations, given by the formula

$$G_v(\tau) = \{x \in \mathbf{R}^\tau \bigm| \tau \cdot x \le v(\tau)\}, \quad \tau \in \sigma_F.$$

Further, we identify any fuzzy TU cooperative game with its coalition function v.

Introduce an important characteristic of the fuzzy TU game v, which allows to present a proper analog of standard balanced TU game. To this end, following (Vasil'ev, 2012) we extend first the notion of balanced family to the case of fuzzy coalitions.

Definition 8. A finite subset $\{\tau^k\}_{k\in K} \subseteq \sigma_F$ is an F-balanced collection, if for some numbers (balancing weights) $\lambda_k \ge 0$, $k \in K$, it holds: $\sum_{k\in K} \lambda_k \tau^k = e_N$.

A fuzzy analog of classical balanced game looks as follows.

Definition 9. A fuzzy TU cooperative game v is said to be a V-balanced game, if

$$\sum_{k\in K} \lambda_k v(\tau^k) \le v(e_N)$$

for any balanced family $\{\tau^k\}_{k\in K}$ with corresponding balancing weights $\{\lambda_k\}_{k\in K}$.

Remark 1. Note, that for fuzzy TU games the following analog of classic original holds: fuzzy cooperative game G_v is F-balanced iff v is V-balanced. By applying this analog, Theorem 1, and a simple extension of the classic description of the core of the standard TU game in terms of linear inequalities (given below), one can obtain a useful criterion for the core $C(v)$ of fuzzy TU cooperative game v to be nonempty. Another approach that doesn't exploit Theorem 1 is given below, in the proof of Theorem 3.

We start with a simple description of the core $C(v)$ of fuzzy TU game v, mentioned above.

Proposition 1. *The core $C(G_v)$ of fuzzy TU cooperative game, generated by characteristic function $v : \sigma_F \to \mathbf{R}$, has the following representation*

$$C(G_v) = \{\ x \in \mathbf{R}^N \mid e_N \cdot x = v(e_N),\ \tau \cdot x \ge v(\tau), \quad \tau \in \sigma_F\}. \tag{1}$$

Proof of Proposition 1 follows directly from the more explicit definition of domination, given by conditions $(C.1), (C.2)$ below; it is left for the readers. Before to turn to the criterion itself we make several notes, useful in description of the core of fuzzy cooperative game. First of all, we propose a more detailed definition of domination in fuzzy TU games.

Definition 10. A fuzzy coalition τ dominates an imputation $x \in G_v(e_N)$, if there exists a vector $y \in \mathbf{R}^\tau$ such that

$(C.1)$ $\tau \cdot y \le v(\tau)$;

$(C.2)$ $y_i > x_i,\ i \in N(\tau)$.

In the sequel, we apply the following standard shortenings: any "singleton" $e_{\{i\}}$ is denoted by e_i, respectively, the value $v(e_{\{i\}})$ is denoted by v_i. In the notations given we introduce a set of "individually rational " imputations of the game v :

$$I(v) := \{x \in \mathbf{R}^N \mid e_N \cdot x = v(e_N),\ x_i \geq v_i, \quad i \in N\}. \tag{2}$$

By applying Proposition 1, duality theorem of linear programming and compactness of the set $I(v)$, we obtain a "fuzzy analog" of the famous Bondareva-Shapley theorem (Bondareva, 1962; Shapley, 1967) on the core of a standard cooperative game. Note once more that in TU case we establish a criterion of non-emptyness (not just sufficient conditions); repeat also that the proof given below doesn't exploit Theorem 1.

Theorem 3. *The core $C(G_v)$ of a fuzzy TU cooperative game G_v is nonempty if and only if v is V-balanced function.*

Proof. Let $C(v) \neq \emptyset$ for some fuzzy TU game v. To prove v is V-balanced we introduce an auxiliary construction related to the representation of the core $C(v)$, given in Proposition 1. Namely, denote by Σ_F collection of all the F-balanced (finite) families of fuzzy coalitions, and for any $\sigma \in \Sigma_F$ define

$$C_\sigma(v) := \{x \in \mathbf{R}^N \mid e_N \cdot x = v(e_N),\ \tau \cdot x \geq v(\tau),\ \tau \in \sigma\}. \tag{3}$$

Remind, that due to Proposition 1 we have

$$C(v) = \{x \in \mathbf{R}^N \mid e_N \cdot x = v(e_N),\ \tau \cdot x \geq v(\tau),\ \tau \in \sigma_F\}.$$

From (1) and (2) it follows: $C(v) \subseteq C_\sigma(v)$ for any $\sigma \in \Sigma_F$. Therefore, non-emptyness of $C(v)$ yields:

$$C_\sigma(v) \neq \emptyset \quad \text{for any } \sigma \in \Sigma_F. \tag{4}$$

Let now $\sigma = \{\tau^k\}_{k \in K}$ be an arbitrary F-balanced family, and $\mu = \{\mu_k\}_{k \in K}$ are corresponding weights. To prove inequality $\sum_{k \in K} \mu_k v(\tau^k) \leq v(e_N)$, we consider the following linear programming problem

$$e_N \cdot x \to \min$$
$$\tau^k \cdot x \geq v(\tau^k), \quad k \in K. \tag{A_σ}$$

