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Multistage Game Model with Time-claiming Alternatives *

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Abstract The new model of multistage game with perfect information, on a closed time interval is considered. On each stage of the game player chooses one of the alternatives and time to perform them. The payoffs depend upon trajectory and the time, at which game terminates. As a solution of this game subgame perfect ϵ - Nash equilibrium is taken.

Keywords: Perfect information, Nash equilibrium, Time-claiming alternative.

1. Introduction

The following finite stage game with perfect information is considered. In each vertex of the game tree belonging to the set of personal moves of player the finite number of basic alternatives is fixed and for each given basic alternative a closed time interval is defined. The elements of this time interval are interpreted as time instants at which the basic alternative can be realized in a given vertex. Each basic alternative in the multistage game with Time-claiming alternatives is associated with an infinite number of alternatives, the basic alternative with corresponding time values we shall call bunch of alternatives.

As usual the strategy of player is a mapping which corresponds to each vertex from the set of personal moves of the player the pair consisting from the index of basic alternative and time necessary to realize this alternative. If the n-tuple of strategies is chosen by players the trajectory of the game path can be uniquely defined. This path consists from the sequence of basic alternatives and corresponding time parameters chosen by players. Payoff function of player for each trajectory of the game continuously depends upon the time when the game terminates and it is a uniformly bounded function. However it is proved that payoff function of the player not necessary continuously depends upon his strategy (part of his strategy, time at which the alternative must be perform). This makes impossible the existence of subgame perfect Nash equilibrium. The example of this case is presented and the existence of subgame perfect ϵ - Nash equilibrium is proved.

This type of games arises in game-theoretical modeling of many real life and business situations.

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2. Difference between classical multistage game with perfect information and multistage game with Time-claiming alternatives

Description of multistage game with Time-claiming alternatives has some differences from multistage game with perfect information.

Multistage game with perfect information is defined on the tree like graph. Denote the game as Γ and graph as $G = (X, F)$, where X is a finite set of vertices and F is a multivalued mapping from X to X ($\forall z \in X, F_z \subset X$).

Consider a partition of the set of vertices X:

$$X_1 \cdots X_n, X_{n+1}, X = \cup_{i=1}^{n+1} X_i, X_k \cap X_l = \emptyset, k \neq l,$$

where $F_z = \emptyset$ for $z \in X_{n+1}$. Set X_i, $i = 1 \cdots n$ is a set of personal moves of player i and set X_{n+1} is a set of final positions. On the set of final positions X_{n+1} payoffs $H_1(z) \cdots H_n(z)$, $z \in X_{n+1}$ are defined. We call a strategy of player i, mapping u_i which to each position $z \in X_i$ uniquely correspond the position $y \in F_z$. Denote the set of all possible strategies of player i as U_i. We call an ordered set $u = (u_1 \cdots u_i \cdots u_n)$, where $u_i \in U_i$ the situation in the game Γ. Define the payoff function K_i, for each player $i = 1 \cdots n$ in the game Γ as follows:

$$K_i(u_1 \cdots u_i \cdots u_n) = H_i(z_l), i = 1 \cdots n,$$

where $z_l \in X_{n+1}$ is a final position which corresponds to the situation $(u_1 \cdots u_i \cdots u_n)$ in the game Γ. Function K_i, $i = 1 \cdots n$ is defined on the set of situations $U = \prod_{i=1}^{n} U_i$. Thus multistage game with perfect information has the following form

$$\Gamma = (N, \{U_i\}_{i \in N}, \{K_i\}_{i \in N}),$$

where N is a set of players.

Multistage game with Time-claiming alternatives is defined on the tree like extended form of the graph G. Denote the extended tree like graph as $\overline{G}(Y, \Phi)$, where Y is the set of positions and $Y = X \times [t_0, T]$, where X is the set of positions from the graph G and $[t_0, T]$ is a closed time interval, in which the game takes place. Φ is a multivalued mapping from Y to Y ($\forall \overline{z} \in Y$, $\Phi_{\overline{z}} \subset Y$), $\Phi_{\overline{z}} = F_z \times [t, T]$, where $\overline{z} = (z, t)$, $z \in X$, $t \in [t_0, T]$.

Denote a pair of positions $(\overline{z}, \overline{z}')$, where $\overline{z} = (z, t)$, $\overline{z}' = (z', t')$, $\overline{z}' \in \Phi_{\overline{z}} = F_z \times [t; T]$ for any possible pair of values t, t' ($\forall t' \in [t, T], \forall t \in [t_0, T]$), as an arc $p = (\overline{z}, \overline{z}')$. Denote a sequence of arcs $p = (p_1, p_2 \cdots p_k \cdots)$, where $p_1 = (\overline{z}_1, \overline{z}_1')$, $p_2 = (\overline{z}_2, \overline{z}_2') \cdots p_k = (\overline{z}_k, \overline{z}_k'), \cdots$ and $\overline{z}_1' = \overline{z}_2$, $\overline{z}_2' = \overline{z}_3 \cdots \overline{z}_k' = \overline{z}_{k+1} \cdots$ for any possible values t_i, t_j' ($\forall t_j' \in [t_i, T], \forall t_i \in [t_0, T]$), $i, j = 1, 2 \cdots k \cdots$, as a path in the graph $\overline{\Gamma}$. In the multistage game with Time-claiming alternatives the length $l(p)$ of the path $p = (p_1, p_2 \cdots p_k)$ is a number of arcs in the path or number of different positions z minus one from graph Γ, $l(p) = k$. Consider a partition of the set of positions Y:

$$Y_1 \cdots Y_n, Y_{n+1}, Y = \cup_{i=1}^{n+1} Y_i, Y_k \cap Y_l = \emptyset, k \neq l,$$

where $\Phi_{\overline{z}} = \emptyset$ for $\overline{z} \in Y_{n+1}$. Set $Y_i = X_i \times [t_0, T]$ is the set of personal moves of player i and $Y_{n+1} = X_{n+1} \times [t_0, T]$. On the set of final positions Y_{n+1} payoffs $\overline{H}_1(\overline{z}) \cdots \overline{H}_n(\overline{z})$, $\overline{z} = (z, t) \in Y_{n+1}$ are defined. Functions $\overline{H}_i(\overline{z}) = \overline{H}_i(z, t)$, $i = 1 \cdots n$ continuously depend upon the parameter t and are uniformly bounded functions on the closed interval $[t_0, T]$. We call strategy of player i mapping $\overline{u}_i$

which to each position $\overline{z} \in Y_i$ correspond the position $\overline{z}' \in \Phi_{\overline{z}}$. Denote the set of all possible strategies of player i as $\overline{U}_i$. We call an ordered set $\overline{u} = (\overline{u}_1 \cdots \overline{u}_i \cdots \overline{u}_n)$, where $\overline{u}_i \in \overline{U}_i$, situation in the game $\overline{\Gamma}$. Define the payoff function $\overline{K}_i$ for each player $i = 1 \cdots n$ in the game $\overline{\Gamma}$, as follows:

$$\overline{K}_i(\overline{u}_1 \cdots \overline{u}_i \cdots \overline{u}_n) = \overline{H}_i(\overline{z}_l), i = 1 \cdots n,$$

where $\overline{z}_l \in Y_{n+1}$ is a final position which corresponds to the situation $(\overline{u}_1 \cdots \overline{u}_i \cdots \overline{u}_n)$ in the game $\overline{\Gamma}$. Function $\overline{K}_i$, $i = 1 \cdots n$ is defined on the set of situation $\overline{U} = \prod_{i=1}^{n} \overline{U}_i$. The length of all paths in the game $\overline{\Gamma}$ is uniformly bounded, because the set of positions is finite. We call the length of the game $\overline{\Gamma}$ the length of the longest path in the game $\overline{\Gamma}$. By construction the lengths of the games Γ and $\overline{\Gamma}$ are the same.

