Contributions to Game Theory and Management, VIII, 278–288

Decomposition Theorem and its Applications

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Abstract In the article a complete proof of decomposition theorem is given. This theorem concerns the so called canonical extension of the order relation on the set of probabilistic measures. Here we study a structure for an extension of the order relation given on some set A on the generated vector space $\mathbb{R}^A$. Corresponding description of the extension with help of stochastic matrices is found (Theorem 2). Decomposition theorem reveals the most significant properties of the canonical extension of orders. In particular the consequences of the theorem are two important statements:
1. The coincidence of the canonical extension of any order with its convex hull and
2. Truth of Choquet condition for the canonical extension (see the corollary 1 and the corollary 2 in the section 3.1).
The complete proof of decomposition theorem is quite complicated. As the first step for the proof of this theorem we prove an assertion of existence of optimal surplus vector (Theorem 1). This theorem having game-theoretical interpretation also can be formulated in economic terms (Remark 1). A geometric interpretation of decomposition theorem is given (example 1).

Keywords: extension of order on the set of probabilistic measures, extension of order on the generated vector space, decomposition theorem, stochastic matrix.

1. Introduction

In our previous works examined some mathematical models for decision making in which the goal structure by partial order relations is given. Examples of such models are games with ordered outcomes in the normal form; games on graphs with ordered final positions or ordered plays; games against nature with ordered outcomes; many-criterion optimization models with partial ordered set of criteria. Since such models have several chance mechanisms (mixed strategies in games in the normal form; chance moves in games on graphs; probability distributions on the set of states in games against nature etc.) then to evaluate chosen strategies, we need to construct for given order relation its extension to the set of probability measures. In our works we have used so called *the canonical extension of order to the set of probability measures* which was introduced by the author (see Rozen, 1976). The canonical extension is based on the fact that any order relation ω on a set A is approximated by the set $C(\omega)$ consisting of all isotonic functions from the ordered set $\langle A, \omega \rangle$ into $\mathbb{R}$, that is, the following equivalence holds:

$$a_1 \overset{\omega}{\leq} a_2 \Leftrightarrow (\forall f \in C(\omega))\, f(a_1) \leq f(a_2)\,. \tag{1}$$

The canonical extension $\tilde{\omega}$ of the order ω on the set of probabilistic measures can be given by the formula similar to (1):

$$\mu_1 \overset{\bar{\omega}}{\leq} \mu_2 \Leftrightarrow (\forall f \in C(\omega))\, \bar{f}(\mu_1) \leq \bar{f}(\mu_2)\,. \tag{2}$$

Remark that in (2) μ_1, μ_2 are arbitrary probabilistic measures and $\bar{f}$ is the usual extension of the function to the set of probabilistic measures, i.e. $\bar{f}(\mu) = (f, \mu)$ (the right part is the standard scalar product). We use the construction of canonical extension of an order on the set of probabilistic measures to prove an existence of equilibrium points and also for description of Nash equilibrium points in games with ordered outcomes (see Rozen, 2010, Rozen, 2011).

In this article, we study the so-called decomposition theorem which stands a structure for an extension of order relation given on some set A on vector space $\mathbb{R}^A$. Corresponding description of this extension with help of stochastic matrices is found.

Consider an arbitrary n-element set $A = \{a_1, \dots, a_n\}$ and some order relation ω on A. We denote by $\mathbb{R}^A$ the set of all n-component vectors $x = \left(x^1, \dots, x^n\right)$ whose components state in 1–1 correspondence with elements of the set A. The set $\mathbb{R}^A$ forms a vector space under so-component operation of addition of vectors and multiplication of vectors on real numbers. Put for any vector $x \in \mathbb{R}^A$ and any subset $B \subseteq A$

$x(B) \overset{df}{=} \sum\limits_{a_k \in B} x^k.$

We denote by $C(\omega)$ the set of all isotonic function φ from the ordered set $\langle A, \omega \rangle$ into $\mathbb{R}$ and by $C_+(\omega)$ the set of all isotonic function φ from ordered set $\langle A, \omega \rangle$ into $\mathbb{R}_+$.

Then we define binary relations $\bar{\omega}$ and $\bar{\omega}_+$ on $\mathbb{R}^A$ in the following manner:

$$x \overset{\bar{\omega}}{\leq} y \Leftrightarrow (\forall \varphi \in C(\omega))\, (\varphi, x) \leq (\varphi, y)\,, \tag{3}$$

$$x \overset{\bar{\omega}_+}{\leq} y \Leftrightarrow (\forall \varphi \in C_+(\omega))\, (\varphi, x) \leq (\varphi, y)\,. \tag{4}$$

where $x, y \in \mathbb{R}^A$ and $\varphi(x) = \sum\limits_{i=1}^{n} \varphi(a_i)\, x^i$.

