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A Time-consistent Solution Formula for Linear-Quadratic Discrete-time Dynamic Games with Nontransferable Payoffs⋆

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Abstract The solution formula for the payoff distribution procedure of a bargaining problem in a cooperative differential game with nontransferable payoffs that lead to a time consistent outcome is proposed by Leon A. Petrosyan and D.W.K. Yeung (2014). In this paper we study this formula for linear-quadratic discrete-time dynamic games with nontransferable payoffs. Pareto-optimal solution is studied as optimality principle. The time consistency and irrational behavior proof condition of this solution are investigated. As an example the government debt stabilization game is considered.

Keywords: linear-quadratic games, discrete-time games, games with nontransferable payoffs, Pareto-optimal solution, time consistency, PDP, irrational behavior proof condition.

1. Introduction

Consider N-person discrete-time dynamic game $\Gamma(k_0, x_0)$ which is described by the state equation

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{n} B_i(k)u_i(k), \qquad k_0 \le k \le K < \infty, \quad x(k_0) = x_0, \tag{1}$$

where

– x is m-dimensional state of system,
– u_i is a r-dimensional control variable of player i,
– $x(k_0) = x_0$ is the arbitrarily chosen initial state of the system,
– $A(k), B_i(k)$ are matrices of appropriate dimensions.

The quadratic cost function of player $i \in N$ is

$$J_i(k_0, x_0, u) = \sum_{k=k_0}^{K-1} \Big(x^T(k)P_i(k)x(k) + u_i^T(k)R_i(k)u_i(k) \Big) + x^T(K)P_i(K)x(K), \qquad \forall i = 1, \dots, n, \tag{2}$$

– $P_i(k) = P_i^T(k), \quad R_i(k) = R_i^T(k)$,
– $P_i(k)$ – positive semidefinite matrices, $R_i(k)$ – positive definite matrices, $i \in N$.

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Suppose that payoffs are nontransferable.

We will assume that the players use feedback strategies $u_i(k,x) = M_i(k)x$ to minimize their costs.

Suppose that players agree to use a Pareto-optimal solution as optimality principle and use vector of weights

$$\alpha = (\alpha_1, \ldots, \alpha_n): \quad \sum_{i=1}^{n} \alpha_i = 1, \quad 0 < \alpha_i < 1$$

on their costs to obtain a Pareto-optimal outcome.

According to (Engwerda, 2005) find optimal cooperative strategies of players solving

$$\min_{(u_1,\ldots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(k_0, x_0, u). \tag{3}$$

Let

$$(u_1^\alpha, \ldots, u_n^\alpha) = \arg \min_{(u_1,\ldots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(k_0, x_0, u), \tag{4}$$

$$J^\alpha(k_0, x_0, u) = \sum_{i=1}^{n} \alpha_i J_i(k_0, x_0, u),$$

$$P^\alpha(k) = \sum_{i=1}^{n} \alpha_i P_i(k),$$

$$R^\alpha(k) = \begin{pmatrix} \alpha_1 R_1(k) & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & \alpha_2 R_2(k) & \ldots & \mathbb{O} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{O} & \mathbb{O} & \ldots & \alpha_n R_n(k) \end{pmatrix}.$$

Then

$$J^\alpha(k_0, x_0, u) = \sum_{k=k_0}^{K-1} (x^T(k) P^\alpha(k) x(k) + u(k) R^\alpha(k) u(k)) + x^T(K) P^\alpha(K) x(K). \tag{5}$$

Finding of Pareto-optimal solution is reduced to linear-quadratic optimal control problem (1)-(5) with one control variable $u(k)$.