3. Definition of multistage game with Time-claiming alternatives

The multistage game with Time-claiming alternatives has the following normal form:

$$\overline{\Gamma} = (N, \{\overline{U}_i\}_{i \in N}, \{\overline{K}_i\}_{i \in N})$$

Suppose all paths in the game $\overline{\Gamma}$ have the same length l, then the game proceeds as follows:

1. Let $\overline{z}_0 = (z_0, t_0) \in Y_{i_1} = X_{i_1} \times [t_0, T]$ then player i_1 chooses

$$\overline{z}_1 = (z_1, t_1) \in \Phi_{\overline{z}_0} = F_{z_0} \times [t_0, T],\ t_1 < T$$

2. if $\overline{z}_1 = (z_1, t_1) \in Y_{i_2} = X_{i_2} \times [t_0, T]$ then player i_2 chooses

$$\overline{z}_2 = (z_2, t_2) \in \Phi_{\overline{z}_1} = F_{z_1} \times [t_1, T],\ t_2 < T$$

...

k. if $\overline{z}_{k-1} = (z_{k-1}, t_{k-1}) \in Y_{i_k} = X_{i_k} \times [t_0, T]$ then player i_k chooses

$$\overline{z}_k = (z_k, t_k) \in \Phi_{\overline{z}_{k-1}} = F_{z_{k-1}} \times [t_{k-1}, T],\ t_k < T$$

...

l. if $\overline{z}_{l-1} = (z_{l-1}, t_{l-1}) \in Y_{i_l} = X_{i_l} \times [t_0, T]$ then player i_l chooses
$\overline{z}_l = (z_l, t_l) \in \Phi_{\overline{z}_{l-1}} = F_{z_{l-1}} \times [t_{l-1}, T]$ $(F_{z_{l-1}} \subset X_{n+1})$ and the game terminates.

The game can terminate if player i chooses $\overline{z} = (z, t)$, where $z \in X_{n+1}$ (where t is any instant from the closed interval $[t_0, T]$)

4. Existence of subgame perfect ϵ - Nash equilibrium in multistage game with Time-claiming alternatives

The following theorem can be proved.

Theorem 1. *In any multistage game $\overline{\Gamma}$ there exists a subgame perfect ϵ-equilibrium $\overline{u}^*$*

It is shown in the example that in multistage game with Time-claiming alternatives sometimes is impossible to construct a subgame perfect equilibrium, because of noncontinuous dependence of payoff function of players upon time (part of the strategy). During the backward induction process in multistage game with perfect information on each stage player is maximizing his payoff choosing the basic alternative and in this game model player in addition chooses time to perform this basic alternative. Consider subgame $\overline{\Gamma}_{(z,t)}$ $(z \in X, t \in [t_0, T])$. Suppose that in this subgame there exist a subgame perfect Nash equilibrium $\overline{u}^*_{(z,t)} = (\overline{u}^*_{(z,t),1} \cdots \overline{u}^*_{(z,t),n})$. Let $\overline{K}_i(z, t; \overline{u}^*_{(z,t),1} \cdots \overline{u}^*_{(z,t),n})$, $i \in N$ be the payoff function of player i in this Nash equilibrium. Since the NE $\overline{u}^*_{(z,t)}$ is fixed this function depends only on initial conditions of the subgame, and we can write

$$V_i(z, t) = \overline{K}_i(z, t; \overline{u}^*_{(z,t),1} \cdots \overline{u}^*_{(z,t),n}), i \in N.$$

Functions $V_i(z, t), i \in N$ we shall call value functions. Also introduce the following function

$$\begin{aligned} V_i(z_k, t_k)(t) = \max_{z \in F_{z_k}} \{V_i(z, t), t \in [t_k, T)\} = V_i(\hat{z}, t), \text{ where } x_k \in X_i \\ V_j(z_k, t_k)(t) = V_j(\hat{z}, t), j \neq i \\ V_i(z_k, t_k)(t) = H_i(x_k, t_k) \forall t \in [t_k, T], \text{ if } x_k \in X_{n+1}, i \in N \\ V_i(z_k, T)(t) = H_i(z_k, T), i \in N \\ V_i(z_k, t_k) = \sup_{t \in [t_k, T]} V_i(z_k, t_k)(t) \end{aligned} \quad (1)$$

Problem takes place when player chooses time to perform basic alternative, because the value in the game $\overline{\Gamma}$ $(V_i(z_k, t_k)(t))$ sometimes cannot be reached (see example). Function $\overline{H}_i(z, t), z \in X_{n+1}$ continuously depends upon the parameter t and is uniformly bounded in the closed interval $[t_0, T] \supset [t_i, T]$. Then function $V_i(z_k, t_k)(t)$ is also uniformly bounded on the closed interval and $\sup_{t \in [t_k, T]} V_i(z_k, t_k)(t)$ exists. Therefore in the game $\overline{\Gamma}$ we can use subgame perfect ϵ - Nash equilibrium as a solution in the game.

5. Example

Consider an example from (Papayoanou, 2010). Game Γ takes place on the graph $G = (X, F)$ below (see Fig. 1). The set of players N consists of two players, $N = \{Alpha, Beta\}$. On the first stage of the game Γ, in the position z_0 player *Alpha* has two alternatives z_1, z_3 (Accept or Reject financial proposal from player *Beta*). If player *Alpha* chooses the alternative z_3 then the game terminates, if player *Alpha* chooses the alternative z_1 then the game proceeds and player *Beta* makes his move. On the second stage of the game Γ, in the position z_1 player *Beta* has two alternatives z_2, z_6 (Offer to Alpha a better financial proposal or Compete with player *Alpha*). If player *Beta* chooses the alternative z_6 then the game terminates, if player *Beta* chooses the alternative z_2 then the game proceeds and player *Alpha* again makes a move. On the third stage of the game Γ, in the position z_2 player *Alpha* has two alternatives z_4, z_5 (Accept the better financial proposal from player *Beta* or Compete with player *Beta*). In either cases the game terminates, payoffs are defined for both players at the end of the game (see Fig. 1).

