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On a Large Population Partnership Formation Game with Continuous Time

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Abstract This paper presents a model of partnership formation in which there are two classes of player (called for convenience male and female). There is a continuum of players and n types of male and female. Each player begins searching at time zero and the mating season is of length μ. Each player searches until he/she finds a mutually acceptable prospective partner and then this pair both leave the pool of searchers. Hence, as the season progresses, the proportion of players still searching for a partner decreases and the distribution of types changes appropriately. The rate at which prospective partners are found is a non-decreasing function of the proportion of players still searching. The value of pairing with a type i partner at time t is assumed to be $v_i e^{-\gamma t}$, where γ is the discount rate and $v_1 > v_2 > \ldots > v_n$. At a Nash equilibrium, each searcher accepts a prospective partner if and only if the value obtained from such a partnership (ignoring previously incurred discounts) is greater or equal to the expected value obtained from further search. Some general results are given. In addition, we derive the form of the equilibrium when there are two types and present two examples.

Keywords: partnership formation, dynamic game, stopping problem, game with a continuum of players.

1. Introduction

In the economics literature, such games are often termed job search games and have developed from the classical problem of one-sided choice (see Stigler, 1961). It is assumed that a job searcher observes a sequence of offers with values from a known distribution (employers are not choosy). The cost of observing a job offer is assumed to be constant. Janetos (1980) was the first to consider such a model in the context of mate choice. These ideas were later developed by Real (1990).

In many species, both sexes are choosy. Parker (1983) was the first to consider a model of two-sided mate choice. McNamara and Collins (1990) presented a model under which searchers explicitly observe a sequence of prospective partners, unlike in Parker's model. However, the conclusions are very similar (players are split into a finite number of types, such that type i males only mate with type i females). Real (1991) looked at associative mating in more detail.

The models described above assume that the distribution of the values of prospective partners is constant over time. If mating is non-seasonal, then this distribution tends to a steady state distribution, which depends on the strategies used within the population (see Burdett and Coles, 1999, Smith, 2006). However, if mating is seasonal, this distribution changes over time. Collins and McNamara

(1993), as well as Ramsey (2008), consider such models of one-sided choice. Johnstone (1997) gives numerical results for a model of two-sided choice in which time is discrete. Searchers generally become less choosy as the season progresses, but searchers of low quality may become more choosy just before the end of the season in the hope of obtaining a high quality mate in the last time period, when no searcher is choosy. Alpern and Reyniers (2005), as well as Alpern and Katrantzi (2009), apply a more analytic approach to such problems, while Mazalov and Falko (2008) prove some general results. In the last three models, time is discrete and the values of prospective partners have a continuous distribution. In the model presented here, time is continuous and the values of prospective partners have a discrete distribution.

Section 2 presents the model and gives some general results. Section 3 considers realizations of the game in which there are just two types of prospective partner and presents two examples. The form of a Nash equilibrium is presented, together with a theorem regarding the number of Nash equilibria that can exist. Section 4 gives a brief conclusion.

2. The Model

Consider a large population in which there are two classes of players and the numbers of individuals in these classes are equal. Each individual in this population wishes to form a partnership with a player of the other class. For convenience, these classes will be referred to as males and females. However, these classes could be also interpreted as e.g. employers (job positions) and job seekers. It is assumed that partnership formation is seasonal, each player begins searching for a partner at the same time and the period of time available for searching (the time horizon) is μ. Unless specifically specified otherwise, it is assumed that μ is finite.

There are n types of both sexes. The value of the i-th type is v_i (independently of sex), where $v_1 > v_2 > \ldots > v_n > 0$. The proportion of players who are of type i (again, independently of sex) is denoted by p_i. Each player searches until he/she finds a partner, at which time the pair involved leave the pool of searchers. Hence, the number of males searching always equals the number of females searching. Also, the size of the pool of searchers and the distribution of their values change over time according to the profile of strategies, π, used in the population. A profile π defines the types of prospective partners each searcher finds acceptable at all times t, $0 \leq t < \mu$. The payoff of a searcher obtaining a partner of type i at time t is $v_i e^{-\gamma t}$, where $\gamma, \gamma > 0$, is the discount rate. If a player does not find a partner, then his/her payoff is defined to be 0. Hence, we consider a game which is symmetric with respect to sex. According to such a model, only the relative values of prospective partners are important. Hence, without loss of generality, we may assume that $v_n = 1$.