According to (Başar and Olsder, 1999) there exists the unique control in class of admissible $\{u_i^\alpha(k) = M_i^\alpha(k)x, \quad i = 1, \ldots, n\}$, minimizing $J^\alpha(k_0, x_0, u)$, where $M_i^\alpha(k)$ – i-th block of matrix $M^\alpha(k)$,$\{M^\alpha(k), \Theta^\alpha(k)\}$ – solution of system of matrix equations

$$\begin{cases} (A(k) + B(k)M^\alpha(k))^T \Theta^\alpha(k+1)(A(k) + B(k)M^\alpha(k)) - \Theta^\alpha(k) + \\ + P^\alpha(k) + M^\alpha(k)^T R^\alpha(k) M^\alpha(k) = 0, \\ M^\alpha(k) = -(R^\alpha(k) + B^T(k)\Theta^\alpha(k+1)B(k))^{-1} B^T(k) \Theta^\alpha k + 1) A(k), \\ k = 1, \ldots, K-1, \\ \Theta^\alpha(K) = P^\alpha(K), \end{cases} \tag{6}$$

where $\Theta^\alpha(k)$ – is symmetric. Here $B(k) = (B_1(k), \dots, B_n(k))$.

The cooperative state trajectory $x^\alpha(k)$ one can find by substituting the cooperative strategies $\{u_i^\alpha(k)\}$ in (1) and solving the system:

$$x(k+1) = A(k)x(k) + B(k)u^\alpha(k). \tag{7}$$

And payoffs of players are:

$$J_i^\alpha(k_0, x_0, u^\alpha) = \sum_{k=k_0}^{K-1} \Big((x^\alpha(k))^T P_i(k) x^\alpha(k) + (u_i^\alpha(k))^T R_i(k) u_i^\alpha(k) \Big) + \\ + (x^\alpha(K))^T P_i(K) x^\alpha(K). \tag{8}$$

2. Time-consistency

Suppose that there exists such α, that inequalities

$$J_i^\alpha(k_0, x_0, u^\alpha) \le V_i(k_0, x_0), \quad i = 1, \dots, n. \tag{9}$$

requiring for individual rationality in the cooperative game are satisfied at initial time. Here $V_i(k_0, x_0)$ – is Nash outcome of player i in game $\Gamma(k_0, x_0)$.

But if there exists $k > k_0$ such that for some i:

$$J_i^\alpha(k, x^\alpha(k), u^\alpha) > V_i(k, x^\alpha(k)),$$

then time-inconsistency of the individual rationality condition is appear.

To overcome the time inconsistency problem in the game with nontransferable payoffs the notion of Payoff Distribution Procedure (PDP) was introduced by L.A. Petrosyan (1997).

The solution formula for the payoff distribution procedure in a cooperative differential game with nontransferable payoffs that leads to a time consistent outcome is proposed by D.W.K. Yeung and Leon A. Petrosyan (2014).

In this paper the PDP and time-consistency of Pareto-optimal solution are detailed for linear-quadratic discrete-time dynamic games.

Definition 1. Vector $\beta(k) = (\beta_1(k), \dots, \beta_n(k))$, $k_0 \le k \le K-1$ is a PDP (Petrosyan, 1997) if

$$\sum_{k=k_0}^{K-1} \Big((x^\alpha(k))^T P_i(k) x^\alpha(k) + (u_i^\alpha(k))^T R_i(k) u_i^\alpha(k) \Big) = \sum_{k=k_0}^{K-1} \beta_i(k), \quad i = 1, \dots, n. \tag{10}$$

Definition 2. Pareto-optimal solution is called time-consistent (Petrosyan, 1993; 1997) if there exists a PDP such that the condition of individual rationality is satisfied

$$\sum_{k=l}^{K-1} \beta_i(k) + (x^\alpha(K))^T P_i(K) x^\alpha(K) \le V_i(l, x^\alpha(l)), \\ \forall l, k_0 \le l \le K, \quad i = 1, \dots, n, \tag{11}$$

where $V_i(l, x^\alpha(l))$ – is Nash outcome of player i in subgame $\Gamma(l, x^\alpha(l))$.