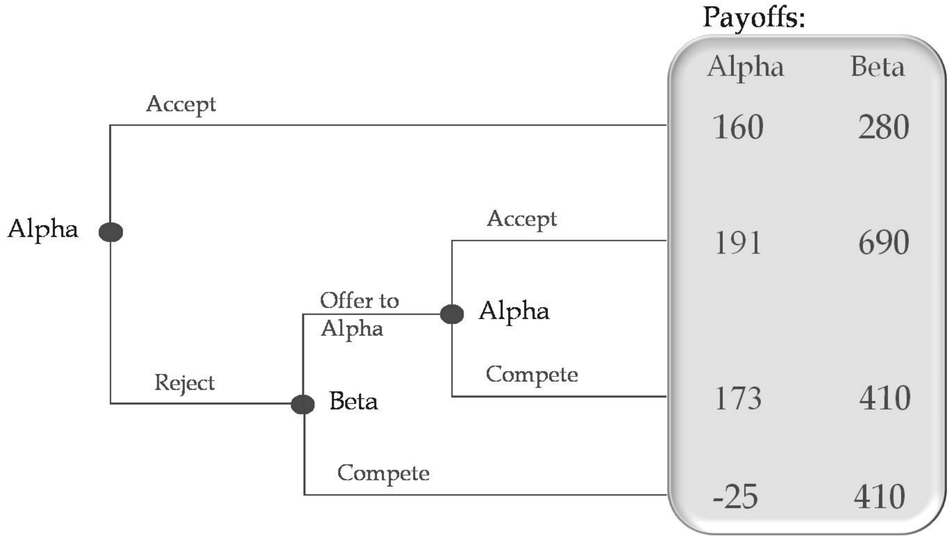

Fig. 1: Example

Often it is important to determine the time necessary to make a decision. Consider related example, which was proposed using the approach of multistage games with time claiming alternatives. In this example it is possible to simulate more properly and efficiently this economic situation, because the model of Multistage game with Time-claiming alternatives considers also a time necessary to make a decision (move). Game $\overline{\Gamma}$ takes place on the graph $\overline{G} = (Y, \Phi)$ below (see Fig. 2). The set of players N has not changed. On the first stage of the game $\overline{\Gamma}$, in the position $\overline{z}_0 = (z_0, t_0)$ player *Alpha* has two bunches of alternatives $\overline{z}_1 = (z_1, t_1), \overline{z}_3 = (z_3, t_3)$, where $t_1, t_3 \in [t_0, T]$. In this model player *Alpha* is not just choosing the alternative Accept or Reject financial proposal from player *Beta*, but also he is selecting the time instant to choose the alternative (basic alternative). If player *Alpha* chooses the alternative $\overline{z}_3 = (z_3, t_3)$ then the game terminates at the moment of time t_3 and player *Alpha* gets the payoff

$$H_{Alpha}(z_3, t_3) = \frac{160}{1 + t_3},$$

player *Beta* gets the payoff

$$H_{Beta}(z_3, t_3) = \frac{280}{1 + t_3}.$$

If player *Alpha* chooses the alternative $\overline{z}_1 = (z_1, t_1)$ then the game proceeds and player *Beta* makes a move. On the second stage of the game $\overline{\Gamma}$, in the position $\overline{z}_1 = (z_1, t_1)$ player *Beta* has two bunches of alternatives $\overline{z}_2 = (z_2, t_2), \overline{z}_6 = (z_6, t_6)$, where $t_2, t_6 \in [t_1, T]$ (Offer to Alpha a better financial proposal or Compete with player *Alpha* and choose time instant to make it). If player *Beta* chooses the alternative $\overline{z}_6 = (z_6, t_6)$ then the game terminates at the moment t_6 and player *Alpha* gets the payoff

$$H_{Alpha}(z_6, t_6) = \ln(t_6 + 1) - 25,$$

player *Beta* gets the payoff

$$H_{Beta}(z_6, t_6) = -20(t_6 - 4)^2 + 730.$$

If player *Beta* chooses the alternative $\overline{z}_2 = (z_2, t_2)$ then the game proceeds and player *Alpha* again makes his move. On the third stage of the game $\overline{\Gamma}$, in the position $\overline{z}_2 = (z_2, t_2)$ player *Alpha* has two bunches of alternatives $\overline{z}_4 = (z_4, t_4), \overline{z}_5 = (z_5, t_5)$, where $t_4, t_5 \in [t_2, T]$ (Accept the better financial proposal from player *Beta* or Compete with player *Beta* and choose time instant to make it). In either cases the game terminates. If player *Alpha* chooses the alternative $\overline{z}_4$ then he gets the payoff

$$H_{Alpha}(z_4, t_4) = -0,35(t_4 - 6)^2 + 204,$$

player *Beta* gets the payoff

$$H_{Beta}(z_4, t_4) = -{t_4}^2 + 690.$$

If player *Alpha* chooses the alternative $\overline{z}_5$ then he gets the payoff

$$H_{Alpha}(z_5, t_5) = -(t_5 - 6)^2 + 209,$$

player *Beta* gets the payoff (see Fig. 2)

$$H_{Beta}(z_5, t_5) = -{t_5}^2 + 410.$$

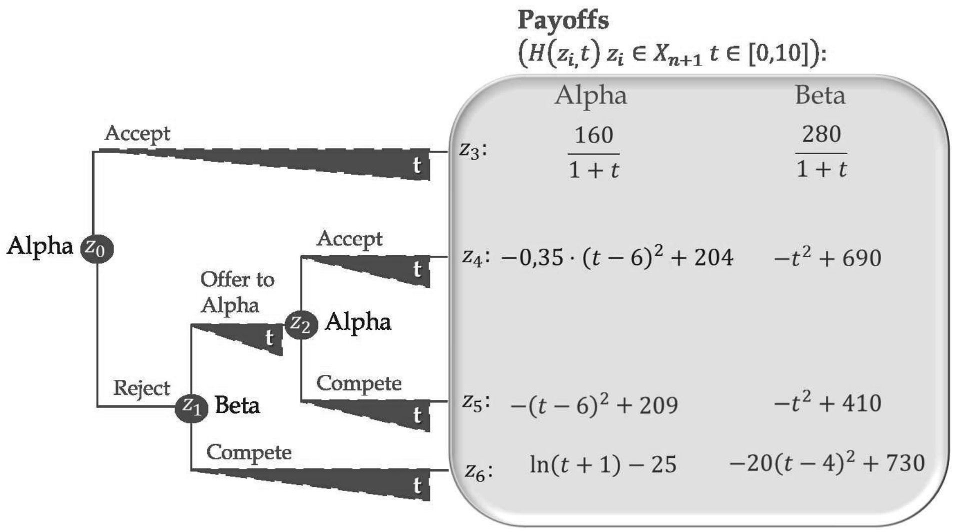

Fig. 2: Example

Payoff functions depending upon the time on which the game $\overline{\Gamma}$ terminates in this example have the following form:

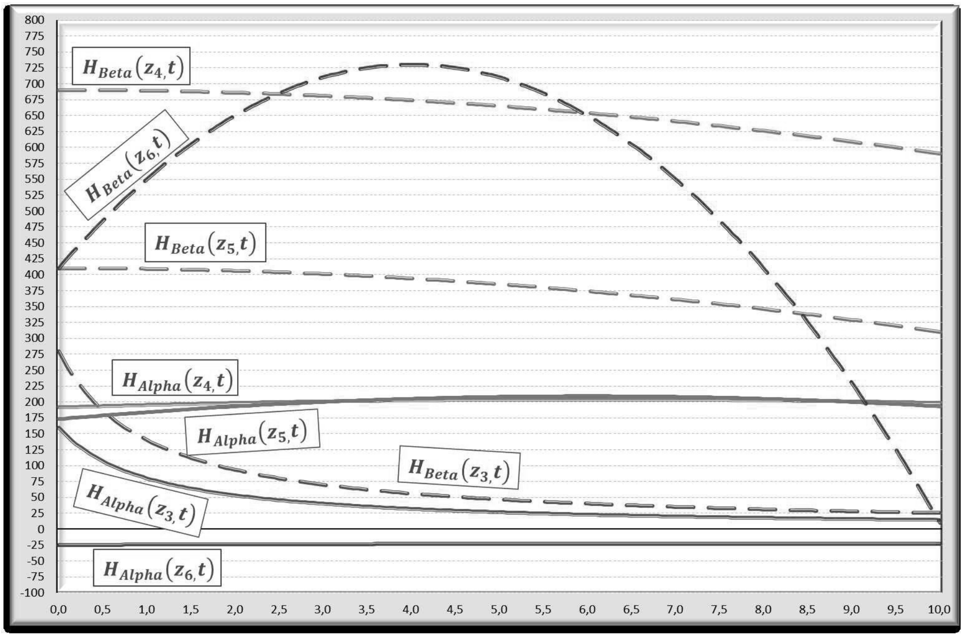

Fig. 3: Example

We use standard backward induction procedure to find the solution of the game. The backward induction in this game is based on the length of the game (which is a number of arcs in the maximal path $l = 3$). We have to solve subgames $\overline{\Gamma}_{(z_i,t_i)}$, $z_i \in X$ for any starting time $t_i \in [0, 10]$. Denote by $V_{Alpha}(k, z_i, t_i)$ and $V_{Beta}(k, z_i, t_i)$, values of subgames $\Gamma_{(z_i,t_i)}$ with length k for player *Alpha* and *Beta* (the exact upper bound of values of players payoffs when a fixed subgame perfect ϵ - Nash equilibrium is used).

Consider subgame $\overline{\Gamma}_{(z_2,t_2)}$, where in the position $\overline{z}_2 = (z_2, t_2)$ player *Alpha* makes a move. z_2 is a starting position of this subgame and $t_2 \in [0, 10]$ is a time chosen by the player *Beta* on the previous stage of the game $\overline{\Gamma}$ (in the position $\overline{z}_1 = (z_1, t_1)$). In this subgame player *Alpha* chooses between two bunches of alternatives $\overline{z}_4 = (z_4, t_4)$, $\overline{z}_5 = (z_5, t_5)$ ($\Phi_{\overline{z}_2} = \{\overline{z}_4, \overline{z}_5\}$). Since the positions $\overline{z}_4, \overline{z}_5 \in Y_{n+1}$ and t_4, t_5 are time instants, when the game $\overline{\Gamma}$ terminates the payoffs are defined in this positions and depend upon t_4, t_5 and are equal to $H_{Alpha}(z_4, t_4)$, $H_{Alpha}(z_5, t_5)$ (see Fig. 4).

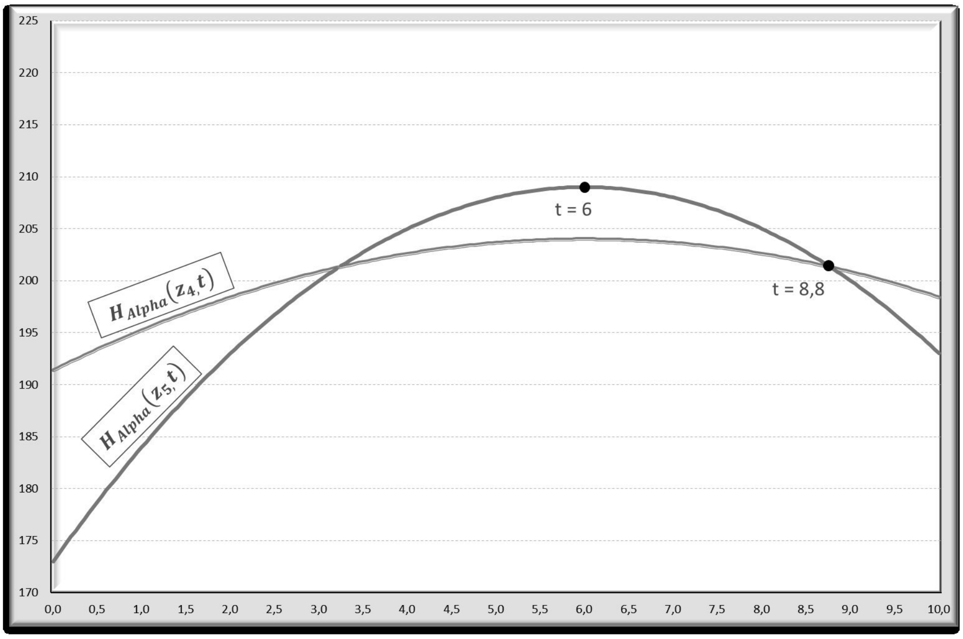

Fig. 4: Example

Since it is impossible for player *Alpha* to choose a moments of time $t_4, t_5 < t_2$, because the subgame $\overline{\Gamma}_{z_2,t_2}$ starts at the moment of time t_2 we must construct a solution for the player *Aplha* for each possible starting time t_2. The value of the subgame $\overline{\Gamma}_{(z_2,t_2)}$ for player *Alpha* is

$$V_{Alpha}(1, z_2, t_2) = \max\{ \sup_{t_4 \in [t_2,10]} (-0,35(t_4-6)^2+204); \sup_{t_5 \in [t_2,10]} (-(t_5-6)^2+209)\}.$$

Consider different cases, where $t_2 \in [0;6], t_2 \in (6;8,8], t_2 \in (8,8;10]$:

1. $t_2 \in [0;6]$, where $t = 6$ is determined from the condition $\overline{H}_{Alpha}(z_4;6) = \sup_{t \in [0,10]} \overline{H}_{Alpha}(z_4,t)$ then the optimal behavior of player *Alpha* in position $\overline{z_2} = (z_2,t_2)$ is defined by formula $\overline{u}_{Alpha}(z_2,t_2) = (z_5;t_5)$, where $t_5 = 6$.