We derive results on equilibria which are symmetric with respect to sex, i.e. players use strategies which are only dependent on their type and not their sex. At such an equilibrium, the distribution of the types of players who are still searching is also independent of sex. Thus, define the proportion of players who are of type i and still searching at time t to be $p_i(t)$. Note that $p_i(0) \equiv p_i$. Let $p(t)$ be the proportion of players who are still searching at time t, i.e. $p(t) = \sum_{i=1}^{n} p_i(t)$. Also, let $q_i(t)$ be the probability that a player is of type i given that he/she is searching at time t, i.e. $q_i(t) = \frac{p_i(t)}{p(t)}$. Note that these functions (and the reward functions defined below) depend on the strategy profile used in the population. However, this dependency will not be reflected in the notation, unless we need to distinguish between the dynamics of the game under different strategy profiles.

The rate at which individuals find prospective partners is given by the function λ, a non-decreasing function of the proportion of players still searching. Prospective partners are chosen at random from the pool of searchers, i.e. a prospective partner encountered at time t is of type i with probability $q_i(t)$. It is assumed that $p \leq \lambda(p) \equiv \lambda[p(t)] \leq 1$ and time is scaled so that $\lambda(1) = 1$. These assumptions take into account that finding prospective partners is likely to become harder as the number of searchers decreases. We consider the following two extreme cases: i) $\lambda(p) = p, \forall p \in [0, 1]$, ii) $\lambda(p) = 1, \forall p \in [0, 1]$. The first case corresponds to random mixing. A searcher encounters members of the other sex at a constant rate of 1 and the individual encountered is chosen at random from all the individuals of the other sex (i.e. is still searching for a partner with probability p). The second case corresponds to what is termed the "singles bar model", where those still searching concentrate their search on individuals of the other sex who have not yet got a partner. Under these assumptions, μ is the expected number of individuals of the opposite sex that an individual meets during the search period (in the single bars model, the number of prospective partners that a searcher expects to meet).

At a Nash equilibrium, no player can increase their expected reward by unilaterally changing strategy. Non-intuitive Nash equilibria may exist. For example, the following strategy profile is always a Nash equilibrium: type 1 searchers only accept prospective partners of type 2 and type 2 searchers only accept prospective partners of type 1. However, such a profile would not evolve via selection. Hence, we assume that an equilibrium must satisfy the optimality criterion (see McNamara and Collins, 1990) stating that a searcher accepts a prospective partner if and only if his/her value is at least as great as the searcher's expected reward from future search (ignoring discounts already incurred). Let this expected future reward of a type i player searching at time t be given by $r_i(t)$. Hence, at time t a type i player should accept a prospective partner of value $\geq r_i(t)$. Such a strategy is called a threshold strategy. Let $A_i(t)$ be the set of mutually acceptable types of prospective partners of a type i player at time t. Define $\overline{v}_i(t)$ to be the expected value of a prospective partner who is mutually acceptable to a type i searcher and still searching at time t. Hence,

$$\overline{v}_i(t) = \frac{\sum_{j \in A_i(t)} v_j p_j(t)}{\sum_{j \in A_i(t)} p_j(t)}. \tag{1}$$

Now consider the dynamics of the game under such a strategy profile. Suppose a player of type i is searching at time t. The probability that such a player finds a partner in the time interval $[t, t+\delta]$ is given by $\delta\lambda(p)\sum_{j \in A_i(t)} q_j(t)$. It follows that

$$p_i(t+\delta) = p_i(t)[1 - \delta\lambda(p) \sum_{j \in A_i(t)} q_j(t)] + O(\delta^2)$$

$$\frac{p_i(t+\delta) - p_i(t)}{\delta} = -p_i(t)\lambda(p) \sum_{j \in A_i(t)} q_j(t) + O(\delta).$$