According to (Başar and Olsder , 1999) if Nash equlibrium $\{u_i^{NE} = M_i^{NE}(k)x,$ $i = 1, \dots, n\}$ exists in game $\Gamma(l, x^\alpha(l))$, then it can be found by solving the system of matrix equations

$$\begin{cases} (A(k) + \sum_{i=1}^{n} B_i(k) M_i^{NE}(k))^T \Theta_i(k+1)(A(k) + \sum_{i=1}^{n} B_i(k) M_i^{NE}(k)) - \\ - \Theta_i(k) + P_i(k) + M_i^{NE}(k)^T R_i(k) M_i^{NE}(k) = 0, \\ M_i^{NE}(k) = -(R_i(k) + B_i^T(k)\Theta_i(k+1)B_i(k))^{-1} B_i^T(k)\Theta_i(k+1) \times \\ \times (A(k) + \sum_{j \neq i} B_j(k) M_j^{NE}(k)), \quad k = k_0, \dots, K-1, \\ \Theta_i(K) = P_i(K), \quad i = 1, \dots, n. \end{cases} \tag{12}$$

And

$$J_i(k, x^\alpha(k), u^{NE}) = (x^\alpha(k))^T \Theta_i(k) x^\alpha(k), \quad i = 1, \dots, n.$$

In (Yeung and Petrosyan, 2014) the formula for PDP, which guarantees a time-consistency in differential game with nontransferable payoffs, is considered. The following theorem gives an analog of this formula.

Theorem 1. *Let inequalities*

$$J_i^\alpha(k_0, x_0, u^\alpha) \leq V_i(k_0, x_0), \quad i = 1, \dots, n,$$

are satisfied for some Pareto-optimal solution. Then PDP $\beta(k)$ computed by formula

$$\beta_i(k) = \frac{J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0)}{K - k_0} - V_i(k+1, x^\alpha(k+1)) + V_i(k, x^\alpha(k))$$
$$i = 1, \dots, n, \quad k = k_0, \dots, K-1 \tag{13}$$

guarantees time-consistency of this Pareto-optimal solution along the cooperative trajectory $x^\alpha(k)$.

Proof. Show that $\beta(k)$ is a PDP:

$$\sum_{k=k_0}^{K-1} \beta_i(k) =$$
$$\sum_{k=k_0}^{K-1} \left(\frac{J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0)}{K - k_0} - V_i(k+1, x^\alpha(k+1)) + V_i(k, x^\alpha(k)) \right) =$$
$$= J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0) - V_i(K, x^\alpha(K)) + V_i(k_0, x_0) =$$
$$= \sum_{k=k_0}^{K-1} \left((x^\alpha(k))^T P_i(k) x^\alpha(k) + (u_i^\alpha(k))^T R_i(k) u_i^\alpha(k) \right) +$$
$$+ (x^\alpha(K))^T P_i(K) x^\alpha(K) - (x^\alpha(K))^T P_i(K) x^\alpha(K) =$$
$$= \sum_{k=k_0}^{K-1} \left((x^\alpha(k))^T P_i(k) x^\alpha(k) + (u_i^\alpha(k))^T R_i(k) u_i^\alpha(k) \right). \tag{14}$$

So $\beta(k)$ satisfies definition 1.

Now show that the condition of individual rationality is satisfied. Using (13) we obtain

$$\sum_{k=l}^{K-1} \beta_i(k) =$$
$$\sum_{k=l}^{K-1} \left(\frac{J_i^{\alpha}(k_0, x_0, u^{\alpha}) - V_i(k_0, x_0)}{K - k_0} - V_i(k+1, x^{\alpha}(k+1)) + V_i(k, x^{\alpha}(k)) \right) =$$
$$= \frac{K-l}{K-k_0}(J_i^{\alpha}(k_0, x_0, u^{\alpha}) - V_i(k_0, x_0)) - V_i(K, x^{\alpha}(K)) + V_i(l, x^{\alpha}(l)). \quad (15)$$