Associated dual problem is as follows:

$$\sum_{k \in K} \lambda_k v(\tau^k) \to \max$$

$$\sum_{k \in K} \lambda_k \tau^k = e_N, \quad \lambda_k \geq 0,\ k \in K. \tag{A^*_σ}$$

It is clear, that the problems (A_σ) and (A^*_σ) have feasible solutions. Hence, due to the duality theorem both problems have optimality solutions, and their optimality values coincide. Designating this common value v^*, let us mention that $v^* \leq v(e_N)$ (which follows immediately from the existence of the solution x of problem (A_σ)

satisfying equality $e_N \cdot x = v(e_N)$). Hence, for the weights $\mu = \{\mu_k\}_{k\in K}$ (which constitute a feasible solution of the problem (A^*_σ)) it hold $\sum_{k\in K} \mu_k v(\tau^k) \le v^* \le v(e_N)$. Consequently, because of arbitrariness of F-balanced family σ, we get required: fuzzy TU cooperative game with nonempty core is V-balanced.

Prove now that V-balancedness of fuzzy TU cooperative game v provides solvability of the system

$$e_N \cdot x = v(e_N),\ \tau \cdot x \ge v(\tau), \quad \tau \in \sigma_F$$

(remind, that by Proposition 1 solutions of this system constitute the core $C(v)$ of the game v). Denote, as before, by σ_0 "the image of N" in σ_F: $\sigma_0 := \{e_i \mid i \in N\}$, and put

$$\widehat{\Sigma}_F := \{\sigma \subseteq \sigma_F \mid \sigma_0 \subseteq \sigma,\ |\sigma| < \infty\}.$$

It is clear that collection $\widehat{\Sigma}_F$ belongs to Σ_F. Further, due to the V-balancedness of v we have that the sets C_σ (defined by formula (3)) are nonempty and compact for any $\sigma \in \widehat{\Sigma}_F$. In fact, by F-balancedness of family σ_0 and V-balancedness of game v we get: the set of individually rational imputations $I(v)$ is nonempty (because of inequality $\sum_{i\in N} v_i \le v(e_N)$), closed and bounded (due to the inequalities $v_i \le x_i \le v(e_N) - \sum_{j\in N\setminus i} v_j$, $i \in N$, satisfied by any $x \in I(v)$). Hence, each set $C_\sigma(v)$, $\sigma \in \widehat{\Sigma_F}$, being a closed subset of $I(v)$ is a compact set. Besides, taking into account F-balancedness of σ belonging to $\widehat{\Sigma}_F$ we obtain: $C_\sigma(v)$ is nonempty for any $\sigma \in \Sigma_F$. To check this assertion, fix an arbitrary system $\sigma \in \widehat{\Sigma}_F$, and consider corresponding dual pair of problems (A_σ) and (A_{σ^*}). Due to the definition of $\widehat{\Sigma}_F$ and V-balancedness of v we have: there exists a feasible solution of the dual problem (A^*_σ) and values of its objective function on the set of its feasible solutions is bounded from above by the number $v(e_N)$. But then, by duality theorem of linear programming we have that there exists an optimal solution x^* of the problem (A_σ) such that $e_N \cdot x^* \le v(e_N)$. Consider a vector $\bar{x} = x^* + y$ such that y is an arbitrary element from $\mathbf{R}^N_+$, satisfying equality $e_N \cdot y = v(e_N) - v^*$. It is clear, that $e_N \cdot \bar{x} = e_N \cdot x^* e_N \cdot y = v(e_N)$ and $\tau \cdot \bar{x} = \tau \cdot x^* + \tau \cdot y \ge v(\tau)$ for any $\tau \in \sigma$. Therefore. we obtain: $\bar{x}$ belongs to $C_\sigma(v)$.

To complete the proof we apply the following well-known fact: any family of compact sets with the finite intersection property has nonempty intersection[1]. To apply this result let us remind first that we have already proved that $C_\sigma(v)$ is nonempty compact set for any family $\sigma \in \widehat{\Sigma}_F$. Therefore, by the intersection theorem mentioned above, to complete the proof of our theorem it is enough to establish the following properties of collection $\widehat{C} = \{C_\sigma(v)\}_{\sigma\in\widehat{\Sigma}_F}$:

(c 1) $\widehat{C}$ has finite intersection property;

(c 2) $\bigcap_{\sigma\in\widehat{\Sigma}_F} C_\sigma(v) = C(v)$.

Note, that property (c 1) follows from the fact that for any finite collection of the families $\sigma_k \in \widehat{\Sigma}_F,\ k = 1, \dots, m$, it holds

$$\bigcap_{k=1}^{m} C_{\sigma_k}(v) = C_{\widehat{\sigma}}(v),$$

[1] Remind (Hildenbrand and Kirman, 1991), that a family $\{A_\xi\}_{\xi\in\Xi}$ has finite intersection property, if any its finite subfamily has nonempty intersection.

where $\widehat{\sigma} = \cup_{k=1}^{m}\sigma_k$. Since, as we have already mentioned, V-balancedness of v implies non-emptyness of all the sets $C_\sigma(v)$, $\sigma \in \widehat{\Sigma}_F$, by (4) we obtain that any finite subcollection of collection $\widehat{C}$ has nonempty intersection: $\bigcap_{k=1}^{m} C_{\sigma_k}(v) \neq \emptyset$ for any finite collection of the families $\sigma_k \in \widehat{\Sigma}_F,\ k = 1, \ldots, m$.