Letting $\delta \to 0$, we obtain the differential equation

$$\frac{dp_i(t)}{dt} = -p_i(t)\lambda(p) \sum_{j \in A_i(t)} q_j(t). \tag{2}$$

In the case of the random mixing model, this leads to

$$\frac{dp_i(t)}{dt} = -p_i(t) \sum_{j \in A_i(t)} p_j(t). \tag{3}$$

In the case of the singles bar model, this gives

$$\frac{dp_i(t)}{dt} = -\frac{p_i(t) \sum_{j \in A_i(t)} p_j(t)}{p(t)}. \tag{4}$$

Let f_i be the density function of the random variable T_i, the time at which a type i player finds a mutually acceptable partner (by definition $T_i = \mu$ when such a player does not find a partner). Let $\alpha_i(t)$ be the rate at which type i individuals find acceptable partners. It follows that

$$\alpha_i(t) = \frac{\lambda[p(t)] \sum_{j \in A_i(t)} p_j(t)}{p(t)}.$$

It follows that for $0 < t < \mu$,

$$f_i(t) = \alpha_i(t) \exp\left[-\int_0^t \alpha_i(s) ds\right]$$
$$P(T_i = \mu) = 1 - \int_0^\mu \alpha_i(t) \exp\left[-\int_0^t \alpha_i(s) ds\right] dt = \exp\left[-\int_0^\mu \alpha_i(t) dt\right].$$

The future expected reward of a searcher of type i at time t (ignoring previous discounts) is given by

$$r_i(t) = \int_t^\mu \overline{v}_i(s) \alpha_i(s) \exp\left[-\int_t^s \gamma + \alpha_i(\tau) d\tau\right] ds. \tag{5}$$

In particular, under the random mixing model, using Equations (1) and (5), since $\lambda[p(t)] = p(t)$, we obtain

$$r_i(t) = \int_t^\mu \left[\sum_{j \in A_i(s)} v_j p_j(s)\right] \exp\left[-\int_t^s \gamma + \sum_{j \in A_i(\tau)} p_j(\tau) d\tau\right] ds. \tag{6}$$

Similarly, under the singles bar model, we obtain

$$r_i(t) = \int_t^\mu \left[\frac{\sum_{j \in A_i(s)} v_j p_j(s)}{\sum_{j=1}^n p_j(s)}\right] \exp\left[-\int_t^s \gamma + \frac{\sum_{j \in A_i(\tau)} p_j(\tau)}{\sum_{j=1}^n p_j(\tau)} d\tau\right] ds. \tag{7}$$

We now present some general results.

Theorem 1. *If each player uses a threshold strategy and $i < j$, then $r_i(t) \geq r_j(t)$.*

Proof. An individual of type i still searching at time t can obtain a future expected reward of $r_j(t)$ by accepting a prospective partner at time s, where $s \geq t$, if and only if such a prospective partner is mutually acceptable to a type j searcher (note that such a prospective partner finds a type i individual acceptable, since $v_i > v_j$).

Theorem 2. *At any equilibrium, there exists some $t_0 < \mu$, such that all searchers accept any prospective partner when $t \geq t_0$.*

Proof. From the definition of $r_i(t)$, independently of the strategy profile used, $r_i(t)$ is a continuous function and $\lim_{t \to \mu} r_i(t) = 0$ for each i, $1 \leq i \leq n$. It follows that there exists some $t_0 < \mu$ for which $r_i(t) \leq v_n$ for $t \geq t_0$ and each i, $1 \leq i \leq n$.