We can see that in (15)

$$\frac{K-l}{K-k_0}(J_i^{\alpha}(k_0, x_0, u^{\alpha}) - V_i(k_0, x_0)) \leq 0.$$

Taking into account the system (12) and the fact, that matrices $P_i(K)$ are positive semidefinite defined, we have

$$-V_i(K, x^{\alpha}(K)) = -(x^{\alpha}(K))^T P_i(K) x^{\alpha}(K) \leq 0,$$

so

$$\frac{K-l}{K-k_0}(J_i^{\alpha}(k_0, x_0, u^{\alpha}) - V_i(k_0, x_0)) - V_i(K, x^{\alpha}(K)) + V_i(l, x^{\alpha}(l)) \leq$$
$$\leq V_i(l, x^{\alpha}(l)).$$

It means that

$$\sum_{k=l}^{K-1} \beta_i(k) + (x^{\alpha}(K))^T P_i(K) x^{\alpha}(K) \leq V_i(l, x^{\alpha}(l)),$$
$$\forall l, k_0 \leq l \leq K, \quad i = 1, \ldots, n.$$

□

2.1. Irrational Behavior Proof Condition

The condition under which even if irrational behaviors appear later in the game the concerned player would still be performing better under the cooperative scheme was considered in (Yeung, 2006).

The irrational behavior proof condition for differential games with nontransferable payoffs is proposed in (Belitskaia, 2012).

In this paper this condition is concretized for linear-quadratic discrete-time dynamic games with nontransferable payoffs.

Definition 3. Pareto-optimal solution $(J_1^{\alpha}(k_0, x_0, u^{\alpha}), \ldots, J_n^{\alpha}(k_0, x_0, u^{\alpha}))$ satisfies the irrational behavior proof condition (Yeung, 2006) in the game $\Gamma(k_0, x_0)$, if the following inequalities hold

$$\sum_{k=k_0}^{l} \beta_i(k) + V_i(l+1, x^{\alpha}(l+1)) \leq V_i(k_0, x_0), \quad i = 1, \ldots, n \quad (16)$$

for $k_0 \leq l \leq K-1$, where $\beta(k) = (\beta_1(k), \dots, \beta_n(k))$ is time-consistent PDP of $(J_1^\alpha(k_0, x_0, u^\alpha), \dots, J_n^\alpha(k_0, x_0, u^\alpha))$.

So if for all $i = 1, \dots, n$ the following inequalities holds

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) \leq 0, \quad k \geq k_0, \tag{17}$$

then the Pareto-optimal solution satisfies the irrational behavior proof condition.

Theorem 2. *If in linear-quadratic discrete-time dynamic games with nontransferable payoffs for some Pareto-optimal solutions and its PDP the following inequalities hold*

$$\beta_i(k) + (x^\alpha(k))^T \Bigg((A(k) + B(k)M^\alpha(k))^T \Theta_i(k+1)(A(k) + B(k)M^\alpha(k)) - \Theta_i(k) \Bigg) x^\alpha(k) \leq 0, \quad k_0 \leq k \leq K-1, \tag{18}$$

where $M^\alpha(k)$ – solution of the system (6), $\Theta_i(k)$ – solution of the system (12), $x^\alpha(k)$ – cooperative optimal trajectory, then the irrational behavior proof condition for this Pareto-optimal solutions is satisfied.

Proof. The proof of the theorem follows directly from the condition (17), system (12) and the state equation (1). □

Proposition 1. *Let inequalities*

$$J_i^\alpha(k_0, x_0, u^\alpha) \leq V_i(k_0, x_0), \quad i = 1, \dots, n,$$

are satisfied for some Pareto-optimal solution and PDP $\beta(k)$ of this solution is calculated using formula (13), then the irrational behavior proof condition for this Pareto-optimal solution is satisfied.