Thus, the collection $\widehat{C}$ meets the requirement (c 1). Taking into account that members of this collection are nonempty and compact, to conclude the proof we check the property (c 2). In turn, it is enough to show that for any coalition $\tau \in \sigma\backslash\sigma_0$ belongs to some family σ from $\widehat{\Sigma}_F$. It is clear that to do so we can take simply $\sigma = \sigma_0 \cup \{\tau\}$. □

The non-emptyness criterion, given in Theorem 3 admits refinements in several directions. To mention just one we apply well-known Helly intersection theorem (Rockafellar, 1970) in order to diminish the number of coalitions needed to check V-balancedness of a game v (cf. Proposition 1 above). In the sequel we apply a notation

$$\widehat{\Sigma}_F^{2n} := \{\sigma \in \widehat{\Sigma}_F \bigm| |\sigma| = 2n\}.$$

Theorem 4. *Fuzzy TU cooperative game v has nonempty core $C(v)$ iff for any system $\{\tau^k\}_{k=1}^{2n} \in \widehat{\Sigma}_F^{2n}$ and corresponding system of balancing weights $\{\lambda_k\}_{k=1}^{2n}$ it holds*

$$\sum_{k=1}^{2n} \lambda_k v(\tau^k) \leq v(e_N).$$

5. On S^*-representation of fuzzy TU cooperative games

Denote by σ_F^* the set of so-called "normed fuzzy coalitions"

$$\sigma_F^* = \{\tau \in \sigma_F \mid \sum_N \tau_i = 1\}.$$

Associate with any fuzzy game $v : \sigma_F \to \mathbf{R}$ its "image", given by the following function on the subset of σ_F :

$$v^*(\tau^*) := \sup\ \{v(t\tau^*)/t \bigm| t \in (0, 1/\|\tau^*\|_\infty]\},\ \tau^* \in \sigma_F^*, \qquad (5)$$

with the domain of definition to be the simplex σ_F^* (one dimension less than for σ_F).

Definition 11. We say that function v^* defined above is S^*-representation of the fuzzy game v.

Remark 2. It is clear, that for any homogeneous game v its S^*-representation coincides with restriction v to the simplex ${\sigma_F}^*$: $v^* \equiv v_{|\sigma_F^*}$. Remind, that fuzzy TU game v is called homogeneous (of degree 1), if $v(t\tau) = tv(\tau)$ for any $t \geq 0$ such that $t\tau$ belongs to σ_F.

Below,we pay strong attention to those fuzzy games that have nonempty cores. We mention first some properties of these games in terms of their S^*-representation.

Definition 12. A fuzzy TU game v is said to be S^*-regular, if its S^*-representation v^* satisfies the following requirements:

(S^*.1) $v(\tau^*) < \infty$ for any $\tau^* \in \sigma_F^*$,

(S^*.2) $v^*(e_N/n) \le v(e_N)/n$.

We show that S^*-regularity is a necessary condition for the non-emptyness of the core.

Proposition 2. *If a fuzzy TU cooperative game v has nonempty core, then v is S^*-regular.*

Proof. Let $C(v)$ be nonempty, and at the same time we have $v^*(\tau^*) = \infty$ for some $\tau^* \in \sigma_F^*$. Fix an arbitrary $x \in C(v)$ and show that this imputation is dominated by coalition $t\tau^*$ under $t > 0$ small enough. To do so put $a := \|x\|_\infty = \max\{|x_i| \mid i \in N\}$ and choose $t > 0$ such that inequality

$$v(t\tau*)/t > a \tag{6}$$

holds (it is clear that such t exists due to the definition of v^* and supposition $v^*(\tau^*) = \infty$ given). Define now $\tau := t\tau^*$, $b := v(\tau)/t$ and show that the vector $y \in \mathbf{R}^\tau$ with $y_i = b$, $i \in N(\tau)$, belongs to $G_v(\tau)$, and moreover, it dominates x via coalition τ in the game v. Indeed, equalities $y \cdot t\tau^* = v(t\tau^*) \sum_{i \in N(t\tau^*)} \tau_i^* = v(\tau)$, following from the construction of y and τ, proves inclusion $y \in G_v(\tau)$. Further, from inequality (6) it follows that for any $i \in N(\tau)$ it holds $y_i > v(\tau)/t > a$. Thus, y belongs to $G_v(t\tau^*)$ and, besides, $y_i > a \ge x_i$ for any $i \in N(t\tau^*)$. Hence, we prove that x is dominated by coalition $\tau = t\tau^*$ (via imputation y), which contradicts to the initial assumption $x \in C(v)$. The contradiction obtained demonstrates property (S^*1) for any v with $C(v) \neq \emptyset$.