Theorem 3. *At equilibrium, type i individuals always find prospective partners of type i acceptable.*

Proof. Suppose that type i players do not find prospective partners of type i acceptable for $t \in (t_1, t_2)$. Thus $r_i(t) > v_i$ for $t \in (t_1, t_2)$. Hence, from Theorem 1, $r_j(t) > v_i$ for $j \in \{1, 2, \ldots, i\}$ and $t \in (t_1, t_2)$. Thus, at such an equilibrium, a type i player does not find a partner in the time interval (t_1, t_2). From Theorem 2, we may assume that type i players start accepting prospective partners of type i at time t_2. From the continuity of $r_i(t)$, it follows that $r_i(t_2) = v_i$. Also, for $t \in (t_1, t_2)$,

$$r_i(t) = e^{-\gamma(t_2 - t)} r_i(t) < v_i.$$

This is a contradiction and hence at equilibrium type i individuals always accept prospective partners of type i.

3. Equilibria in Games with Two Types of Individual

Assume that there are two types of prospective partner of values $v_1 > 1$ and $v_2 = 1$. From Theorem 3 and the optimality condition, type 2 individuals always find any prospective partner acceptable. Consider a strategy profile in which type 1 searchers accept a prospective partner of type 2 if and only if $t \geq t_1$ and type 2 searchers always accept any prospective partner. To highlight the dependency of an individual's future expected reward on the strategy profile used, the expected future reward of a type i individual at time t is denoted $r_i(t; t_1)$. Other notation is changed analogously. We have $r_1(t; t_1) = r_2(t; t_1), \forall t \geq t_1$. Also, from Equation (2),

$$\frac{dp_i(t; t_1)}{dt} = -p_i(t; t_1)\lambda[p(t; t_1)],$$

where $p(t; t_1) = p_1(t; t_1) + p_2(t; t_1)$. Let $R(t; t_1) = \frac{p_1'(t; t_1)}{p_2'(t; t_1)}$. For $t \geq t_1$, $R(t; t_1) = \frac{p_1(t; t_1)}{p_2(t; t_1)}$ and thus $p_1(t; t_1) = k(t_1)p_2(t; t_1)$, where $k(t_1) = \frac{p_1(t_1; t_1)}{p_2(t_1; t_1)}$. Hence, for $t > t_1$, the expected value of a prospective partner, $u(t_1)$, is independent of the time at which an encounter occurs. We have

$$u(t_1) = \frac{p_1(t_1; t_1)v_1 + p_2(t_1; t_1)}{p_1(t_1; t_1) + p_2(t_1; t_1)} = \frac{k(t_1)v_1 + 1}{k(t_1) + 1}. \tag{8}$$

Hence, the future expected reward of a player still searching at time t, $t \geq t_1$ is $u(t_1)$ times the expected discount factor from time t onwards (this discount factor is $e^{\gamma(t - \tau)}$ when the next prospective partner is met at time τ, $\tau < \mu$, and 0 otherwise). It should be noted that the time at which the first prospective partner after time t is seen, denoted here by τ, has the following density function

$$f(\tau) = \lambda[p(\tau; t_1)] \exp\left\{ -\int_t^{\tau} \lambda[p(s; t_1)] ds \right\}.$$

It follows that

$$r_1(t;t_1) = r_2(t;t_1) = u(t_1)\int_t^{\mu}\lambda[p(\tau;t_1)]\exp\left\{-\gamma(\tau-t)-\int_t^{\tau}\lambda[p(s;t_1)]ds\right\}d\tau. \quad (9)$$

Since $p(t;t_1)$ is decreasing and λ non-increasing in t, $r_1(t;t_1)$ is decreasing in t. Hence, if $r_1(t_1;t_1) \leq 1$, then the optimal response of any individual still searching at time t, where $t \geq t_1$, is to accept the next prospective partner. Hence, the strategy profile considered above can only describe an equilibrium profile with $t_1 > 0$ when $r_1(t_1;t_1) = 1$. Random pairing (i.e. each individual accepts the first prospective partner encountered) is an equilibrium profile if and only if $r_1(0;0) \leq 1$.