The relation $\bar{\omega}$ is called *the canonical extension of order* ω *and* $\bar{\omega}_+$ *the positive canonical extension of order* ω *on vector space* $\mathbb{R}^A$.

We now indicate some properties of extensions $\bar{\omega}$ and $\bar{\omega}_+$ (for the proof see Rozen, 2014).

1. Relations $\bar{\omega}$ and $\bar{\omega}_+$ are extensions of the order ω on vector space $\mathbb{R}^A$ (i.e. $\bar{\omega}$ and $\bar{\omega}_+$ are closed conic orders on $\mathbb{R}^A$, whose restrictions on A coincide with ω). Denote by $K(\bar{\omega})$ and $K(\bar{\omega}_+)$ positive cones of these orders respectively. By definition an arbitrary vector $x \in \mathbb{R}^A$ belongs to $K(\bar{\omega}_+)$ if and only if $(\forall \varphi \in C_+(\omega))\, (\varphi, x) \geq 0$ and belongs to $K(\bar{\omega})$ if and only if $(\forall \varphi \in C(\omega))\, (\varphi, x) \geq 0$ holds.

2. The relations $\bar{\omega}$ and $\bar{\omega}_+$ are convex ones. Recall that the convexity of an arbitrary relation ρ on vector space $\mathbb{R}^A$ means that the conditions

$$(x_1, y_1) \in \rho, (x_2, y_2) \in \rho, \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$$

imply

$$(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2) \in \rho.$$

3. The formulas (3) and (4) are not effective since the sets $C(\omega)$ and $C_+(\omega)$ are infinite even for finite A. An effective method for definition of these extensions are given in the following. The canonic extension $\bar{\omega}_+$ can be presented by the equivalence

$$x \overset{\bar{\omega}_+}{\leq} y \Leftrightarrow (\forall B \in M(\omega))\, x(B) \leq y(B), \tag{5}$$

and the canonic extension $\bar{\omega}$ by the equivalence

$$x \overset{\bar{\omega}}{\leq} y \Leftrightarrow (\forall B \in M(\omega))\, x(B) \leq y(B), x(A) = y(A), \tag{6}$$

where $x, y \in \mathbb{R}^A$ and $M(\omega)$ is the set of all majorantly stable subsets in ordered set $\langle A, \omega \rangle$ (recall that subset $B \subseteq A$ is called majorantly stable if conditions $a \in B, a' \overset{\bar{\omega}}{\geq} a$ imply $a' \in B$).

4. Positive cones $K(\bar{\omega}_+)$ and $K(\bar{\omega})$ of orders $\bar{\omega}_+$ and $\bar{\omega}$ can be effective presented by the following:

$$K(\bar{\omega}_+) = \left\{x \in \mathbb{R}^A \colon\ (\forall B \in M(\omega))\, x(B) \geq 0\right\}, \tag{7}$$

$$K(\bar{\omega}) = \left\{x \in \mathbb{R}^A \colon\ (\forall B \in M(\omega))\, x(B) \geq 0, x(A) = 0\right\}. \tag{8}$$

2. An optimal surplus vector

2.1. Existence of optimal surplus vector

Consider an arbitrary n-element set $A = \{a_1, \ldots, a_n\}$ and some order relation ω on A.

Definition 1. A *support* for vector $x \in \mathbb{R}^A$ is called a majorant stable subset $B^* \subseteq A$ such that for any majorantly subset $B \supseteq B^*$ the condition $x(B) \geq x(A)$ holds.

Remark that subset $B^* = A$ is the trivial support for any vector $x \in \mathbb{R}^A$. If all components of vector $x \in \mathbb{R}^A$ are positive then it has the trivial support only. But for vector with negative or null components can be non-trivial support.

Definition 2. We say that a vector of $\mathbb{R}^A$ is *a central one* if the sum of its components is equal to zero.