$$V_{Alpha}(1, z_2, t_2) = \max\{ \sup_{t_4 \in [t_2,10]} (-0,35(t_4-6)^2+204); \sup_{t_5 \in [t_2,10]} (-(t_5-6)^2+ +209)\} = \sup_{t_5 \in [t_2,10]} (-(t_5-6)^2+209) = -(6-6)^2+209 = 209$$

$$V_{Beta}(1, z_2, t_2) = -6^2 + 410 = 374$$

(Since in the position $\overline{z}_2 = (z_2,t_2)$ player *Beta* is not making a move)

2. $t_2 \in (6;8,8]$, where $t = 8,8$ is determined from the condition $\overline{H}_{Alpha}(z_4;8,8) = \overline{H}_{Alpha}(z_5;8,8)$ then the optimal behavior of player *Alpha* in position $\overline{z_2} =$

(z_2, t_2) is defined by formula $\overline{u}_{Alpha}(z_2; t_2) = (z_5; t_5)$, where $t_5 = t_2$.

$$V_{Alpha}(1, z_2, t_2) = \max\{ \sup_{t_4 \in [t_2, 10]} (-0,35(t_4 - 6)^2 + 204); \sup_{t_5 \in [t_2, 10]} (-(t_5 - 6)^2 + +209)\} = \sup_{t_5 \in [t_2, 10]} (-(t_5 - 6)^2 + 209) = -(t_2 - 6)^2 + 209)$$

$$V_{Beta}(1, z_2, t_2) = -t_2^2 + 410$$

3. $t_2 \in (8, 8; 10]$ then the optimal behavior of player *Alpha* in position $\overline{z_2} = (z_2, t_2)$ is defined by formula $\overline{u}_{Alpha}(z_2; t_2) = (z_4; t_4)$, where $t_4 = t_2$.

$$V_{Alpha}(1, z_2, t_2) = \max\{ \sup_{t_4 \in [t_2, 10]} (-0,35(t_4 - 6)^2 + 204); \sup_{t_5 \in [t_2, 10]} (-(t_5 - 6)^2 + +209)\} = \sup_{t_4 \in [t_2, 10]} (-0,35(t_4 - 6)^2 + 204) = -0,35(t_2 - 6)^2 + 204$$

$$V_{Beta}(1, z_2, t_2) = -t_2^2 + 690$$

Introduce the following function:

$$V_{Alpha}(1, z_2, t_2)(t) = \max\{-0,35(t_4 - 6)^2 + 204; -(t_5 - 6)^2 + 209, \text{ where } t \in [t_2, 10]\},$$

$$\text{it is easy to see that } \sup_{t \in [t_2, 10]} V_{Alpha}(1, z_2, t_2)(t) =$$

$$= \sup_{t \in [t_2, 10]} [\max\{-0,35(t_4 - 6)^2 + 204; -(t_5 - 6)^2 + 209, \text{ where } t \in [t_2, 10]\}] =$$

$$= \max\{ \sup_{t_4 \in [t_2, 10]} (-0,35(t_4 - 6)^2 + 204); \sup_{t_5 \in [t_2, 10]} (-(t_5 - 6)^2 + 209)\} =$$

$$= V_{Alpha}(1, z_2, t_2),$$

where t corresponds to the moments of time t_4 or t_5: if $V_{Alpha}(1, z_2, t_2)(t) = -(t - 6)^2 + 209$ then t corresponds to t_5, if $V_{Alpha}(1, z_2, t_2)(t) = -0,35(t - 6)^2 + 204$ then t corresponds to t_4. On the Fig. 5 we can see function $V_{Alpha}(1, z_2, t_2)(t)$, where $t_2 = 0$, if subgame $\overline{\Gamma}_{(z_2, t_2)}$ started at the moment of time $t_2 = 0$. If the subgame $\overline{\Gamma}_{(z_2, t_2)}$ starts for example at the moment of time $t_2 = 5$, then we must consider all values of function $V_{Alpha}(1, z_2, t_2)(t)$ on the Fig. 5 for $t \in [5, 10]$. In this sense the function on the Fig. 5 corresponds to the solution constructed earlier.

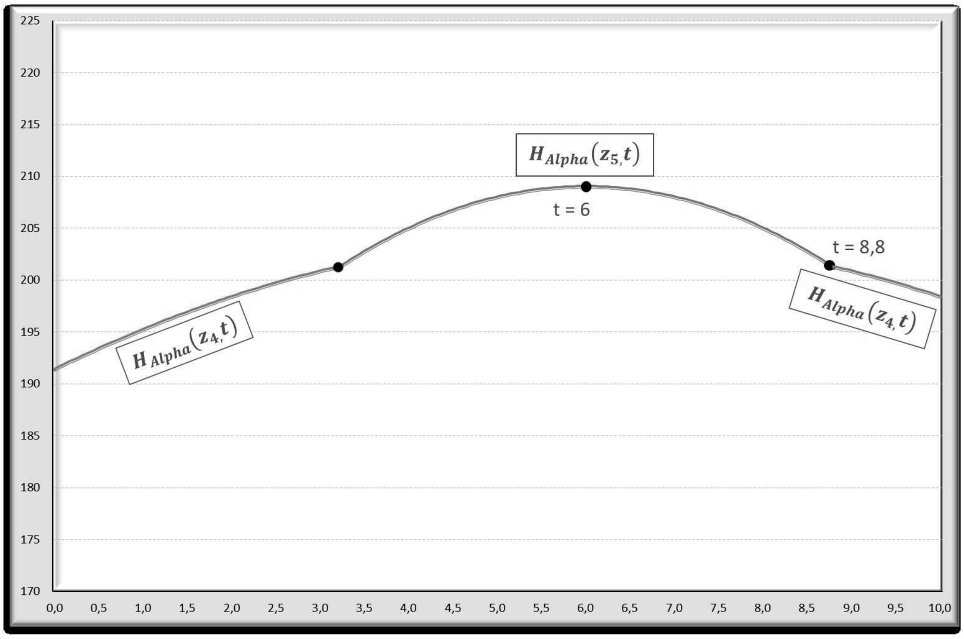

Fig. 5: Example

Consider subgame $\overline{\Gamma}_{(z_1,t_1)}$, where in the position $\overline{z}_1 = (z_1, t_1)$ player *Beta* makes a move. z_1 is a starting position and $t_1 \in [0, 10]$ is a time chosen by the player *Alpha* on the previous stage of the game $\overline{\Gamma}$ (in the position $\overline{z}_0 = (z_0, t_0)$). In this subgame player *Beta* chooses between the alternatives $\overline{z}_2 = (z_2, t_2)$, $\overline{z}_6 = (z_6, t_6)$ ($\Phi_{\overline{z}_1} = \{\overline{z}_2, \overline{z}_6\}$). Since position $\overline{z}_6 \in Y_{n+1}$ and t_6 is the time instant, when the game $\overline{\Gamma}$ terminates the payoff in this position is equal to $H_{Beta}(z_6, t_6)$. $\overline{z} = (z_2, t_2) \notin Y_{n+1}$, which means that if player *Beta* in the position $\overline{z}_1 = (z_1, t_1)$ chooses a basic alternative $z_2 \in X$ and any time instant $t_2 \in [t_1, T]$, then player *Beta* must take in account which alternative is going to choose player *Alpha* on the next stage in the position $\overline{z} = (z_2, t_2)$ for each moment of time t_2 and calculate his payoff accordingly, $V_{Beta}(1, z_2, t_2)$ (see Fig. 6).