To prove (S^*2) suppose, to the contrary, that $v^*(e_N^*) > v(e_N)/n$. Then, according to the definition of $v^*(e_N^*)$ we get: there exists $t > 0$ such that te_N^* belongs to σ_F^* and, moreover, by (5) we obtain

$$v(te_N^*)/t > v(e_N)/n.$$

Applying the last inequality we prove that the coalition $\tau := t\tau^*$ dominates each imputation from $G_v(e_N)$. Indeed, let x be any element from $G_v(e_N)$. Without loss of generality, we may assume that $x \cdot e_N = v(e_N)$. Choose number c such that $x + \bar{c}$ with $\bar{c} = (c, \ldots, c)$ meets the requirement

$$(x + \bar{c}) \cdot te_N^* = v(te_N^*). \tag{7}$$

Remove parentheses in (7), and taking account equality $x \cdot e_N = v(e_N)$ we get

$$v(te_N^*) = x \cdot te_N^* + \bar{c} \cdot te_N^* = \frac{t}{n} v(e_N) + t\bar{c} \cdot e_N^*.$$

Consequently, we obtain the following equalities for the vector $\bar{c}$:

$$c \cdot e_N^* = \big[v(te_N^*) - \frac{t}{n} v(e_N)\big]/t = \big[v(te_N^*)/t - v(e_N)/n\big].$$

Applying this equalities and inequality (6) to evaluate the sign of number c we get: $c = \bar{c} \cdot e_N^* = v(te_N^*)/t - v(e_N)/n > 0$.

Hence, due to (7) we get: vector $z := x + \bar{c}$ belongs to $G_v(te_N^*)$ and, by positivity of c, satisfies inequalities $z_i = x_i + c > x_i$, $i \in N$. Therefore, fuzzy coalition te_N^* dominates imputation $x \in G_v(e_N)$. Since x was taken as an arbitrary element of $G_v(e_N)$, the latter conclusion means that the core of the game v is empty. But this contradicts our assumption $C(v) \neq$. Hence, any TU fuzzy cooperative game with nonempty core meets requirement (S^*2). □

Remark 3. It follows immediately from the very definition of function v^* that assumption $(S^*.2)$ is valid whenever equality

$$v^*(e_N^*) = v(e_N)/n$$

takes place.

To avoid misrepresentations, let us stress that in the sequel, symbol v^* denotes the game G_{v^*}, defined on σ_F^* by the formula

$$G_{v^*}(\tau) := \{x \in \mathbf{R}^\tau \mid \tau \cdot x \leq v^*(\tau)\}, \quad \tau \in \sigma_F^*.$$

In accordance with the definitions, given above, we say that coalition $\tau \in \sigma_F^*$ improves upon (dominates) an imputation $x \in G_{v^*}(e_N^*)$, if there exists $y \in \mathbf{R}^\tau$ such that $y \cdot \tau \leq v(\tau)$ and $y_i > x_i$ for each $i \in N(\tau)$. An imputation $x \in G_{v^*}(e_N^*)$ that is not improved upon (dominated) by any coalition $\tau \in \sigma_F^*$ is called *non-dominated*. As before, the set of all members of $G_{v^*}(e_N^*)$ that are non-dominated is said to be the core of the game v^*. The core of the game v^*, defined on the simplex σ_F^*, we denote by $C(v^*)$ (like in situations, when games are defined on the hypercube $[0,1]^N$). To exclude ambiguity and to mark games, defined on σ_F^*, we apply asterisk, as before (if necessary).

Similar to games defined on the hypercube, it is quite easy to get the following analog of Proposition 1, providing a useful representation of the core $C(v^*)$.

Proposition 3. *For any fuzzy TU cooperative game $v^* : \sigma_F^* \to \mathbf{R}$ it holds*

$$C(v^*) = \{x \in \mathbf{R}^N \mid e_N^* \cdot x = v^*(e_N^*),\ \tau^* \cdot x \geq v^*(\tau^*),\ \tau^* \in \sigma_F^*\}.$$

By applying this representation and argumentation similar to that used in the proof of Theorem 3, we get the following analog for games defined on the simplex σ_F^*.

Theorem 5. *The core $C(v^*)$ of a game $v^* : \sigma_F^* \to \mathbf{R}$ is nonempty if and only if for any representation of the center of gravity $e_N^* = e_N/n$ of simplex σ_F^* as a convex combination $e_N^* = \sum_{k \in K} \lambda_k^* \tau_*^k$ of some elements τ_*^k, $k \in K$, of the simplex σ_F^* it holds*

$$v^*(e_N^*) \geq \sum_{k \in K} \lambda_k^* v^*(\tau_*^k).$$

Proof. Due to Proposition 3 we can apply the same argumentation, as in the proof of Theorem 3. Hence, we restrict ourselves to consideration of the non-emptyness criterion for the set $\{x \in \mathbf{R}^N \mid e_N^* \cdot x = v^*(e_N^*),\ \tau^* \cdot x \geq v(\tau^*),\ \tau^* \in \sigma_F^*\}$, and specific character of the definition domain of the game v^*. Namely, doing the same way, as in proof of Theorem 3 we get: the core $C(v^*)$ of fuzzy TU game $v^* : \sigma_F^* \to \mathbf{R}$ is nonempty whenever $\sum_{k\imath K} \lambda_k^* v^*(\tau_*^k) \leq v^*(e_N^*)$ for any collection $\{\tau_*^k\}_{k\in K} \subseteq \sigma_F^*$ with corresponding nonnegative weights $\{\lambda_k^*\}_{k\in K}$, satisfying equality $\sum_{k\in K} \lambda_k^* \tau_*^k = e_N^*$. By summing up the components on the left and right hand-sides of this last vector equality we obtain: $\sum_{k\in K} \lambda_k^* \sum_{i\in N} (\tau_*^k)_i = 1$. Taking account that all the points τ_*^k belong to the standard simplex, we get equality $\sum_{k\in K} \lambda_*^k = 1$, which completes the proof of Theorem 5. □

