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An Axiomatization of the Interval Shapley Value and on Some Interval Solution Concepts

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Abstract The Shapley value, one of the most common solution concepts in Operations Research applications of cooperative game theory, is defined and axiomatically characterized in different game-theoretical models. In this paper, we focus on the Shapley value for cooperative games where the set of players is finite and the coalition values are compact intervals of real numbers. In this study, we study the properties of the interval Shapley value on the class of size monotonic interval games, and axiomatically characterize its restriction to a special subclass of cooperative interval games by using fairness property, efficiency and the null player property. Further, we introduce the interval Banzhaf value and the interval egalitarian rule. Finally, the paper ends with a conclusion and an outlook to future studies.
Keywords: Shapley value, Banzhaf value, egalitarian rule, interval uncertainty, fairness property.

1. Introduction

The Shapley value, which is proposed by Lloyd Shapley in his 1953 Phd dissertation, is one of the most common single-valued solution concepts in cooperative game theory. The Shapley value associates to each game $v \in G^N$ one payoff vector in $\mathbb{R}^N$. The Shapley value is introduced and characterized for cooperative games with a finite player set and where coalition values are real numbers, it has captured much attention being extended in new game theoretical models and widely applied for solving reward/cost sharing problems in Operations Research (OR) and economic situations, sociology, computer science, etc.

The Shapley value probably is the most eminent one-point solution concept for transferable utility games (TU games). Ever since its original characterization by Shapley himself, much effort has been put in the endeavor to provide alternative characterizations. Various axiomatizations of the Shapley value have been given. In the literature, the Shapley value and its characterizations are: Shapley (1953), Owen (1972), Young (1985), Hart and Mas-Colell (1989), Monderer et al. (1992) and van den Brink (2002). Uncertainty on coalition values, which is a challenge of the modern world, led to new models of cooperative games and corresponding Shapley-like values (Branzei et al. (2003), Timmer et al. (2003), Alparslan Gök et al. (2009a)).

This paper focuses on the characterization of the interval Shapley value which was introduced by Alparslan Gök et al. (2009a). A cooperative interval game (Alparslan Gök et al. (2009c)) is an ordered pair $< N, w >$ where $N = \{1, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function which assigns to each coalition $S \in 2^N$ a closed and bounded interval $w(S) \in I(\mathbb{R})$, such that $w(\emptyset) = [0, 0]$. The interval Shapley value associates to each cooperative interval game a payoff vector whose components are compact intervals of real numbers.

The interval Shapley value assigns a payoff vector components which are compact intervals of real numbers to each cooperative interval game. Alparslan Gök et al. (2010) give an axiomatization of the interval Shapley value by using additivity, efficiency, symmetry and dummy player properties on an additive cone of cooperative interval games. Different from this, they focus on two alternative characterizations of the interval Shapley value on the same additive cone of cooperative games arising from several OR and economic situations with interval data. For example, peer group games (Branzei et al. (2010)) are useful in modeling sequential production situations, auctions and flow situations. Airport situations with interval data (Alparslan Gök et al. (2009b)), some connection problems where the cost of the connections are affected by interval uncertainty (Moretti et al. (2011)), lot sizing problems with uncertain demands (Kimms and Drechsel (2009)), and sequencing problems, where parameters are compact intervals of real numbers (Alparslan Gök et al. (2009a)) can be solved using interval games in the additive cone under consideration.

This paper is organized as follows. We give basic notions and facts from the theory of interval calculus in Section 2. The properties of the interval Shapley value is studied on the class of size monotonic interval games and an axiomatic characterization of the interval Shapley value is given on a special subclass of cooperative interval games in Section 3. In Section 4 and Section 5, we introduce the interval Banzhaf value on the class of size monotonic interval games and the interval egalitarian rule on the class of cooperative interval games. Finally, we conclude this paper, including an outlook to future research challenges.

2. Preliminaries

In this section, we recall the notions of cooperative interval games in Alparslan Gök (2009, 2010).

A cooperative interval game is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all nonempty, compact intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the minimal reward which coalition S could receive on its own, and $\overline{w}(S)$ is the maximal reward which coalition S could get. The family of all interval games with player set N is denoted by IG^N.

Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;
(ii) $\alpha I = \left[\alpha \underline{I}, \alpha \overline{I}\right]$.

By (i) and (ii) we see that $I(\mathbb{R})$ has a cone structure.

In this paper we also need a partial subtraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$.

For $w_1, w_2 \in IG^N$ we say that $w_1 \succcurlyeq w_2$ if $w_1(S) \succcurlyeq w_2(S)$, for each $S \in 2^N$.

For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with "$\succcurlyeq$" is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.

The model of interval cooperative games is an extension of the model of classical TU-games. We recall that a classical TU-game $< N, v >$ is defined by $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We denote the size of a coalition $S \subset N$ by $|S|$, the family of such games by G^N, and recall that G^N is a $(2^{|N|} - 1)$-dimensional linear space for which unanimity games form an interesting basis. Let $S \in 2^N \setminus \{\emptyset\}$. The unanimity game based on S, $u_S : 2^N \to \mathbb{R}$ is defined by

$$u_S(T) = \begin{cases} 1, S \subset T, \\ 0, \text{otherwise.} \end{cases}$$

The reader is referred to Peters (2008) and Branzei et al. (2008) for a survey on classical TU-games.

Interval solutions are useful to solve reward/cost sharing problems with interval data using cooperative interval games as a tool. The interval payoff vectors, which are the building blocks for interval solutions, are the vectors whose components belong to $I(\mathbb{R})$. We denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors.

We call a game $< N, w >$ size monotonic if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N.

The interval marginal operators and the interval Shapley value were defined on $SMIG^N$ in Alparslan Gök et al. (2009a) as follows.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$ of $N = \{1, 2, \ldots, n\}$. The interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$, corresponding to σ, associates with each $w \in SMIG^N$ the interval marginal vector $m^\sigma(w)$ of w with respect to σ defined by $m_i^\sigma(w) = w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))$ for each $i \in N$, where $P^\sigma(i) := \left\{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$, and $\sigma^{-1}(i)$ denotes the entrance number of player i.

For size monotonic games $< N, w >$, $w(T) - w(S)$ is defined for all $S, T \in 2^N$ with $S \subset T$, since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$. Now, we notice that for each $w \in SMIG^N$ the interval marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S \cup \{i\}) - \underline{w}(S \cup \{i\}) \geq \overline{w}(S) - \underline{w}(S)$, which can be rewritten as $\overline{w}(S \cup \{i\}) - \overline{w}(S) \geq \underline{w}(S \cup \{i\}) - \underline{w}(S)$. So, $w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$. Note that all the interval marginal vectors of a size monotonic game are efficient interval payoff vectors.

3. Properties of the interval Shapley value

In this section we study some properties of the interval Shapley value on the class of size monotonic interval games.

The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N. \quad (1)$$

We can write (1) as follows

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))). \quad (2)$$

The terms after the summation sign in (2) are of the form $w(S \cup \{i\}) - w(S)$, where S is a subset of N not containing i.

We note that there are exactly $|S|!(n-1-|S|)!$ orderings for which one has $P^\sigma(\{i\}) = S$. The first factor, $|S|!$, corresponds to the number of orderings of S and the second factor, $(n-1-|S|)!$, is just the number of orderings of $N \setminus (S \cup \{i\})$. Using this, we can rewrite (2) as

$$\Phi_i(w) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} (w(S \cup \{i\}) - w(S)).$$

Let us observe that

$$\sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} = 1.$$

Next, we recall that efficiency, null player and additivity axioms for solutions $g : SMIG^N \to I(\mathbb{R})^N$.

Efficiency (E): For every $w \in SMIG^N$ it holds that $\sum_{i \in N} g_i(w) = w(N)$.

Let $w \in SMIG^N$ and $i \in N$. Then, i is called a *null player* if $w(S \cup \{i\}) = w(S)$, for each $S \in 2^{N \setminus \{i\}}$.

Null Player (NP): If $i \in N$ is a null player in an interval game $w \in SMIG^N$, then $g_i(v) = [0, 0]$.