Theorem 1 (existence of optimal surplus vector). *Consider n-elements set $A = \{a_1, \ldots, a_n\}$ and an order relation ω on A. Assume $x = \left(x^1, \ldots, x^n\right) \in K(\bar{\omega}_+)$. Then there exists a vector $v = \left(v^1, \ldots, v^n\right) \in \mathbb{R}^A_+$ such that the difference $y = x - v$ is a central one and $y \in K(\bar{\omega}_+)$. Moreover, we can choose vector v such that the following additional conditions satisfy:*

(a) $0 \leq v^i \leq x^i$ for all $i = 1, \ldots, n$ where $x^i \geq 0$;

(b) $v^i = 0$ for all $i = 1, \ldots, n$ where $x^i < 0$;

(c) Let B^* be a support of vector x. Then $v^i = 0$ for all $i = 1, \ldots, n$ where $a_i \notin B^*$.

Any vector v satisfying indicated in theorem 1 conditions is called an *optimal surplus vector for vector* $x \in K(\bar{\omega}_+)$.

Proof (of theorem 1). Put $\delta = x(A) = \sum_{a_i \in A} x^i$. Since $(\forall \varphi \in C_+(\omega))\,(\varphi, x) \geq 0$ then fixing $\varphi \equiv 1$, we have $\delta \geq 0$. In the case $\delta = 0$, vector x is a central one hence the statement of theorem 1 is evident. Further we consider only $\delta > 0$. Fix some support B^* of vector x. Put $I^+_{B^*} = \{i = 1, \dots, n \colon x^i > 0, a_i \in B^*\}$. Consider the set D of all vectors $v = (v^1, \dots, v^n) \in \mathbb{R}^A_+$ satisfying the conditions:

$$\begin{cases} 0 \leq v^i \leq x^i \ \left(i \in I^+_{B^*}\right), \\ v^i = 0 \ \left(i \in I^+_{B^*}\right), \\ \sum\limits_{i=1}^{n} v^i = \delta. \end{cases} \tag{9}$$

Using the definition of support, we have $\sum_{i \in I^+_{B^*}} x^i \geq \sum_{a_i \in B^*} x^i \geq \delta$ hence $D \neq \emptyset$. Any vector $v \in D$ is called *a surplus vector for vector x with support B^**. Remark that for $v \in D$, the vector $y = x - v$ is a central one since $y(A) = x(A) - v(A) = \delta - \delta = 0$. Hence a vector $v_* \in \mathbb{R}^A$ is an optimal surplus vector if and only if $v_* \in D$ and $(x - v_*) \in K(\bar{\omega}_+)$. To prove the existence a required vector v_* put $f_v(B) = \sum_{a_i \in B} (x^i - v^i)$ for arbitrary $v \in D$ and $B \in M(\omega)$. Consider for arbitrary $v \in D$ the function $g(v) = \min_{B \in M(\omega)} f_v(B)$. Since the function $g(v)$ is continuous one and defined on compact set it has the greatest value on D.

Lemma 1. *Let $v_* \in D$ be the vector which delivers for function $g(v)$ the greatest value on D.Then $g(v_*) \geq 0$.*

Let us show that indicated vector v_* is an optimal surplus vector for vector x with support B^*. Indeed, it is evident that the vector v_* is a surplus vector. To check its optimality we need to prove the inclusion $(x - v_*) \in K(\bar{\omega}_+)$. Using lemma 1 we obtain $\min_{B \in M(\omega)} f_{v_*}(B) = g(v_*) \geq 0$ hence for any majorantly stable subset $B \in M(\omega)$ the inequality $f_{v_*}(B) \geq 0$ holds i.e. $\sum_{a_i \in B} (x^i - v^i_*) \geq 0$. Last inequality in according with (6) means $(x - v_*) \in K(\bar{\omega}_+)$ which was to be proved. □

2.2. Game-theoretical interpretation of optimal surplus vector

It follows from the proof of theorem 1 that an optimal surplus vector is a vector which delivers an external extremum for $\max_{v \in D} \min_{B \in M(\omega)} f_v(B)$. Hence the problem for finding of optimal surplus vector have game-theoretical interpretation. Consider the following antagonistic game in which the set of strategies of player 1 is the set D consisting of all surplus vectors, the set of strategies of player 2 is the set $M(\omega)$ of all majorantly stable subsets of ordered set $\langle A, \omega \rangle$ and payoff function is defined above function $f_v(B)$ $v \in D, B \in M(\omega)$. For any $B \in M(\omega)$ there exists $\max_{v \in D} f_v(B)$ (since $f_v(B)$ considered as a function of v is defined on compact set D and it is continuous one). In our case the set $M(\omega)$ is finite hence there exists $\min_{B \in M(\omega)} \max_{v \in D} f_v(B)$. We obtain:

$$\min_{B\in M(\omega)}\max_{v\in D} f_v(B) \le \max_{v\in D} f_v(A) = \max_{v\in D}\sum_{a_i\in A}\left(x^i - v^i\right) = \max_{v\in D}\sum_{i=1}^{n}\left(x^i - v^i\right) = \\ = \max_{v\in D}\left(\sum_{i=1}^{n} x^i - \sum_{i=1}^{n} v^i\right) = \max_{v\in D}(\delta-\delta) = 0. \tag{10}$$