Assume that for $t < t_1$ searchers only pair with prospective partners of the same type. From Equation (2),

$$\frac{dp_i(t;t_1)}{dt} = -\frac{[p_i(t;t_1)]^2\lambda[p(t;t_1)]}{p_1(t;t_1)+p_2(t;t_1)}, \qquad i \in \{1,2\}.$$

Hence, $R(t;t_1) = \frac{[p_1(t;t_1)]^2}{[p_2(t;t_1)]^2}$. It follows that $p_1(t;t_1) > p_2(t;t_1)$ if and only if $p_1 > p_2$, i.e. $p_1 > 0.5$. Moreover, for $t < t_1$, the ratio $\frac{p_1(t;t_1)}{p_2(t;t_1)}$ is

1 increasing in t when $p_1 < 0.5$,
2 constant (equal to 1) when $p_1 = 0.5$,
3 decreasing in t when $p_1 > 0.5$.

It follows from the above analysis that $u(t_1)$ is

1 increasing in t and $< 0.5(v_1+1)$ when $p_1 < 0.5$,
2 constant, equal to $0.5(v_1+1)$, when $p_1 = 0.5$,
3 decreasing in t and $> 0.5(v_1+1)$ when $p_1 > 0.5$.

The analysis above enables us to formulate the following theorem:

Theorem 4. *Let $p_1 \geq 0.5$. For any λ, there is a unique equilibrium profile. When $r_1(0;0) \leq 1$, at equilibrium each searcher accepts the first prospective mate encountered. Otherwise, type 2 players accept any prospective partner and type 1 players only accept type 2 prospective partners when $t \geq t^*$, where t^* satisfies $r_1(t^*;t^*) = 1$.*

Proof. From Equation (9), we have

$$r_1(t_1;t_1) = u(t_1)\int_{t_1}^{\mu}\lambda[p(\tau;t_1)]\exp\left\{-\gamma(\tau-t_1)-\int_{t_1}^{\tau}\lambda[p(s;t_1)]ds\right\}d\tau.$$

Note that $u(t_1)$ is non-increasing in t_1 and from the form of the system of differential equations defining the dynamics of the game, $p_i(t+t_1;t_1)$ is decreasing in t_1 for $i = 1,2$. Hence, $r_1(t_1;t_1)$ is decreasing in t_1. Since $r_1(t_1;t_1)$ is continuous in t_1 and $r_1(\mu;\mu) = 0$, either $r(0;0) \leq 1$, or there is a unique $t^* > 0$, such that $r(t^*;t^*) = 1$.

When $r(0;0) \leq 1$ and the rest of the population accept the first prospective partner encountered, then the best response of any player is to also accept the first prospective partner encountered. Hence, random mating is an equilibrium profile. Since in this case $r(t_1;t_1) < 1$, $\forall t_1 > 0$, it is the unique equilibrium profile.

Now suppose that $t^* > 0$, $r(t^*;t^*) = 1$ and the population use the corresponding strategy profile. All searchers should accept any prospective partner encountered after time t^*. It remains to show that when $t < t^*$, type 1 players should not accept type 2 prospective partners. Consider a type 1 player who accepts any prospective partner from time t_0, where $t_0 < t^*$. From the above analysis, the expected value of the first prospective partner encountered after time t is non-increasing in t. Hence, the expected future reward of a type 1 player at time t_0, $r_1(t_0; t^*)$, satisfies

$$r_1(t_0; t^*) \geq u(t^*) \int_{t_0}^{\mu} \lambda[p(\tau; t^*)] \exp\left\{-\gamma(\tau - t_0) - \int_{t_0}^{\tau} \lambda[p(s; t^*)]ds\right\} d\tau.$$

Since $\lambda[p(t; t^*)]$ is decreasing in t, the integral expression above is decreasing in t_0 and hence $r_1(t_0; t^*) > r(t^*; t^*) = 1$. Hence, a type 1 individual should not accept type 2 prospective partners when $t < t^*$.

Example 1 (The Singles Bar Model).