Definition 13. A fuzzy TU cooperative game v is said to be a concave w.r.t. coalition e_N^*, if for any convex representation $e_N/n = \sum_{k\in K} \lambda_k^* \tau_*^k$ (with $\tau_*^k \in \sigma_F^*$ and $\lambda_k^* \in \mathbf{R}_+$, $k \in K$, such that $\sum_{k\in K} \lambda_k^* = 1$) it holds

$$\sum_{k\in K} \lambda_k^* v^*(\tau_*^k) \leq v^*(e_N/n). \tag{8}$$

It is quite easy to prove by induction that a game $v^* : \sigma_F^* \to \mathbf{R}$. is a concave function w.r.t. the center of gravity e_N^* whenever the inequalities (8) take place just for two summands: $|K| = 2$. Hence, we get the following simplified core-non-emptyness criterion for the concave w.r.t. the center of gravity e_N^* games v^*.

Corollary 1. *If $v^*(e_N^*) \geq \lambda_1^* v^*(\tau_*^1) + \lambda_2^* v^*(\tau_*^2)$ for any fuzzy coalitions $\tau_*^1, \tau_*^2 \in \sigma_F^*$ and numbers $\lambda_1^*, \lambda_2^* \in \mathbf{R}_+$ such that $\lambda_1^* + \lambda_2^* = 1$ and $e_N^* = \lambda_1^* \tau_*^1 + \lambda_2^* \tau_*^2$, then the core $C(v^*)$ of TU game $v^* : \sigma_F^* \to \mathbf{R}$ is nonempty.*

Remark 4. In case v is homogeneous ($v(t\tau) = tv(\tau)$ for any τ, $t\tau \in \sigma_F$) we can replace v^* by initial game v in inequality (8).

Theorem 6. *For any fuzzy TU game v and its S^*-representation v^* it holds*

$$C(v) = C(v^*).$$

Proof. First prove insertion $C(v) \subseteq C(v^*)$. Let x be an element of the core $C(v)$. Fix some $\tau^* \in \sigma_F^*$ and prove that $x \cdot \tau^* \geq v^*(\tau^*)$. To this end let us mention that due to the inclusion $x \in C(v)$ and Proposition 1 we have $x \cdot t\tau^* \geq v(t\tau^*)$ for any $t > 0$ such that $t\tau^* \in \sigma_F$. Hence, $x \cdot \tau^* \geq v(t\tau^*)/t$ for any $t \in \big(0, 1/\|\tau^*\|_\infty\big]$ and, consequently, it holds

$$x \cdot \tau^* \geq \sup \left\{v(t\tau^*)/t \;\middle|\; t \in \big(0, 1/\|\tau^*\|_\infty\big]\right\} = v^*(\tau^*).$$

Hence, due to Proposition 3 to complete the proof of inclusion $x \in C(v^*)$ we have to state that $x \cdot e_N^* = v^*(e_N^*)$. Since it has already been proved that $x \cdot e_N^* \geq v^*(e_N^*)$ we just check the opposite inequality $x \cdot e_N^* \leq v^*(e_N^*)$. To do so we just mention that due to the assumption $x \in C(v)$ and Proposition 1 we have

$$x \cdot e_N = v(e_N). \tag{9}$$

Further, by our assumption the core $C(v)$ is nonempty. Hence, Proposition 2 and Remark 3 yield equality $v^*(e_N^*) = v(e_N)/n$. Combining this last equality and relation (9), we get required: $x \cdot e_N^* = v^*(e_N^*)$.

To prove the opposite inclusion $C(v^*) \subseteq C(v)$, consider an arbitrary imputation $x \in C(v^*)$, and fix some $\tau \in \sigma_F$. To prove the inequality $x \cdot \tau \geq v(\tau)$ we note first that by definition of σ_F and σ_F^* there exist number $t > 0$ and coalition $\tau^* \in \sigma_F^*$ such that $\tau = t\tau^*$. Further, inclusion $x \in C(v^*)$ and definition of v^* imply inequalities $x \cdot \tau^* \geq v^*(\tau^*) \geq v(t\tau^*)/t$. Hence, we get: $x \cdot \tau^* \geq v(t\tau^*)/t$. After multiplying this last inequality by t we obtain required: $x \cdot \tau \geq v(\tau)$. As to the equality $x \cdot e_N = v(e_N)$, it follows immediately from the equality $e_N^* \cdot x = v^*(e_N^*)$ and Remark 3. □

Corollary 2. *The core $C(v)$ of a fuzzy TU cooperative game v is nonempty if and only if the core $C(v^*)$ of its S^*-representation v^* is nonempty.*

6. Applications to some allocation problems

Below some applications of Theorems 1 and 3 to the cost and profit allocation problems in mathematical economics and game theory are given. We derive several fuzzy core allocation existence conditions for the well-known pure exchange model and airport game. The most surprising result of the considerations proposed is as follows: sufficient conditions for the non-emptyness of the fuzzy core are either the same or very close to the classical conditions for the standard core.