It is shown in the proof of theorem 1 that

$$\max_{v\in D}\min_{B\in M(\omega)} f_v(B) = \max_{v\in D} g(v) = g(v_*) \ge 0. \tag{11}$$

It follows from (10) and (11) the equalities

$$\max_{v\in D}\min_{B\in M(\omega)} f_v(B) = \min_{B\in M(\omega)}\max_{v\in D} f_v(B) = 0. \tag{12}$$

Therefore indicated antagonistic game has a solution in pure strategies and the value of the game is equal to zero. Moreover any optimal surplus vector is an optimal pure strategy of the player 1 in this game.

Remark 1. Theorem 1 which concerns of optimal surplus vector admits the following economic interpretation. Consider n economic products $A = \{1,\dots,n\}$ the set of which is partially ordered under its utility by order relation ω for some person. We consider components of vector $x \in \mathbb{R}^A$ as "asset" or as "liability" for the person depending on the sign its component ($x^i > 0$ means that the person can let for sale the corresponding quantity of i-th product which considered as his "asset" and $x^i < 0$ means that the person need to buy $\left|x^i\right|$ units of i-th product which considered as his "liability"). Any vector $\varphi \in C_+(\omega)$ can be interpreted here as a vector of prices (remark that the price of any product must be non negative and the isotonic condition means that a product with more utility has a higher price). Then the condition $x \in K(\bar{\omega}_+)$ means in this interpretation that for vector x the total value of the "asset" exceeds the total value of "liabilities", i.e. that for vector x no deficit at any price agreed with utilities. In such interpretation, theorem th1 asserts that for any vector $x \in K(\bar{\omega}_+)$ there exists the so-called "vector of excess" v such that its extraction leads to a total equality of "assets" and "liabilities" and moreover the resulting vector $x - v$ has not the deficit also. In addition, the excess shall be withdrawn only from the "asset" and without transfer of "asset" into "liability" (see conditions (a) and (b)). Finally the condition (c) means that it is possible "extraction of excess" from the products which are contained in the support of vector x only.

3. Decomposition Theorem

3.1. Representation of canonic extension of order with a decomposition matrix

We introduce first some auxiliary concepts. Consider a finite set $A = \{a_1,\dots,a_n\}$ and some order relation ω on A. The corresponding ordered set is denoted briefly by $\langle A,\omega\rangle$.

Definition 3. *Probabilistic vector* on A is called a vector $\mu = \left(\mu^1, \dots, \mu^n\right) \in \mathbb{R}^A$ with conditions: $\mu^i \geq 0 \quad (i = 1, \dots, n)$, $\sum\limits_{i=1}^{n} \mu^i = 1$. The set of all probabilistic vectors on A we denote by $S\,(A)$. Spectrum of vector $\mu = \left(\mu^1, \dots, \mu^n\right)$ is $Sp\,\mu = \left\{i = 1, \dots, n\colon \mu^i > 0\right\}$.

Remark 2. Any probabilistic vector on A can be considered as a probabilistic measure on the set A. Particularly, a vector for which one component is equal to 1 an others components equal to 0, is a degenerate probabilistic measure concentrated at the corresponding point. The set $S\,(A)$ is a simplex whose vertices are all degenerate probabilistic vectors.

Remark 3. Introduced above two canonic extensions of order ω on the vector space $\mathbb{R}^A$ are coincides on the simplex $S\,(A)$ (see (5) and (6)).

Definition 4. *A decomposition matrix for ordered set* $\langle A, \omega \rangle$ is called real nonnegative matrix $\mathit{\Delta} = \|d_i^j\|$ with format $n \times n$ satisfying the following conditions:

(1) The sum of elements of any row of matrix $\mathit{\Delta}$ is equal to 1 (i.e. the matrix $\mathit{\Delta}$ is a stochastic one): $\sum_{j=1}^{n} d_i^j = 1$ (for all $i = 1, \dots, n$);

(2) The ratio $d_i^j \neq 0$ implies $a_i \overset{\omega}{\leq} a_j$.

Remark that in the decomposition matrix $\mathit{\Delta} = \|d_i^j\|$ the lower index indicates the row number and the upper index the column number.