Suppose the population follow the strategy profile given by t_1, i.e. type 1 players accept type 2 prospective partners if and only if $t > t_1$ and type 2 players are not choosy. For $t < t_1$, it follows from Equation (4) that

$$p_1'(t; t_1) = -\frac{[p_1(t; t_1)]^2}{p_1(t; t_1) + p_2(t; t_1)}; \qquad p_2'(t; t_1) = -\frac{[p_2(t; t_1)]^2}{p_1(t; t_1) + p_2(t; t_1)}. \tag{10}$$

Setting

$$U(t; t_1) = p_1(t; t_1) - p_2(t; t_1); \qquad V(t; t_1) = \frac{1}{p_2(t; t_1)} - \frac{1}{p_1(t; t_1)},$$

we obtain the system of differential equations: $U'(t; t_1) = -U(t; t_1)$ and $V'(t; t_1) = 0$, together with the boundary conditions $U(0; t_1) = p_1 - p_2$ and $V(0; t_1) = 1/p_2 - 1/p_1$. This leads to the following set of equations for $p_1(t_1; t_1)$ and $p_2(t_1; t_1)$:

$$U(t; t_1) = p_1(t; t_1) - p_2(t; t_1) = (p_1 - p_2)e^{-t}$$
$$V(t; t_1) = \frac{1}{p_2(t; t_1)} - \frac{1}{p_1(t; t_1)} = \frac{1}{p_2} - \frac{1}{p_1}.$$

Solving these equations, we obtain

$$p_1(t; t_1) = \frac{(p_1 - p_2)e^{-t} + \sqrt{(p_1 - p_2)^2 e^{-2t} + 4p_1p_2e^{-t}}}{2} \tag{11}$$

$$p_2(t; t_1) = \frac{\sqrt{(p_1 - p_2)^2 e^{-2t} + 4p_1p_2e^{-t}} - (p_1 - p_2)e^{-t}}{2}. \tag{12}$$

In particular, when $p_1 = p_2 = 0.5$, $p_1(t; t_1) = p_2(t; t_1) = 0.5e^{-0.5t}$. Also,

$$k(t_1) = 1 + \frac{p_1 - p_2}{2p_1p_2}\left[(p_1 - p_2)e^{-t_1} + \sqrt{(p_1 - p_2)^2 e^{-2t_1} + 4p_1p_2e^{-t_1}}\right]. \tag{13}$$

For $t > t_1$, from Equation (4), $p_i'(t; t_1) = -p_i(t; t_1)$, $i = 1, 2$. Using the continuity of the functions $p_1(t; t_1)$ and $p_2(t; t_1)$ to obtain the boundary conditions at $t = t_1$, these differential equations lead to $p_i(t; t_1) = p_i(t_1; t_1)e^{t_1 - t}$, $i = 1, 2$ where $p_1(t_1; t_1)$ and $p_2(t_1; t_1)$ can be evaluated from Equations (11) and (12), respectively.

Having derived the dynamics of the game, we now consider the future expected reward of players at time t_1. From Equation (9), it follows that

$$r(t_1; t_1) = u(t_1) \int_{t_1}^{\mu} \exp[-(\gamma + 1)(\tau - t_1)]d\tau = \frac{u(t_1)[1 - \exp\{-(\gamma + 1)(\mu - t_1)\}]}{1 + \gamma}, \tag{14}$$

where $u(t_1)$ can be found using Equations (8) and (13).

Theorem 5. *In any realization of the game based on the singles bar model with two types of prospective partner, there are at most three Nash equilibria.*

Proof. When $p_1 \geq 0.5$, there is a unique equilibrium. Hence, assume that $p_1 < p_2$ and let $z(t) = u(t)[1 - \exp\{(1+\gamma)(t-\mu)\}]$. Since $z(t)$ is a differentiable function, it suffices to show that if $z'(t_0) = 0$, then $z''(t_0) < 0$. In this case, $z(t)$ has at most one extreme point, which must be a maximum. Hence, there exist at most two solutions of the equation $z(t) = 1 + \gamma$. The theorem then follows from the fact that if t^* defines a Nash equilibrium, then either $t^* = 0$ and $z(0) < 1 + \gamma$ or $z(t^*) = 1 + \gamma$.