Proof. If

$$\beta_i(k) = \frac{J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0)}{K - k_0} - V_i(k+1, x^\alpha(k+1)) + V_i(k, x^\alpha(k)),$$

then

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) = \frac{J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0)}{K - k_0},$$

where $\frac{J_i^\alpha(k_0, x_0, u^\alpha) - V_i(k_0, x_0)}{K - k_0} \leq 0$ for all $i = 1, \dots, n$. So

$$\beta_i(k) + V_i(k+1, x^\alpha(k+1)) - V_i(k, x^\alpha(k)) \leq 0$$

and according to the theorem 2 it guarantees that irrational behavior proof condition for this Pareto-optimal solution is satisfied. □

3. Application to a Monetary and Fiscal Regulation Dynamic Game

As an example consider the government debt stabilization game (van Aarle, Bovenberg and Raith, 1995). Assume that government debt accumulation, $d(k)$, is the sum of interest payments on government debt, $rd(k)$, and primary fiscal deficits, $f(k)$, minus the seignorage (i.e. the issue of base money) $m(k)$. So,

$$d(k+1) = (r+1)d(k) + f(t) - m(t), \qquad d(0) = d_0,$$

here $r > 0$ – the interest rate.

The objective of the fiscal authority is:

$$J_1 = \sum_{k=0}^{K-1} \left(\frac{1}{1+\rho}\right)^k ((f(k) - \overline{f})^2 + \eta(m(k) - \overline{m})^2 + \lambda(d(k) - \overline{d})^2).$$

The objective of monetary authorities is:

$$J_2 = \sum_{k=0}^{K-1} \left(\frac{1}{1+\rho}\right)^k ((m(k) - \overline{m})^2 + \gamma(d(k) - \overline{d})^2).$$

Following (Rincon-Zapatero et. al., 2000) we consider the finite-horizon game, where players wish to minimize the deviations from fixed target $\overline{f}$, $\overline{m}$, $\overline{d}$. We suppose that the two institutions wish a balanced budget, that is to say,

$$r\overline{d} + \overline{f} - \overline{m} = 0.$$

In this case the game can be formulated as follows:

$$x_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (d(k) - \overline{d}),$$

$$x_2(k) = -\overline{d}\left(\frac{1}{1+\rho}\right)^{\frac{k+1}{2}},$$

$$u_1(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (f(k) - \overline{f}),$$

$$u_2(k) = \left(\frac{1}{1+\rho}\right)^{\frac{k}{2}} (m(k) - \overline{m}).$$

Then our system can be rewritten as

$$x(k+1) = Ax(k) + \sum_{i=1}^{2} B_i u_i(k),$$

$$A = \begin{pmatrix} (r+1)\left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} & 1 \\ 0 & \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix}, B_2 = \begin{pmatrix} \left(-\frac{1}{1+\rho}\right)^{\frac{1}{2}} \\ 0 \end{pmatrix},$$

$$J_i = \sum_{k=k_0}^{K-1} (x^T(k)P_i x(k) + \sum_{j=1}^{2} u_j^T(k) R_{ij} u_j(k)), \quad \forall i = 1,\ldots,n,$$

$$P_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad R_{11} = 1, \quad R_{12} = \eta, \quad R_{21} = 0, \quad R_{22} = 1.$$

Following (Basar and Olsder, 1999) to find the Nash equilibrium we solve the system

$$\begin{cases} (A + \sum_{i=1}^{2} B_i M_i^{NE}(k))^T \Theta_i(k+1)(A + \sum_{i=1}^{2} B_i M_i^{NE}(k)) - \\ - \Theta_i(k) + P_i + M_j^{NE}(k)^T R_{ij} M_j^{NE}(k) + \\ + M_i^{NE}(k)^T R_{ii} M_i^{NE}(k) = 0, \\ M_i^{NE}(k) = -(R_{ii} + B_i^T \Theta_i(k+1) B_i)^{-1} B_i^T \Theta_i(k+1) \times \\ \times (A + B_j M_j^{NE}(k)), \quad j \neq i. \\ \Theta_i(K) = 0, \quad i = 1, 2 \end{cases}$$

For $\lambda = 1, \eta = 2, \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}} = \frac{1}{4}, r = 0,3, \gamma = 2, \alpha = 0.47311, K = 9, k_0 = 0, d_0 = 100, \bar{d} = 1, x_0 = \begin{pmatrix} 99 \\ -1/4 \end{pmatrix}$ tables 1 and 2 give us the decision of this system.