Theorem 2 (decomposition theorem). *Consider n-element ordered set $\langle A, \omega \rangle$ and fix $\mu \in S\,(A)$. A vector $\nu \in S\,(A)$ is a majorant for vector μ under the canonic extension $\bar{\omega}$ if and only if it can be presented in the form: $\nu = \mu\mathit{\Delta}$ where $\mathit{\Delta}$ is a decomposition matrix for the ordered set $\langle A, \omega \rangle$.*

Proof (of theorem 2). Necessity. Suppose $\nu \overset{\tilde{\omega}}{\geq} \mu$. We need to prove the existence of decomposition matrix $\mathit{\Delta}$ for ordered set $\langle A, \omega \rangle$ such that $\nu = \mu\mathit{\Delta}$. The proof is by induction under the number of elements of spectrum of the probabilistic vector μ.

The base of induction. Assume $\nu \overset{\tilde{\omega}}{\geq} \mu$ where μ is a degenerate probabilistic measure concentrated at one point $a_i \in A$. It easy to show that in this case the matrix $\mathit{\Delta}$ which is obtained from the identity matrix with format $n \times n$ by replacing of its i-th row on the row $\left(\nu^1, \dots, \nu^n\right)$ of component of vector ν is a decomposition matrix of the ordered set $\langle A, \omega \rangle$. Moreover the equality $\mu\mathit{\Delta} = \nu$ holds.

Induction step. Assume the required assertion is true for any probabilistic vector with spectrum of $k - 1$ elements $(k = 2, \dots, n)$ and suppose $\nu \overset{\tilde{\omega}}{\geq} \mu$, where $|Sp\,\mu| = k$. Without loss of generality we can put $Sp\,\mu = \{1, \dots, k\}$. Let us write a decomposition of the vector μ in the form $\mu = \mu^1 \tau\,(a_1) + \ldots + \mu^k \tau\,(a_k)$, $\tau\,(a_i)$ is a the degenerate probability measure concentrated at one point a_i $(i = 1, \dots, k)$ and $\mu^1 > 0, \dots, \mu^k > 0, \mu^1 + \ldots + \mu^k = 1$. We have

$$\mu = \mu^1 \tau\,(a_1) + \left(1 - \mu^1\right) \left(\frac{\mu^2}{1 - \mu^1} \tau\,(a_2) + \ldots + \frac{\mu^k}{1 - \mu^1} \tau\,(a_k) \right). \tag{13}$$

Using (13) we obtain $\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2$ where

$$\alpha_1 = \mu^1, \alpha_2 = 1 - \mu^1, \mu_1 = \tau\,(a_1)\,, \mu_2 = \frac{\mu^2}{1 - \mu^1} \tau\,(a_2) + \ldots + \frac{\mu^k}{1 - \mu^1} \tau\,(a_k)\,.$$

It follows from theorem 1 that there exist probabilistic vectors $\nu_1, \nu_2 \in S(A)$ such that

1) $\alpha_1 \nu_1 + \alpha_2 \nu_2 = \nu$;
2) $\nu_1 \overset{\bar{\omega}}{\geq} \mu_1$;
3) $\nu_2 \overset{\bar{\omega}}{\geq} \mu_2$.

Using the assertion of base of induction for vector μ_1 and induction assumption for vector μ_2, we obtain the existence of decomposition matrices Δ_1, Δ_2 of the ordered set $\langle A, \omega \rangle$ with conditions $\nu_1 = \mu_1 \Delta_1, \nu_2 = \mu_2 \Delta_2$. Let Δ be a matrix which is obtained from the matrix Δ_2 by replacing of its 1-th row on the 1-th row of matrix Δ_1. It is easy to check that Δ is a decomposition matrix for the ordered set $\langle A, \omega \rangle$ and the following equalities hold:

$$\nu_1 = \mu_1 \Delta, \quad \nu_2 = \mu_2 \Delta. \tag{14}$$

Multiplying the first equality in (14) by α_1, the second by α_2, summing up and using equality 1), we get $\nu = \mu \Delta$, which was to be proved.