The concept of Nash equilibrium obtained by backward induction suggests that on the first step of the subgame player who makes a move on this stage expects that in all subsequent subgames players use a fixed predetermined Nash equilibrium. In our case it means that player *Beta* in the subgame $\overline{\Gamma}_{(z_2,t_2)}$ expects that player *Alpha* in the following subgame uses Nash equilibrium strategy, i.e. he expects to get the payoff $V_{Alpha}(1, z_2, t_2)$ (which is a result of using Nash equilibrium strategy by the player *Alpha*).

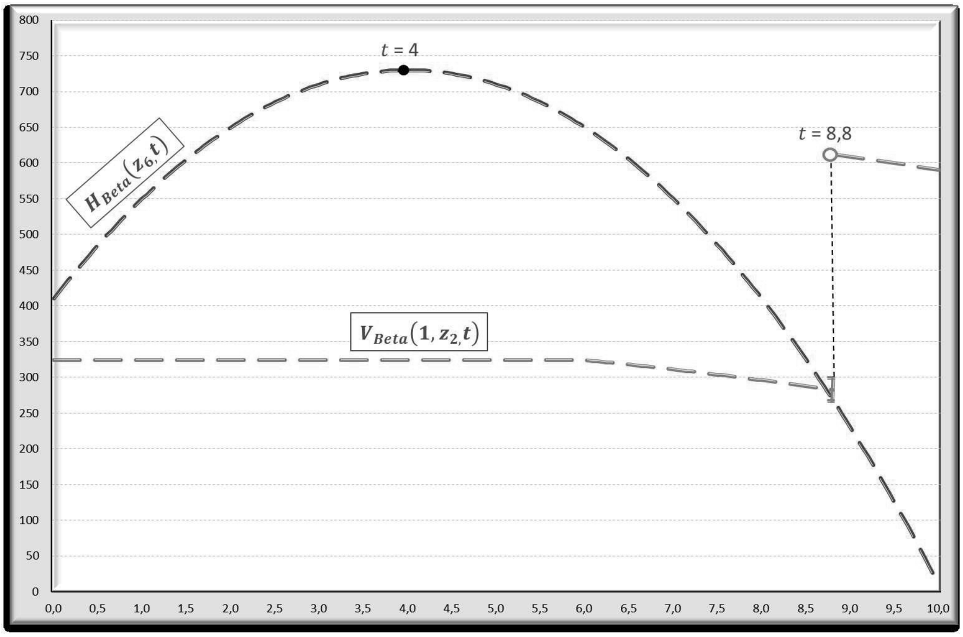

Fig. 6: Example

The decision of player *Beta* in this subgame depends upon the time t_1. The value of the subgame $\overline{\Gamma}_{(z_1,t_1)}$ for player *Beta* is

$$V_{Beta}(2, z_1, t_1) = \max\{ \sup_{t_2 \in [t_1,10]} V_{Beta}(1, z_2, t_2); \sup_{t_6 \in [t_1,10]} (-20(t_6 - 4)^2 + 730)\}.$$

Consider different cases, where $t_1 \in [0; 4], t_1 \in (4; 8, 8], t_1 \in (8, 8; 10]$:

1. $t_1 \in [0; 4]$, where $t = 4$ is determined from the condition $\overline{H}_{Beta}(z_6; 4) = \max_{t \in [0,10]} \overline{H}_{Beta}(z_6, t)$ then the optimal behavior of player *Beta* in position $\overline{z_1} = (z_1, t_1)$ is defined by formula $\overline{u}_{Beta}(z_1, t_1) = (z_6; t_6)$, where $t_6 = 4$. Then subgame $\overline{\Gamma}_{(z_1,t_1)}$ terminates, because $\overline{z}_6 \in Y_{n+1}$.

$$V_{Beta}(2, z_1, t_1) = \max\{ \sup_{t_2 \in [t_1,10]} V_{Beta}(1, z_2, t_2); \sup_{t_6 \in [t_1,10]} (-20(t_6 - 4)^2 + 730)\} =$$
$$= \sup_{t_6 \in [t_1,10]} (-20(t_6 - 4)^2 + 730) = -20(4 - 4)^2 + 730 = 730$$
$$V_{Alpha}(2, z_1, t_1) = ln(4 + 1) - 25 = -23, 4$$

2. $t_1 \in (4; 8, 8]$ then the optimal behavior of player *Beta* in position $\overline{z_1} = (z_1, t_1)$ is defined by formula $\overline{u}_{Beta}(z_1; t_1) = (z_6; t_6)$, where $t_6 = t_1$. Then subgame $\overline{\Gamma}_{(z_1,t_1)}$ terminates, because $\overline{z}_6 \in Y_{n+1}$.

$$V_{Beta}(2, z_1, t_1) = \max\{ \sup_{t_2 \in [t_1,10]} V_{Beta}(1, z_2, t_2); \sup_{t_6 \in [t_1,10]} (-20(t_6 - 4)^2 + 730)\} =$$
$$= \sup_{t_6 \in [t_1,10]} (-20(t_6 - 4)^2 + 730) = -20(t_1 - 4)^2 + 730$$
$$V_{Alpha}(2, z_1, t_1) = ln(t_1 + 1) - 25$$

3. $t_1 \in (8,8;10]$ then the optimal behavior of player *Beta* in position $\overline{z_1} = (z_1, t_1)$ is defined by formula $\overline{u}_{Beta}(z_1;t_1) = (z_2;t_2)$, where $t_2 = t_1$. Then on the next stage $\overline{u}_{Alpha}(z_2;t_2) = (z_4;t_4)$, where $t_4 = t_2$.

$$V_{Beta}(2,z_1,t_1) = \max\{ \sup_{t_2\in[t_1,10]} V_{Beta}(1,z_2,t_2); \sup_{t_6\in[t_1,10]} (-20(t_6-4)^2+730)\} =$$

$$= \sup_{t_2\in[t_1,10]} V_{Beta}(1,z_2,t_2) = V_{Beta}(1,z_2,t_1)$$

$$V_{Alpha}(2,z_1,t_1) = V_{Alpha}(1,z_2,t_1) = -0,35(t_1-6)^2+204$$

The solution here is constructed using the solution for the player *Alpha* on the previous step. Consider the following function:

$$V_{Beta}(2,z_1,t_1)(t) = \max\{V_{Beta}(1,z_2,t); -20(t-4)^2+730, \text{ where } t\in[t_1,10]\},$$

$$\text{it is easy to see that } \sup_{t\in[t_1,10]} V_{Beta}(2,z_1,t_1)(t) = V_{Beta}(2,z_1,t_1),$$

where t is a moment of time t_2 or t_6: if $V_{Beta}(2,z_1,t_1)(t) = V_{Beta}(1,z_2,t)$ then t corresponds to t_2, if $V_{Beta}(2,z_1,t_1)(t) = -20(t-4)^2+730$ then t corresponds to t_6. On the Fig. 5 we can see function $V_{Beta}(2,z_1,t_1)(t)$, where $t_1 = 0$, if subgame $\overline{\Gamma}_{(z_1,t_1)}$ started at the moment of time $t_1 = 0$. If the subgame $\overline{\Gamma}_{(z_1,t_1)}$ starts for example at the moment of time $t_1 = 5$, then we must consider all values of the function $V_{Beta}(2,z_1,t_1)(t)$ on the Fig. 5 for $t\in[5,10]$. In this sense the function on the Fig. 5 corresponds to the solution constructed earlier.