Table 1: $\Theta_1(k)$

k	$\Theta_1(k)$
$k=0$	$\begin{pmatrix} 1.13085953605222 & 0.438592990966073 \\ 0.438592990966073 & 1.55703024555485 \end{pmatrix}$
$k=1$	$\begin{pmatrix} 1.13085953460631 & 0.438592951590143 \\ 0.438592951590143 & 1.55702942344064 \end{pmatrix}$
$k=2$	$\begin{pmatrix} 1.13085950611707 & 0.438592392155762 \\ 0.438592392155762 & 1.55701985468374 \end{pmatrix}$
$k=3$	$\begin{pmatrix} 1.13085903588365 & 0.438585138453740 \\ 0.438585138453740 & 1.55691667615166 \end{pmatrix}$
$k=4$	$\begin{pmatrix} 1.13085210271981 & 0.438499026695478 \\ 0.438499026695478 & 1.55589985156105 \end{pmatrix}$
$k=5$	$\begin{pmatrix} 1.13075887119456 & 0.437572367917709 \\ 0.437572367917709 & 1.54697503639703 \end{pmatrix}$
$k=6$	$\begin{pmatrix} 1.12961452817083 & 0.428782686890835 \\ 0.428782686890835 & 1.48063905917239 \end{pmatrix}$
$k=7$	$\begin{pmatrix} 1.11703601107493 & 0.360110803313158 \\ 0.360110803313158 & 1.10803324099723 \end{pmatrix}$
$k=8$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Table 2: $\Theta_2(k)$

k	$\Theta_2(k)$
$k=0$	$\begin{pmatrix} 2.17958048176956 & 0.588717975058136 \\ 0.588717975058136 & 2.04805681717168 \end{pmatrix}$
$k=1$	$\begin{pmatrix} 2.17958048379181 & 0.588717995877213 \\ 0.588717995877213 & 2.04805687587772 \end{pmatrix}$
$k=2$	$\begin{pmatrix} 2.17958050404908 & 0.588718140256045 \\ 0.588718140256045 & 2.04805605539460 \end{pmatrix}$
$k=3$	$\begin{pmatrix} 2.17958066030114 & 0.588718424730071 \\ 0.588718424730071 & 2.04803140716259]] \end{pmatrix}$
$k=4$	$\begin{pmatrix} 2.17958103086527 & 0.588704450465071 \\ 0.588704450465071 & 2.04762648496451 \end{pmatrix}$
$k=5$	$\begin{pmatrix} 2.17956252319188 & 0.588360921208274 \\ 0.588360921208274 & 2.04249323471908 \end{pmatrix}$
$k=6$	$\begin{pmatrix} 2.17902244454632 & 0.582968296364818 \\ 0.582968296364818 & 1.99034950051843 \end{pmatrix}$
$k=7$	$\begin{pmatrix} 2.16853185591212 & 0.518559556726842 \\ 0.518559556726842 & 1.59556786703601 \end{pmatrix}$
$k=8$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

We solve the following system to find the Pareto solution

$$\begin{cases} (A+BM^{\alpha}(k))^T\Theta^{\alpha}(k+1)(A+BM^{\alpha}(k))-\Theta^{\alpha}(k)+ \\ +P^{\alpha}+M^{\alpha}(k)^TR^{\alpha}M^{\alpha}(k)=0, \\ M^{\alpha}(k)=-(R^{\alpha}+B^T\Theta^{\alpha}(k+1)B)^{-1}B^T\Theta^{\alpha}k+1)A, \\ k=1,\dots,K-1, \\ \Theta^{\alpha}(K)=0. \end{cases}$$

Tables 3 and 4 give $M_i^{\alpha}(k)$ – decision of system. We can find the corresponding optimal trajectory $x^{\alpha}(k)$ in table 5.