Sufficiently. Assume $\nu = \mu \Delta$ for some vectors $\mu, \nu \in S(A)$ where $\Delta = \|d_i^j\|$ is a decomposition matrix for the ordered set $\langle A, \omega \rangle$. Let us formulate the following

Rule (rule for decomposition). *The vector $\nu = \mu \Delta$ can be obtained from the vector μ by the formal replacement $\tau(a_i) \to \sum\limits_{i=1}^{n} d_i^j \, \tau(a_j)$ in the equality $\mu = \sum\limits_{i=1}^{n} \mu^i \tau(a_i)$ (in fact indicated replacement means that instead of element a_i we stand a convex linear combinations of elements $a_1, \ldots, a_n$ with coefficients which are in i-th row of decomposition matrix).*

Indeed

$$\sum_{i=1}^{n} \mu^i \sum_{j=1}^{n} d_i^j \tau(a_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\mu^i d_i^j\right) \tau(a_j) = \sum_{j=1}^{n} \tau(a_j) \sum_{i=1}^{n} \mu^i d_i^j = \sum_{j=1}^{n} \nu^j \tau(a_j) = \nu.$$

Now we proof the sufficient condition in theorem 2. Fix arbitrary index $i = 1, \ldots, n,$ let index j takes values such that $d_i^j \neq 0$. According to condition (2) for decomposition matrix we have $a_i \overset{\omega}{\leq} a_j$ hence $\tau(a_i) \overset{\bar{\omega}}{\leq} \tau(a_j)$. Using the convexity of the order $\bar{\omega}$ (see the section 1.) and the condition $\sum\limits_{j=1}^{n} d_i^j = 1$ we obtain

$$\tau(a_i) \overset{\bar{\omega}}{\leq} \sum_{j=1}^{n} d_i^j \tau(a_j). \tag{15}$$

Multiplying both parts of (15) by $\mu^i \geq 0$, summing over all $i = 1, \ldots, n$ and using once more the convexity of the order $\bar{\omega}$ we get in accordance with rule for decomposition: $\mu \overset{\tilde{\omega}}{\leq} \nu$ i.e. $\nu \overset{\tilde{\omega}}{\geq} \mu$ which completes a proof of decomposition theorem. □

3.2. Some consequences of decomposition theorem

We now indicate some important consequences of decomposition theorem. According with property 2 (see the section 1.) the relation $\bar{\omega}$ on the set $S(A)$ consisting of probabilistic vectors is a convex one. The following proposition strengthens this result. Note preliminarily that we can identify the order relation ω given on the set A with its image under the mapping $\tau\colon A \to S(A)$. In this case we consider ω as a relation which on vertices of simplex $S(A)$ is given.

Corollary 1. *The canonic extension $\bar{\omega}$ coincides with smallest (under inclusion) convex relations on simplex $S(A)$ which contain the order relation ω.*

Proof (of corollary 1). Denote by $\hat{\omega}$ the smallest (under inclusion) convex relations on simplex $S(A)$ which contain the order relation ω. We need to prove the equality $\hat{\omega} = \bar{\omega}$. Indeed, since the relation $\bar{\omega}$ is a convex one and $\omega \subseteq \bar{\omega}$, we have the inclusion $\hat{\omega} \subseteq \bar{\omega}$. The converse inclusion follows from decomposition theorem. Indeed, assume $\mu \overset{\bar{\omega}}{\leq} \nu$ and let $\varDelta = \|d_i^j\|$ be a decomposition matrix for the ordered set $\langle A, \omega \rangle$ such that $\nu = \mu\varDelta$. Fix an index $i = 1, \ldots, n$. According with property (2) for decomposition matrices we have $a_i \overset{\omega}{\leq} a_j$ provided $d_i^j \neq 0$ hence $\tau(a_i) \overset{\hat{\omega}}{\leq} \tau(a_j)$. Multiplying final ratios by $d_i^j > 0$, summing over all $j = 1, \ldots, n$ and using the convexity of the order $\hat{\omega}$ we get $\tau(a_i) \overset{\hat{\omega}}{\leq} \sum\limits_{d_i^j > 0} d_i^j \tau(a_j) = \sum\limits_{j=1}^{n} d_i^j \tau(a_j)$. Then multiplying last ratios by $\mu^i \geq 0$, summing over all $i = 1, \ldots, n$ and using the convexity of the order $\hat{\omega}$ we get $\sum\limits_{i=1}^{n} \mu^i \tau(a_i) \overset{\hat{\omega}}{\leq} \sum\limits_{j=1}^{n} d_i^j \tau(a_j)$. Since the left part of last ratio is μ and the right part in accordance with rule for decomposition is equal to ν, we get $\mu \overset{\hat{\omega}}{\leq} \nu$ which was to be proved. □

Corollary 2 (Choquet condition). *Assume $\nu \overset{\bar{\omega}}{\geq} \alpha_1\mu_1 + \alpha_2\mu_2$ for some probabilistic vectors $\mu_1, \mu_2, \nu \in S(A)$ and real numbers $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$. Then there exist probabilistic vectors $\nu_1, \nu_2 \in S(A)$ such that*

1) $\alpha_1\nu_1 + \alpha_2\nu_2 = \nu$;

2) $\nu_1 \overset{\bar{\omega}}{\geq} \mu_1$;

3) $\nu_2 \overset{\bar{\omega}}{\geq} \mu_2$.