6.1. Fuzzy core of a pure exchange model

We show in this subsection that an application of Theorem1 to the fuzzy domination in a pure exchange model yields quite unexpected result: rather weak standard conditions guaranteeing non-emptyness of the classical core of the economy provides non-emptyness of its fuzzy core, as well (even though the fuzzy core, normally, constitute a very "small" subset of standard core of the economy under consideration (Ekeland, 1979)).

Remind (Ekeland, 1979; Hildenbrand and Kirman, 1991), that pure exchange model $\mathcal{E}$ is given by the following data

$$\mathcal{E} = \langle N, \{X_i, u_i, w^i\}_{i \in N} \rangle,$$

where $N = \{1, \dots, n\}$ is a set of economic agents (participants), and $X_i \subseteq \mathbf{R}^l,\ w^i \in \mathbf{R}^l,\ u_i : X_i \to \mathbf{R}$ are their consumption sets, initial endowments and utility functions, respectively. An integer $l \geq 1$ denotes the number of commodities involved in exchange.

Consider fuzzy NTU cooperative game $G_{\mathcal{E}}$ in strategic form, associated with the exchange model $\mathcal{E}$. According to (Aubin, 1993; Ekeland, 1979), coalitional strategies of $G_{\mathcal{E}}$ (feasible τ-allocations of $\mathcal{E}$) are given by the formula

$$X_{\mathcal{E}}(\tau) = \{(x^i)_{i \in N(\tau)} \in \prod_{i \in N(\tau)} X_i \mid \sum_{i \in N(\tau)} \tau_i x^i = \sum_{i \in N(\tau)} \tau_i w^i\}, \quad \tau \in \sigma_F.$$

In particular, feasible e_N-allocations compose the set of all available distributions of the total endowment $\sum_{i \in N} w^i$ of the economy $\mathcal{E}$:

$$X_{\mathcal{E}}(e_N) = \{(x^i)_{i \in N} \in \prod_{i \in N} X_i \mid \sum_{i \in N} x^i = \sum_{i \in N} w^i\},$$

and feasible $e_{\{i\}}$-allocations constitute singletons $\{w^i\}$ in case w^i belongs to X_i, otherwise $X_{\mathcal{E}}(e_{\{i\}})$ is empty.

Recall standard definition of fuzzy blocking (domination via fuzzy coalitions) used in mathematical economics (Aubin, 1993; Ekeland, 1979), and formulate classical definition of fuzzy core in pure exchange economy of type $\mathcal{E}$.

Definition 14. A coalition $\tau \in \sigma_F$ blocks an allocation $x = (x^i)_{i \in N} \in X_{\mathcal{E}}(e_N)$, if there exists

$$\tilde{x} = (\tilde{x}^i)_{i \in N(\tau)} \in X_{\mathcal{E}}(\tau)$$

such that $u_i(\tilde{x}^i) > u_i(x^i)$ for any $i \in N(\tau)$. Collection of allocations from $X_{\mathcal{E}}(e_N)$ that are not blocked by any coalition from σ_F is denoted by $C_F(\mathcal{E})$ and is said to be the fuzzy core of the model $\mathcal{E}$.

Describe now a fuzzy NTU cooperative game $G_F^{\mathcal{E}}$, associated with the pure exchange model $\mathcal{E}$ in order to simplify applications of Theorem 1 to the investigation of the fuzzy core non-emptyness problem, relating to the economy $\mathcal{E}$.

Definition 15. Fuzzy NTU cooperative game $G_F^{\mathcal{E}}$ associated with the pure exchange model $\mathcal{E}$ is said to be NTU cooperative game defined by the formula

$$G_F^{\mathcal{E}}(\tau) = \{\omega \in \mathbf{R}^\tau \mid \exists (x^i)_{i \in N(\tau)} \in X_{\mathcal{E}}(\tau)[\omega_i \le u_i(x^i),\ i \in N(\tau)]\}, \quad \tau \in \sigma_F.$$

We omit a straightforward proof of the following technical proposition.

Proposition 4. *For any pure exchange model $\mathcal{E}$ the core $C(G_{\mathcal{E}}^F)$ is nonempty if and only if the fuzzy core $C_F(\mathcal{E})$ is nonempty.*

Rather natural and simple conditions providing that the game $G_{\mathcal{E}}^F$ is F-balanced are as follows.

Proposition 5. *Suppose that consumption sets X_i of the model $\mathcal{E}$ are convex and contain corresponding initial endowments w^i, and its utility functions u_i are quasi-concave. Then the game $G_{\mathcal{E}}^F$ is F-balanced.*

Proof. Let $\{\tau^k\}_{k \in K}$ be some F-balanced collection of fuzzy coalitions, and $\{\lambda_k\}_{k \in K}$ are their weights. Fix an arbitrary vector $\omega \in \mathbf{R}^N$ satisfying relations: $\omega_{N_k} \in G_{\mathcal{E}}^F(\tau^k)$ for each $k \in K$, where $N_k = N(\tau^k)$, $k \in K$. Due to definition of correspondence $G_{\mathcal{E}}^F$, for any $k \in K$ there exists coalitional allocation $(x^{k,i})_{i \in N_k} \in X(\tau^k)$ such that

$$\omega_i \le u_i(x^{k,i}), \quad i \in N_k. \tag{10}$$

Put $\mu_{ki} = \lambda_k \tau_i^k$, $i \in N_k$, $k \in K$, and define allocation $\bar{x} = (\bar{x}^i)_{i \in N}$ by the formula