Additivity (A): For every $w, w' \in SMIG^N$ it holds that $g(w + w') = g(w) + g(w')$.

Now, we mention the fairness property. Fairness property is introduced in van den Brink (2002).

Let $w \in SMIG^N$ and $i, j \in N$. Then, i and j are called *symmetric players*, if $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S)$, for each S with $i, j \notin S$.

Symmetry (S): If $i, j \in N$ are symmetric players in $w \in SMIG^N$, then $g_i(w) = g_j(w)$.

Fairness state that if to an interval game $w \in SMIG^N$ we add an interval game $w' \in SMIG^N$ in which players i and j are symmetric, then the interval payoffs of players i and j change by the same amount.

Fairness (F): If $i, j \in N$ are symmetric players in $w \in SMIG^N$, then

$$g_i(w' + w) - g_i(w') = g_j(w' + w) - g_j(w') \text{ for all } w' \in SMIG^N.$$

Fairness requires two players' interval payoffs to change by the same amount whenever a game is added where these players are symmetric. This property is quite plausible because adding such a game does not affect the differential of these players productivities measured by interval marginal contributions.

4. An axiomatic characterization of interval Shapley value

In this section, we give the characterization of interval Shapley value by using fairness property on the special subclass of cooperative interval games. Firstly, we mention about the relationship between our axioms.

Every interval solution that fulfills symmetry and additivity also satisfies fairness. Further, every interval solution that fulfills the null player property and fairness also satisfies symmetry. These propositions are an extension of the propositions obtained by Rene to cooperative interval games.

Proposition 1. *If $g : SMIG^N \to I(\mathbb{R})^N$ fulfills symmetry and additivity, then g also satisfies fairness.*

Proof. We know that $g : SMIG^N \to I(\mathbb{R})^N$ fulfill symmetry and additivity. Let $g : SMIG^N \to I(\mathbb{R})^N$ satisfy symmetry and additivity. If $i, j \in N$ are symmetric in $w' \in SMIG^N$, then for every $w \in SMIG^N$ it holds that

$$\begin{aligned} g_i(w'+w) - g_i(w') &= g_i(w') + g_i(w) - g_i(w') && \text{(from } \mathbf{A}\text{)} \\ &= g_i(w) && \text{(from } \mathbf{S}\text{)} \\ &= g_j(w) \\ &= g_j(w) + g_j(w') - g_j(w') && \text{(from } \mathbf{A}\text{)} \\ &= g_j(w'+w) - g_j(w'). \end{aligned}$$

Thus, g fulfills fairness.

Proposition 2. *If $g : SMIG^N \to I(\mathbb{R})^N$ fulfills the null player property and fairness, then f also satisfies symmetry.*

Proof. Let $g : SMIG^N \to I(\mathbb{R})^N$ fulfill the null player property and fairness. For the null game $w_0 \in SMIG^N$ given by $w_0(S) = [0,0]$ for all $S \subset N$, the null player property implies that $g_i(w_0) = [0,0]$ for all $i \in N$. If $i, j \in N$ are symmetric players in $w \in SMIG^N$, then

$$\begin{aligned} g_i(w) &= g_i(w_0+w) - g_i(w_0) \quad \text{(from } \mathbf{F}\text{)} \\ &= g_j(w_0+w) - g_j(w_0) \\ &= g_j(w). \end{aligned}$$

Thus, g fulfills symmetry.

We know that the interval Shapley value is characterized by efficiency, the null player property, symmetry and additivity. By Proposition 1 it also guarantees fairness. Now, we give the main result of this paper.

Theorem 1. *An interval solution $g : SMIG^N \to I(\mathbb{R})^N$ is equal to the interval Shapley value if and only if it fulfills efficiency, the null player property and fairness.*

Proof. The proof is a straightforward generalization from the classical case and can be obtained by following the steps of Theorem 2.5 in van den Brink (2002).

Logical independence of the three axioms of Theorem 1 can be illustrated by the following two well-known solutions. Interval Banzhaf value fulfills the null player property and fairness but it does not satisfy efficiency. Interval egalitarian rule fulfills effciency and fairness but it does not satisfy the null player property.