Indeed, let $\varDelta$ be the decomposition matrix of the ordered set $\langle A, \omega \rangle$ with condition $\nu = (\alpha_1\mu_1 + \alpha_2\mu_2)\varDelta$. Then $\nu = \alpha_1(\mu_1\varDelta) + \alpha_2(\mu_2\varDelta)$. Put $\nu_1 = \mu_1\varDelta, \nu_2 = \mu_2\varDelta$. Obviously, the condition 1) of corollary 2 holds and conditions 2) and 3) hold in accordance with decomposition theorem.

Remark 4. Choquet condition is important for some questions of mathematical economics (see for example Kiruta et al., 1980).

3.3. An example: a geometric interpretation of decomposition theorem

Example 1. Consider the ordered set $\langle A, \omega \rangle$ where $A = \{a_1, a_2, a_3\}$ and the order relation ω by its diagram is given (Fig. 1). Let the following matrix $\varDelta$ be a decomposition matrix for the ordered set $\langle A, \omega \rangle$

$$\varDelta = \begin{pmatrix} 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}$$

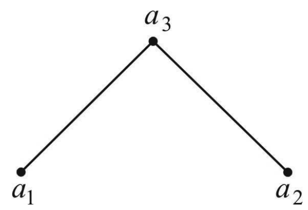

Fig. 1: Diagram of ordered set

Moreover, the rows and columns of this matrix correspond to elements of the set A. Clearly-geometrically the decomposition matrix Δ can be presented with three intervals of unit length corresponding to elements a_1, a_2, a_3 and each of this intervals covered by intervals whose lengths are defined by the corresponding row of the decomposition matrix (Fig. 2). In accordance with the properties of decomposition matrices, an interval corresponding to element a_j can be included in the coverage of interval corresponding to element a_i provided $a_i \overset{\omega}{\leq} a_j$. Such a coverage shall name *the agreed floor under the order* ω.

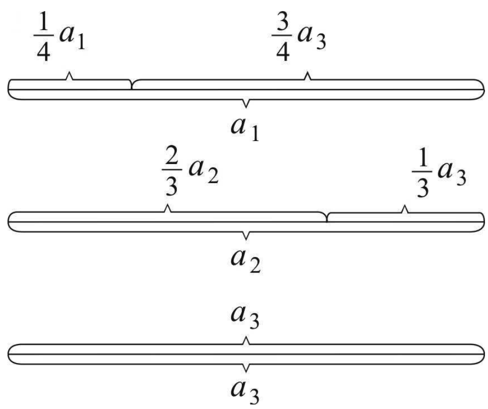

Fig. 2: Geometrical interpretation of decomposition matrix

Fix arbitrary probabilistic vector $\mu \in S(A)$, for example, $\mu = (1/2, 1/3, 1/6)$, and find the product of this vector by the decomposition matrix Δ. We obtain

$$\mu\Delta = \left(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}\right) \begin{pmatrix} 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} = \left(\tfrac{1}{8}, \tfrac{2}{9}, \tfrac{3}{8} + \tfrac{1}{9} + \tfrac{1}{6}\right).$$

Next, we represent the probability vector μ in the form of unit interval, consisting of three subintervals whose length are $\mu(a_1) = 1/2, \mu(a_2) = 1/3, \mu(a_3) = 1/6$. A probabilistic vector ν can be presented in a similar way. *A coverage of vector* μ *by some vector* ν means a partition of the unit interval corresponding to vector ν in disjoint subintervals whose union coincides with union interval corresponding to vector μ. Clearly-geometrically, the equality $\nu = \mu\Delta$ means that a coverage of the vector μ by the vector ν can be obtained from the Fig. 2 with contraction of its parts in 2, 3 and 6 times, respectively (see Fig. 3).

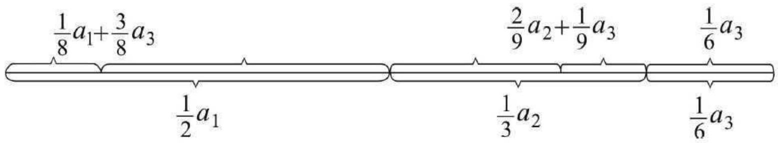

Fig. 3

Obviously, such coverage remains to be agreed floor under the order ω.