$$\bar{x}^i = \sum_{k \in K_i} \mu_{ki} x^{k,i}, \quad i \in N, \tag{11}$$

where, as before, $K_i = \{k \in K \mid i \in N_k\}$, $i \in N$. Taking into account equality $\sum_{k \in K} \lambda_k \tau^k = e^N$ and definition of the numbers μ_{ki} we get: $\mu_{ki} \ge 0$ for each $k \in K$ and $i \in N$ and, besides, $\sum_{k \in K_i} \mu_{ki} = 1$ for each $i \in N$. This relations together with convexity of X_i, $i \in N$, and formula (11) imply inclusions: $\bar{x}^i \in X_i$ for each $i \in N$. Applying once more the fact that the bundles $\bar{x}^i$ are convex combinations of the elements of corresponding consumption sets X_i, due to the quasi-concavity of utility functions u_i, from (10) it follows: $u_i(\bar{x}^i) \ge \omega_i$ for any $i \in N$. To complete the proof of inclusion $\omega \in G_{\mathcal{E}}^F(N)$ it remains to check that allocation $\bar{x} = (\bar{x}^i)_{i \in N}$ satisfies equality $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i$. Carrying out elementary transformations needed, from (11) we get (for the left hand-side of the equality checked):

$$\sum_{i \in N} \bar{x}^i = \sum_{i \in N} \sum_{k \in K_i} \mu_{ki} x^{k,i} = \sum_{k \in K} \sum_{i \in N_k} \mu_{ki} x^{k,i} = \sum_{k \in K} \lambda_k \Big(\sum_{i \in N_k} \tau_i^k x^{k,i} \Big).$$

But the latter sum, due to the equality $\sum_{i\in N_k}\tau_i^k x^{k,i} = \sum_{i\in N_k}\tau_i^k w^i$ takes the form $\sum_{k\in K}\lambda_k(\sum_{i\in N_k}\tau_i^k w^i)$. Change places in the summation formula for the last expression we obtain required: $\sum_{i\in N}(\sum_{k\in K_i}\mu_{ki})w^i = \sum_{i\in N} w^i$. □

Passing through the standard argumentation (see, e.g., (Aubin, 1993, Ekeland, 1979), we present rather simple assumptions providing that all the other requirements of Theorem 1 (besides the F-balancedness) are valid for the game $G_F^{\mathcal{E}}$.

Proposition 6. *Suppose that consumption sets X_i, $i \in N$, of the exchange model $\mathcal{E}$ are closed, bounded from below and contain corresponding initial endowments $w^i, /, i \in N$; and utility functions u_i, $i \in N$, are continuous on corresponding consumption sets. Then the sets $G_{\mathcal{E}}^F(\tau)$ are nonempty, close and comprehensive from below for any fuzzy coalition $\tau \in \sigma_F$, and, besides, the set $\widehat{G}_{\mathcal{E}}^F(e_N)$ is nonempty and bounded from above.*

By applying Theorem 1 and Propositions 5 and 6, we conclude that under the same assumptions that provide non-emptyness of the standard core in the pure exchange model $\mathcal{E}$ we guarantee realizability of so considerably more subtle optimality principle as fuzzy core $C_F(\mathcal{E})$.

Theorem 7. *Suppose initial endowments w^i belong to the corresponding consumption sets X_i for any $i \in N$ and, besides, consumption sets X_i, $i \in N$, are convex, closed and bounded from below, utility functions u_i, $i \in N$, are continuous and quasiconcave. Then fuzzy core $C_F(\mathcal{E})$ of pure exchange model $\mathcal{E}$ is nonempty.*

6.2. Non-dominated cost allocations of airport game

Remind that in terms of (Peleg and Sudhölter, 2003), a cost allocation problem is an ordinary game (N, c), where c is the cost function $c : 2^N \to \mathbf{R}$, representing the least cost of serving the members of a standard coalition S by the most efficient means. Hence, to solve the cost allocation problem we are to find some vector $x \in \mathbf{R}^N$ such that $e_N \cdot x = c(N)$, $e_S \cdot x \le c(S)$, $S \in 2^N$ (recall, that 2^N denotes the family of all the nonempty subsets of N). To sharpen requirements to the solution of our problem, we may extend function c in appropriate way from 2^N to the hypercube $[0,1]^N$ and consider an analog of the standard core we deal with earlier:

$$D(d_c) = \{x \in \mathbf{R}^N \mid e_N \cdot x = d_c(N),\ \tau \cdot x \le d_c(\tau), \quad \tau \in \sigma_F\}, \tag{12}$$

where $d_c(e_S) = c(S)$ for any $S \in 2^N$. It can easily be checked that Theorem 3 implies the following non-emptyness criterion for $D(d_c) : D(d_c)$ is nonempty whenever d_c is D-balanced. Here, we mean d_c is D-balanced, if for any F-balanced family $\{\tau^k\}_{k\in K}$ and corresponding family of balancing weights $\{\lambda_k\}_{k\in K} \subseteq \mathbf{R}_+$ it holds

$$\sum_{k\in K}\lambda_k d_c(\tau^k) \ge d_c(e_N).$$

To give an example of a cost allocation problem with natural and nontrivial extension of cost function consider a slight modification of airport game from (Peleg and Sudhölter, 2003). Suppose that we have an airport with one runway to be constructed, and that there are n different aircrafts characterized by the positive numbers c_i, $i = 1, \dots, n$, being the costs of building a runway to accommodate the aircraft $i \in N = \{1, \dots, n\}$. The cost function c representing the least cost of serving the members of coalition $S \in 2^N$ is given by

$$c(S) = \max \{c_i \mid i \in S\}, \quad \text{and} \quad c(\emptyset) = 0.$$

Consider an extension d_c of airport game c to the set σ_F, defined by the formula

$$d_c(\tau) = \max \{\tau_i c_i \mid i \in N\}, \quad \tau = (\tau_1, \dots, \tau_n) \in \sigma_F. \tag{13}$$