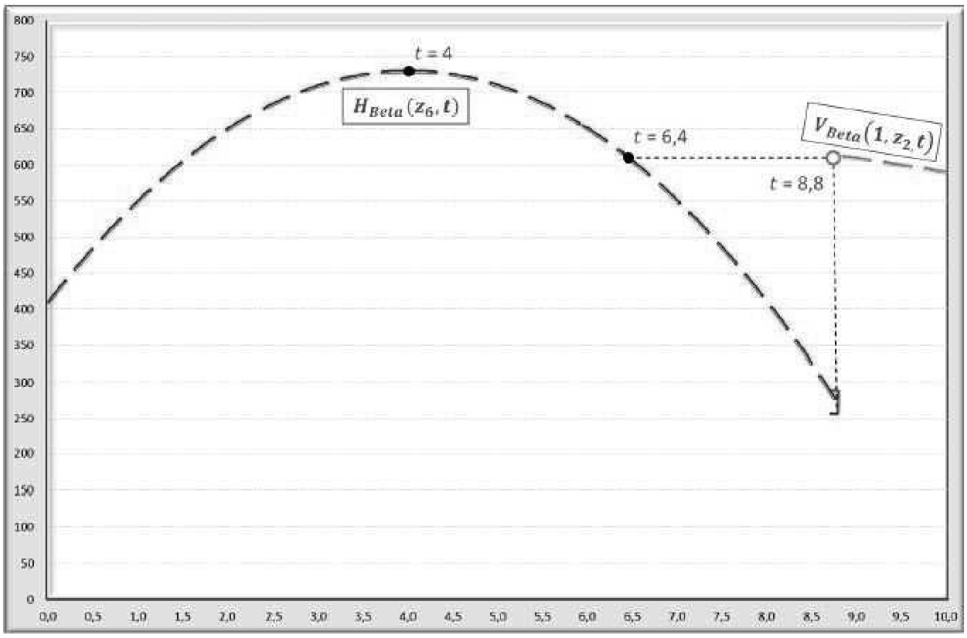

Fig. 7: Example

It is shown in the solution for the player *Beta* that the function $V_{Beta}(2,z_1,t_1)(t)$, where t is a time instant chosen by the player *Beta* (t_2,t_6) noncontinuously depends upon the parameter t. Which means that if subgame $\overline{\Gamma}_{z_1,t_1}$ starts at the moment of time $t_1 \in (6,4;10]$ then

$$V_{Beta}(2, z_1, t_1) = \max\{\sup_{t_2\in[t_1,10]} V_{Beta}(1, z_2, t_2); \sup_{t_6\in[t_1,10]} (-20(t_6 - 4)^2 + 730)\}$$

can not be reached.

Consider subgame $\overline{\Gamma}_{(z_0,t_0)} = \overline{\Gamma}$, where in the position $\overline{z}_0 = (z_0, t_0)$ player *Alpha* makes a move. z_0 is a starting position and $t_0 = 0$ is a starting moment of the whole game. In this subgame player *Alpha* chooses between two alternatives $\overline{z}_1 = (z_1, t_1)$, $\overline{z}_3 = (z_3, t_3)$ ($\Phi_{\overline{z}_0} = \{\overline{z}_1, \overline{z}_3\}$). Since the position $\overline{z}_3 \in Y_{n+1}$, t_3 is the time instant, when the game $\overline{\Gamma}$ terminates the payoff in this position is defined and is equal to $H_{Aplha}(z_3, t_3)$. $\overline{z}_1 = (z_1, t_1) \notin Y_{n+1}$, this means that if player *Alpha* in the position $\overline{z}_0 = (z_0, t_0)$ chooses a basic alternative $z_1 \in X$ and any time instant $t_1 \in [t_0, T]$, then player *Alpha* must determine which alternative is going to choose player *Beta* $\overline{z}_2$ or $\overline{z}_6$ on the next stage and the player *Alpha* on the stage after $\overline{z}_4$ or $\overline{z}_5$. In the position $\overline{z}_1 = (z_1, t_1)$ for each moment of time t_1 player *Alpha* can calculate his payoff according to the optimal behavior of player *Beta* on this stage and optimal behavior of player *Alpha* on the next one, $V_{Alpha}(2, z_1, t_1)$ (see Fig. 6) (see Fig. 8).