Differentiating, we obtain

$$z'(t) = u'(t)[1 - \exp\{(1+\gamma)(t-\mu)\}] - (1+\gamma)u(t)\exp[(1+\gamma)(t-\mu)].$$

Hence, if $z'(t_0) = 0$, then

$$(1+\gamma)u(t_0)\exp[(1+\gamma)(t_0-\mu)] = u'(t_0)[1 - \exp\{(1+\gamma)(t_0-\mu)\}]. \tag{15}$$

Differentiating again, we obtain

$$z''(t) = u''(t) - \exp[(1+\gamma)(t-\mu)][u''(t) + 2(1+\gamma)u'(t) + (1+\gamma)^2 u(t)].$$

Together with Equation (15), this gives

$$z''(t_0) = \{u''(t_0) - [1+\gamma]u'(t_0)\}\{1 - \exp[(1+\gamma)(t_0-\mu)]\} - 2(1+\gamma)u'(t_0)\exp[(1+\gamma)(t_0-\mu)].$$

Since $u'(t)>0$ for $p_1<0.5$, if $u''(t_0)<u'(t_0)$, then $z''(t_0)<0$ for any $\gamma>0$. We also have

$$u'(t) = \frac{(v_1-1)k'(t)}{[k(t)+1]^2}; \quad u''(t) = \frac{[k(t)+1](v_1-1)k''(t) - 2(v_1-1)[k'(t)]^2}{[k(t)+1]^3}, \tag{16}$$

where $k(t) = \frac{p_1(t;t_1)}{p_2(t;t_1)}$ for $t \leq t_1$. Note that

$$k'(t) = \frac{p_2(t;t_1)p_1'(t;t_1) - p_1(t;t_1)p_2'(t;t_1)}{[p_2(t;t_1)]^2} = \frac{p_1(t;t_1)[p_2(t;t_1) - p_1(t;t_1)]}{p_2(t;t_1)[p_2(t;t_1) + p_1(t;t_1)]}, \tag{17}$$

where the second equality follows from the system of differential equations given in (10). Similarly, differentiating again, we obtain

$$k''(t) = \frac{p_1(t;t_1)\{[p_2(t;t_1)]^3 - 3[p_2(t;t_1)]^2 p_1(t;t_1) + [p_1(t;t_1)]^2 p_2(t;t_1) + [p_1(t;t_1)]^3\}}{p_2(t;t_1)[p_1(t;t_1) + p_2(t;t_1)]^3}. \tag{18}$$

Hence, from the pair of equations (16), there are at most three Nash equilibria if

$$[k'(t) - k''(t)][k(t)+1] + 2[k'(t)]^2 \geq 0.$$

From Equations (17) and (18), this is equivalent to

$$\frac{6p_1^2(t;t_1)[p_2(t;t_1) - p_1(t;t_1)]}{p_2(t;t_1)[p_1(t;t_1) + p_2(t;t_1)]^2} \geq 0.$$

This equation is satisfied, since $p_2(t;t_1) \geq p_1(t;t_1) \geq 0$.

Numerical calculations indicate that three Nash equilibria can exist, e.g. when $v_1 = 1.1, \gamma = 0.02, \mu = 50$, there are three Nash equilibria given by $t^{*,1} = 0$ (i.e. each individual accepts the first prospective partner), $t^{*,2} \approx 1.151$ and $t^{*,3} \approx 46.514$.

Example 2 (The Random Mixing Model).

Suppose the population follow the strategy profile given by t_1, i.e. type 1 players accept type 2 prospective partners if and only if $t > t_1$ and type 2 players are not choosy. For $t < t_1$, it follows from Equation (3) that

$$p_1'(t;t_1) = -[p_1(t;t_1)]^2; \qquad p_2'(t;t_1) = -[p_2(t;t_1)]^2.$$

From the boundary conditions, $p_1(0;t_1) = p_1$ and $p_2(0;t_1) = p_2$, we obtain

$$p_1(t;t_1) = \frac{p_1}{p_1 t + 1}; \qquad p_2(t;t_1) = \frac{p_2}{p_2 t + 1}. \tag{19}$$

It follows that

$$u(t_1) = \frac{(v_1+1)p_1p_2t_1 + p_1v_1 + p_2}{1 + 2p_1p_2t_1}; \qquad k(t_1) = \frac{p_1p_2t_1 + p_1}{p_1p_2t_1 + p_2}.$$