Now we are able to interpret the sufficient condition of decomposition theorem. Namely, assume the equality $\nu = \mu\Delta$ holds for some probabilistic vectors $\mu, \nu \in S(A)$ and for decomposition matrix Δ. Then there exists a coverage of μ by vector ν which is agreed floor under the order ω. Fix arbitrary some isotonic mapping $\varphi\colon A \to \mathbb{R}_+$. We need to prove the inequality $(\varphi, \mu) \leq (\varphi, \nu)$. In our example a truth of this inequality *for arbitrary isotonic mapping* $\varphi\colon A \to \mathbb{R}_+$ is shown in the following way. We have

$$(\varphi, \mu) = \mu(a_1)\varphi(a_1) + \mu(a_2)\varphi(a_2) + \mu(a_3)\varphi(a_3) = \frac{1}{2}\varphi(a_1) + \frac{1}{3}\varphi(a_2) + \frac{1}{6}\varphi(a_3);$$

$$(\varphi, \nu) = \nu(a_1)\varphi(a_1) + \nu(a_2)\varphi(a_2) + \nu(a_3)\varphi(a_3) =$$
$$= \left(\frac{1}{8}\varphi(a_1) + \frac{3}{8}\varphi(a_3)\right) + \left(\frac{2}{9}\varphi(a_2) + \frac{1}{9}\varphi(a_3)\right) + \frac{1}{6}\varphi(a_3).$$

Thus clearly-geometrically values (φ, μ) and (φ, ν) are areas of two step figures; because the second figure contains the first figure then the area of the second figure is more than the area of the first figure i.e. $(\varphi, \mu) \leq (\varphi, \nu)$ (the excess area indicated by hatching in Fig. 4).

The proof of necessary condition in decomposition theorem is significantly more complex: for arbitrary two probabilistic vectors $\mu, \nu \in S(A)$ the inequality $(\varphi, \mu) \leq (\varphi, \nu)$ holds *for any isotonic mapping* $\varphi\colon A \to \mathbb{R}_+$ only in the case of existence of a coverage of vector μ by vector ν which is agreed floor under the order ω, that is equivalent the existence of decomposition matrix for the ordered set $\langle A, \omega\rangle$ satisfying the condition $\nu = \mu\Delta$.

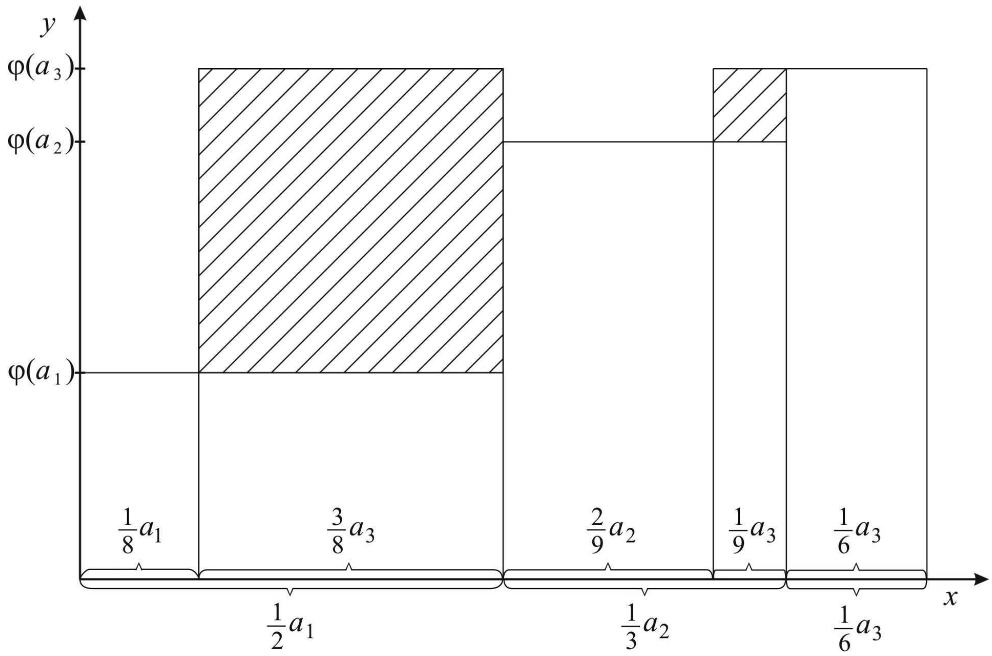

Fig. 4

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