Proposition 7. *Function d_c, given by the formula (13) is D-balanced.*

Proof. Let $\{\tau^k\}_{k\in K} \subseteq \sigma_F$ and $\{\lambda_k\}_{k\in K} \subseteq \mathbf{R}_+$ be an F-balanced family of fuzzy coalitions and corresponding balancing weights, respectively. Denote by $i_0 \in N$ the player with maximal cost: $c_{i_0} = \max \{c_i \mid i \in N\}$. By definition of function d_c we have

$$d_c(\tau^k) \geq \tau^k_{i_0} c_{i_0}, \quad k \in K. \tag{14}$$

Multiplying each inequality in (14) by corresponding balancing weight λ_k, $k \in K$, and summing up the inequalities obtained we get

$$\sum_{k\in K} \lambda_k d_c(\tau^k) \geq (\sum_{k\in K} \lambda_k \tau^k_{i_0}) c_{i_0}.$$

From this inequality (taking into account equations $\sum_{k\in K} \lambda^k \tau^k_i = 1,\ i \in N$)we obtain required: $\sum_{k\in K} \lambda_k d_c(\tau^k) \geq c_{i_0} = d_c(e_N)$. □

Corollary 3. *For any airport game (N, c) its fuzzy D-core, given by the formula (12) is nonempty.*

Remark 5. Airport games give a number of examples of sharp diminishing of D_0-core under the transition to the fuzzy D-core. Recall that the D_0-core is given by the formula

$$D_0(c) := \{x \in \mathbf{R}^N \mid e_N \cdot x = c(N),\ e_S \cdot x \leq c(S), \quad S \in 2^N\},$$

To complete this subsection, we present rather simple 3-person airport game with $c_1 = 2, c_2 = 3, c_3 = 5$, which has the cores discussed differing from each other as strongly, as possible

$$D_0(c) = \text{conv}\ \{(2,1,2), (2,0,3), (0,3,2), (0,0,5)\},$$

$$D(d_c) = \{(0,0,5)\}.$$

7. Conclusion

In conclusion, we consider an accessibility problem for the fuzzy TU cooperative games. Accessibility under discussion is similar to that investigated in (Vasil'ev, 1987) for the classical cooperative games (for more details, concerning the classical case, see (Vasil'ev, 2006)). A natural analog of the classical domination for the fuzzy coalitions is given by definition below.

Definition 16. We say that an imputation $y \in I(v)$ dominates an imputation $x \in I(v)$, if there exists a coalition $\sigma \in \sigma_F$ such that

$$1)\ x_i < y_i,\ i \in N(\tau),\ \text{and}\ 2)\ \tau \cdot y \leq v(\tau).$$

Domination relation, defined by conditions 1), 2) we denote by α_F^F :

$$x\ \alpha_v^F\ y \Leftrightarrow \exists \tau \in \sigma_F[(x_i < y_i,\ i \in N(\tau))\ \&\ (\tau \cdot y \leq v(\tau))], \quad x, y \in I(v).$$

Similar to the definition, given in (Vasil'ev, 1987), we propose the following notion.

Definition 17. A sequence $\{x_r\}_{r=0}^{\infty} \subseteq I(v)$ of imputations of a fuzzy TU game v is said to be α_v^F-monotone, if $x_r\ \alpha_v^F\ x_{r+1}$ for any $r \geq 0$.

Accessibility Problem (discrete version). Suppose, the core $C(v)$ of a fuzzy TU cooperative game v is nonempty, and x is an arbitrary imputation outside the core ($x \in I(v) \setminus C(v)$). Does there exist any α_v^F-monotone convergent sequence of imputations $\{x_r\}_{r=0}^{\infty} \subseteq I(v)$ such that $x_0 = x$, and $\lim_{r\to\infty} x_r$ belongs to the core $C(v)$?

We propose also a continuous version of the α_v^F-monotone sequence in the following way.

Definition 18. We say that a trajectory $x : T \to I(v)$ with $T = [0, 1]$ is an α_v^F-monotone, if there exists a function $\delta : T \to \mathbf{R}_+ \setminus \{0\}$ such that $x(t')\ \alpha_v^F\ x(t)$ for any $t, t' \in T$ with $t' < t, t - t' < \delta(t)$.

Accessibility Problem (continuous version). Suppose, the core $C(v)$ of a fuzzy TU cooperative game v is nonempty, and $x_0 \in I(v)$ is an arbitrary imputation that doesn't belong to the core $C(v)$. Whether there exist an α_v^F-monotone convergent trajectory $x : T \to I(v)$ with $x(0) = x_0$ and $\lim_{t\to 1} x(t)$ belonging to the core $C(v)$?

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