5. The interval Banzhaf value

A cooperative game describes a situation in which a finite set of n players can generate certain payoffs by cooperation. A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. Two well-known solution concepts are the Shapley value as proposed by Shapley (1953), and the Banzhaf value, initially introduced in the context of voting games by Banzhaf (1965). Now, we introduce the Banzhaf value by using interval uncertainty.

The classical Banzhaf value $\beta : G^N \to \mathbb{R}^N$ given by

$$\beta_i (v) := \frac{1}{2^{|N|-1}} \sum_{i \in S} (v (S) - v (S \backslash \{i\}))$$

for all $i \in N$.

The Banzhaf value considers that every player is equally likely to enter to any coalition whereas the Shapley value assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally likely. In addition, the Shapley value is efficient, while the Banzhaf value is not efficient. Thus, the Shapley value distributes the total utility among players while the total amount that players get from Banzhaf's allocation depends on the structure of the TU game.

The interval Banzhaf value is defined for $SMIG^N$ since the interval marginal operators is defined for $SMIG^N$.

The interval Banzhaf value $\beta : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\beta(w) := \frac{1}{2^{|N|-1}} \sum_{i \in S} (w (S) - w (S \backslash \{i\}) , \text{ for each } w \in SMIG^N.$$

Example 1. Let $< N, w >$ be a cooperative interval game with $N = \{1, 2, 3\}$ and $w (1) = w (13) = [7, 7] , \; w (12) = [12, 17] , \; w (123) = [24, 29] ,$ and $w (S) = [0, 0]$ otherwise. The interval Shapley value of this game can be calculated as follows: Then the interval marginal vectors are given in the following table, where $\sigma : N \to N$ is identified with $(\sigma (1) , \sigma (2) , \sigma (3))$. Firstly, for $\sigma_1 = (1, 2, 3)$, we calculate the interval marginal vectors. Then,

$$\begin{aligned}
m_1^{\sigma_1} (w) &= w (1) = [7, 7] , \\
m_2^{\sigma_1} (w) &= w (12) - w (1) = [5, 10] , \\
m_3^{\sigma_1} (w) &= w (123) - w (12) = [12, 12] .
\end{aligned}$$

The others can be calculated similarly, which is shown in Table 1. Table 1 illustrates the interval marginal vectors of the cooperative interval game in Example 1. The average of the six interval marjinal vectors is the interval Shapley value of this game which can be shown as:

$$\Phi(w) = ([\frac{27}{2}, 16], [\frac{13}{2}, 9], [4, 4]).$$

The interval Banzhaf value of this game can be calculated as follows: The interval Banzhaf value is defined as

$$\beta(w) = \frac{1}{2^{|N|-1}} \sum_{i \in S} (w (S) - w (S \backslash \{i\}) .$$

Table 1: Interval marginal vectors

σ	$m_1^{\sigma}(w)$	$m_2^{\sigma}(w)$	$m_3^{\sigma}(w)$
$\sigma_1 = (1,2,3)$	$[7,7]$	$[5,10]$	$[12,12]$
$\sigma_2 = (1,3,2)$	$[7,7]$	$[17,22]$	$[0,0]$
$\sigma_3 = (2,1,3)$	$[12,17]$	$[0,0]$	$[12,12]$
$\sigma_4 = (2,3,1)$	$[24,29]$	$[0,0]$	$[0,0]$
$\sigma_5 = (3,1,2)$	$[7,7]$	$[17,22]$	$[0,0]$
$\sigma_6 = (3,2,1)$	$[24,29]$	$[0,0]$	$[0,0]$

Then,

$$\begin{aligned}\beta_1(w) &= \frac{1}{2^2}\sum_{1\in S}(w(S) - w(S\setminus\{1\})\\ &= \frac{1}{2^2}(w(1) + w(12) - w(2) + w(13) - w(3) + w(123) - w(23))\\ &= \frac{1}{2^2}([7,7] + [12,17] - [0,0] + [7,7] - [0,0] + [24,29] - [0,0])\\ &= \frac{1}{4}([50,60]) = [12\frac{1}{2}, 15].\end{aligned}$$

The others can be calculated similarly:

$$\beta_2(w) = [5\frac{1}{2}, 8],\ \beta_3(w) = [3,3]\,.$$

Then, the interval Banzhaf value can be shown as

$$\beta(w) = ([12\frac{1}{2}, 15], [5\frac{1}{2}, 8], [3,3]).$$

Remark 1. The interval Banzhaf value satisfies the null player property, and fairness but does not fulfill efficiency.