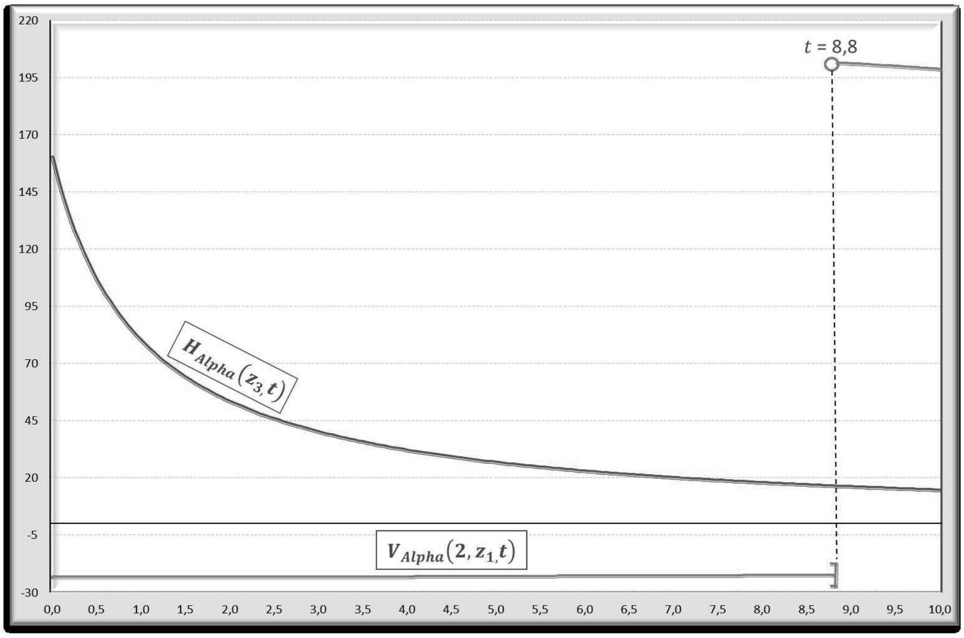

Fig. 8: Example

if player *Alpha* chooses the alternative $\overline{z}_1 = (z_1, t_1)$ ($\overline{u}_{Alpha}(z_0;t_0) = (z_1;t_1)$), where $t_1 \in (6,4;8,8]$ then on the next stage player *Beta* chooses the alternative $\overline{z}_2 = (z_2, t_2)$ ($\overline{u}_{Beta}(z_1;t_1) = (z_2;t_2)$), where $t_2 \in [t_1;10]$. Suppose $t_2 = 8,8 + \epsilon$ (where ϵ is sufficiently small). Then on the stage after player *Alpha* chooses the alternative $\overline{z}_4 = (z_4, t_4)$ ($\overline{u}_{Alpha}(z_2;t_2) = (z_4;t_4)$), where $t_4 = t_2 = 8,8 + \epsilon$. The value of the subgame $\overline{\Gamma}_{(z_0,t_0)} = \overline{\Gamma}$ for player *Alpha* is

$$\begin{aligned} V_{Alpha}(3,z_0,t_0) = \max\{ \sup_{t_3\in[0,10]} (\frac{160}{1+t_3}); \sup_{t_1\in[0,10]} V_{Alpha}(2,z_1,t_1)\} = \\ = \sup_{t_1\in[0,10]} V_{Alpha}(2,z_1,t_1) = V_{Alpha}(2,z_1,8,8) = \\ = -0,35((8,8+\epsilon)-6)^2 + 204 = 201,2-\epsilon. \end{aligned}$$

Supremum is not reached, although it is possible to choose t close enough to $8,8$ ($t+\epsilon$, where ϵ is sufficiently small) in which value of the function is close enough to the supremum.

The value of the subgame $\overline{\Gamma}_{(z_0,t_0)}$ for player *Beta* is

$$V_{Beta}(3,z_0,t_0) = V_{Beta}(2,z_1;8,8) = V_{Beta}(1,z_2,8,8) = 612,6-\epsilon$$

The solution here is constructed using the solution for the player *Alpha* and *Beta* on the previous steps. Consider the following function:

$$\begin{aligned} V_{Alpha}(3,z_0,t_0)(t) = \max\{\frac{160}{1+t}; V_{Alpha}(2,z_1,t), \text{ where } t\in[0,10]\}, \\ \text{it is easy to see that } \sup_{t\in[0,10]} V_{Alpha}(3,z_0,t_0)(t) = \\ = \sup_{t\in[0,10]} [\max\{\frac{160}{1+t}; V_{Alpha}(2,z_1,t), \text{ where } t\in[0,10]\}] = \\ = max\{ \sup_{t_3\in[0,10]} (\frac{160}{1+t_3}); \sup_{t_1\in[0,10]} V_{Alpha}(2,z_1,t_1)\} = V_{Alpha}(3,z_0,t_0), \end{aligned}$$

where t is a moment of time t_1 or t_3: if $V_{Alpha}(3,z_0,t_0)(t) = V_{Alpha}(2,z_1,t)$ then t corresponds to t_1, if $V_{Alpha}(3,z_0,t_0)(t) = \frac{160}{1+t}$ then t corresponds to t_3. Function $V_{Alpha}(3,z_0,t_0)(t)$ has the following form (see Fig. 9).

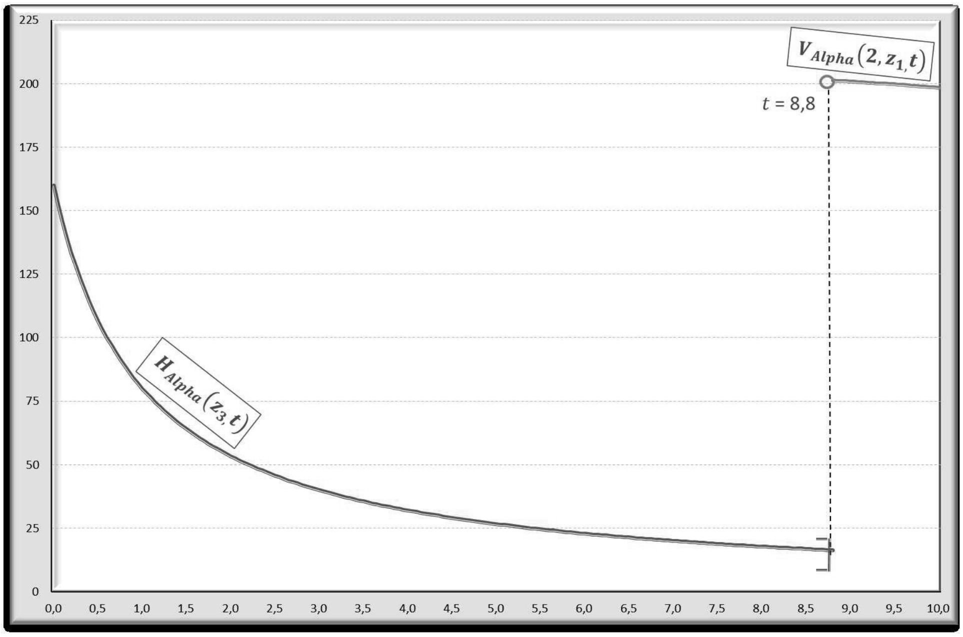

Fig. 9: Example

It is shown in the solution for the player *Alpha* that the function $V_{Alpha}(3, z_0, t_0)(t)$, where t is a time instant chosen by the player *Alpha* (it can be t_3 or t_1 depends on which basic alternative is chosen) noncontinuously depends upon the parameter t and $V_{Alpha}(3, z_0, t_0) = \sup\limits_{t \in [0,10]} V_{Alpha}(3, z_0, t_0)(t)$ cannot be reached.

Subgame perfect ϵ-Nash equilibrium $\overline{u}^*$ has the form:

$$\overline{u}^*_{Aplha} = (\overline{u}^*_{Aplha}(z_0, t_0), \overline{u}^*_{Aplha}(z_2, t_2)) = ((z_1; 8,8 + \epsilon), (z_4; 8,8 + \epsilon));$$
$$\overline{u}^*_{Beta} = \overline{u}^*_{Beta}(z_1, t_1) = (z_2; 8,8 + \epsilon)$$

The payoffs in Subgame perfect ϵ-Nash equilibrium $\overline{u}^*_t$:

$$\overline{K}^*_{Aplha} = 201,2 - \epsilon'; \overline{K}^*_{Beta} = 612,6 - \epsilon',$$

where ϵ' is sufficiently small.

6. Conclusion

The model of multistage game with time claiming alternatives can be successfully used in the business or science applications where time is a decisive parameter in the decision making process.

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