For $t > t_1$, it follows from Equation (3) that

$$p_1'(t;t_1) = -p_1(t;t_1)[p_1(t;t_1) + p_2(t;t_1)]; \qquad p_2'(t;t_1) = -p_2(t;t_1)[p_1(t;t_1) + p_2(t;t_1)].$$

Solving these equations, using the boundary conditions obtained at $t = t_1$, from the continuity of the functions $p_1(t;t_1)$ and $p_2(t;t_1)$, it follows that

$$p_1(t;t_1) = \frac{k(t_1)p_2}{p_2t + p_2k(t_1)(t-t_1) + 1}; \qquad p_2(t;t_1) = \frac{p_2}{p_2t + p_2k(t_1)(t-t_1) + 1}. \tag{20}$$

Having derived the dynamics of the game, we now consider the future expected reward of players at time t_1. From Equation (9), it follows that

$$r_1(t_1;t_1) = u(t_1)\int_{t_1}^{\mu} \frac{[1+k(t_1)](1+p_2t_1)p_2\exp[-\gamma(\tau-t_1)]}{\{[1+k(t_1)]p_2\tau - p_2k(t_1)t_1 + 1\}^2}d\tau. \tag{21}$$

Equation (21) was used to investigate the number of equilibria when $p_1 < 0.5$. From Theorem 4, $t^* = 0$ describes an equilibrium profile when $r_1(0;0) \leq 1$ and the strategy profile corresponding to t^*, $t^* > 0$, is a Nash equilibrium when $r_1(t^*;t^*)=1$. Realizations of the game were found for which both $r_1(0;0) \leq 1$ and there exist two solutions of the equation $r_1(t_1;t_1)=1$, $t_1>0$, i.e. there can be three equilibria. Unfortunately, no proof has been found that this is the maximum number of equilibria, but also no counterexample has been found. For example, when $\gamma=0.02$ and $p=0.1$, there are multiple equilibria for $v_1 \in (1.751, 1.768)$. In particular, when $v_1 = 1.76$, the three Nash equilibria are given by $t^{*,1}=0$, $t^{*,2} \approx 0.197$ and $t^{*,3} \approx 1.686$.

4. Conclusion

This article has presented a model of seasonal mating where the values of males and females have a discrete distribution and time is continuous. The game is symmetric with respect to sex, i.e. the sex ratio equals one and the distribution of the values of players is independent of sex. The rate at which prospective partners are found is non-increasing in the proportion of players still searching for a mate. In particular, two extreme cases were considered. In one case, players concentrate their search on unmated individuals. In the other case, individuals of the opposite sex are met at a constant rate and the probability that such an individual is unmated is equal to the proportion of unmated individuals in the population as a whole.

Some general results and the form of the equilibrium when prospective partners take one of two values were presented. It was proved that under the singles bar model, there are at most three Nash equilibria. Numerical results indicate that this may be true for the random mixing model, but this hypothesis has not been proven. When three Nash equilibria exist, at one of these equilibria, individuals are not choosy (corresponding to $t^{*,1}=0$). At the other two equilibria, corresponding to $t^{*,2}$ and $t^{*,3}$ where $0<t^{*,2}<t^{*,3}$, the most attractive searchers (type 1) are initially choosy. From the equilibrium conditions, when the vast majority of the population accept any prospective partner, then selection strictly favours non-choosy players. Hence, considering evolutionary dynamics, it is expected that the equilibrium corresponding to $t^{*,2}$ will not evolve, i.e. it is not an evolutionarily stable strategy. At the choosy Nash equilibria, the proportion of the pool of searchers that are type 1 increases over time, meaning that it pays type 1 searchers to wait for a type 1 prospective partner to appear. At the non-choosy Nash equilibrium, the proportion of the pool of searchers that are type 1 is constant. Hence, if it does not pay to wait for a type 1 partner initially, it never pays to wait for such a partner.

Future research will derive Nash equilibria when there are more types.

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