In Example 1, the interval Banzhaf value does not satisfy efficiency.

$$\begin{aligned}\sum_{i=1}^{3}\beta_i(w) &= \beta_1(w) + \beta_2(w) + \beta_3(w)\\ &= [12\frac{1}{2}, 15] + [5\frac{1}{2}, 8] + [3,3]\\ &= [21,26] \neq [24,29] = w(N)\end{aligned}$$

6. The interval egalitarian rule

A family of monotonic solutions to general cooperative games (coalitional form games where utility is not assumed to be transferable) is the egalitarian rule. The egalitarian rule is introduced by Kalai and Samet (1985).

The classical egalitarian rule $\gamma : G^N \to \mathbb{R}^N$ is given by

$$\gamma_i(v) := \frac{v(N)}{|N|},$$

for all $i \in N$.

The interval egalitarian rule is defined for IG^N. The interval egalitarian rule $\gamma : IG^N \to I(\mathbb{R})^N$ is defined by

$$\gamma(w) = \frac{w\,(N)}{|N|}, \text{ for each } w \in IG^N.$$

Example 2. Consider again the cooperative interval game in Example 1. Then, the interval egalitarian rule of this game is:

$$\gamma_i(w) = \frac{w\,(N)}{|N|} = \frac{[24,29]}{3} = [8, 9\frac{2}{3}], \quad i = 1,2,3.$$

Remark 2. The interval egalitarian rule satisfies efficiency and fairness but does not fulfill the null player property.

Example 3. Consider the cooperative interval game $< N, w >$ with $N = \{1,2,3\}$ and $w\,(2) = w\,(12) = [7,7]\,,\ w\,(3) = w\,(13) = [12,17]\,, w\,(23) = w\,(N) = [24,29]\,,$ and $w\,(S) = [0,0]$ otherwise. The interval egalitarian rule of this game is:

$$\gamma_i(w) = \frac{w\,(N)}{|N|} = \frac{[24,29]}{3} = [8, 9\frac{2}{3}], \quad i = 1,2,3$$

Here, the null player is player 1 and $\gamma_1(w) = [8, 9\frac{2}{3}]$. Thus, the interval egalitarian rule does not satisfy the null player property.

7. Conclusion and Outlook

We end this paper by giving a concluding remark and an invitation to future research and application. The Shapley value and the Banzhaf value are the most common single-valued solutions in cooperative game theory. In this context, we give the characterization of the interval Shapley value by using fairness property in the smaller class of cooperative interval games. Then, we introduce the interval Banzhaf value and the egalitarian rule by using interval uncertainty.

We notice that whereas the Shapley value and the Banzhaf value are defined and axiomatically characterized for arbitrary cooperative TU-games, the interval Shapley value and the Banzhaf value is defined only for a subclass of cooperative interval games, called size monotonic games. The interval Shapley value is axiomatically characterized but the interval Banzhaf value and the interval egalitarian rule are not axiomatically characterized. There exists a gap to be filled by characterizing the interval Banzhaf value and the interval egalitarian rule. In fact, it is possible to characterize the interval Banzhaf value and the interval egalitarian rule in the next future.

Finally, the partial subtraction operator, introduced in Alparslan Gök et al. (2009a), is an essential tool in interval game theory. But, it is possible to obtain the solutions for cooperative interval games by using the Moore's substraction operator[1]. As a future work it is interesting to find characterizations of the interval Shapley value and the interval Banzhaf value on the whole class of size monotonic games and to calculate the interval Banzhaf value by using the Moore's subtraction operator.

[1] The Moore's subtraction operator (Moore (1979)) is defined by $I \ominus J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$.

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