Table 3: $M_1^{\alpha}(k)$

k	$M_1^{\alpha}(k)$	
$k=0$	$(-0.221353984966506$	$-0.726861227865480)$
$k=1$	$(-0.221353984091185$	$-0.726861207232395)$
$k=2$	$(-0.221353970295295$	$-0.726860921513799)$
$k=3$	$(-0.221353752858969$	$-0.726857045235910)$
$k=4$	$(-0.221350325856959$	$-0.726805906912980)$
$k=5$	$(-0.221296313324356$	$-0.726158014981779)$
$k=6$	$(-0.220445092925548$	$-0.718457766961832)$
$k=7$	$(-0.207046034461112$	$-0.637064721418805)$

Table 4: $M_2^\alpha(k)$

k	$M_2^\alpha(k)$	
$k=0$	(0.0710909462480762	0.233441708708404)
$k=1$	(0.0710909459669545	0.233441702081799)
$k=2$	(0.0710909415362105	0.233441610319252)
$k=3$	(0.0710908717034755	0.233440365398077)
$k=4$	(0.0710897710735695	0.233423941606262)
$k=5$	(0.0710724241888834	0.233215861998106)
$k=6$	(0.0707990428021844	0.230742819083171)
$k=7$	(0.0664957466610752	0.204602297418693)

Table 5: $(x^\alpha(k))^T$

k	$(x^\alpha(k))^T$	
$k=0$	(99	$-0,25$)
$k=1$	(24.7470068859750	-0.0625)
$k=2$	(6.18599779642059	-0.015625)
$k=3$	(1.54630957056064	-0.00390625)
$k=4$	(0.386529670417299	-0.0009765625)
$k=5$	(0.0966208179301434	-0.000244140625)
$k=6$	(0.0241539540978471	-0.00006103515625)
$k=7$	(0.00604480920566207	-0.0000152587890625)
$k=8$	(0.00153913793836859	-0.000003814697265625)
$k=9$	(0.000496405132704166	$-9.53674316406250*10^{-7}$)

It can be shown that

$$J_1^\alpha(k_0, x_0, u^\alpha) = 11061.9281050210,$$

$$J_2^\alpha(k_0, x_0, u^\alpha) = 20960.4249790116.$$

We can see that for the chosen value of parameter α at the moment $k=0$ conditions (9) are satisfied

$$J_1^\alpha(k_0, x_0, u^\alpha) - V_1(k_0, x_0) = -0.0131691644110106,$$

$$J_2^\alpha(k_0, x_0, u^\alpha) - V_2(k_0, x_0) = -372.629786597452,$$

$$J_i^\alpha(k_0, x_0, u^\alpha) \leq V_i(k_0, x_0), \quad i = 1,2,$$

But

$$J_1^\alpha(5, x^\alpha(5), u^\alpha) > V_1(5, x^\alpha(5)), \quad i = 1,2,$$

It means, that time-inconsistency of the individual rationality condition is appear. To avoid this problem, use PDP, calculated by formula (13):

k	$\beta_1(k)$	$\beta_2(k)$
$k=0$	10370.7358550510	19958.6582163535
$k=1$	648.012784901324	1208.29816815828
$k=2$	40.4895523303710	36.6839941266950
$k=3$	2.52860341121872	−36.5240521384429
$k=4$	0.156626650974212	−41.0984318452099
$k=5$	0.00841474752835961	−41.3842599309515
$k=6$	−0.000846196276454088	−41.4021198035311
$k=7$	−0.00142483882727764	−41.4032358165648
$k=8$	−0.00146085094459085	−41.4033054830370

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