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Generalized Nucleolus, Kernels, and Bargainig Sets for Cooperative Games with Restricted Cooperation

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Abstract Generalization of the theory of the bargaining set, the kernel, and the nucleolus for cooperative TU-games, where objections and counter–objections are permited only between the members of a family of coalitions $\mathcal{A}$ and can use only the members of a family of coalitions $\mathcal{B} \supset \mathcal{A}$, is considered. Two versions of objections and two versions of counter–objections generalize the definitions for singletons. These definitions provide 4 types of generalized bargaining sets. For each of them, necessary and sufficient conditions on $\mathcal{A}$ and $\mathcal{B}$ for existence these bargaining sets at each game of the considered class are obtained.
Two types of generalized kernels are defined. For one of them, the conditions that ensure its existence generalize the result for $\mathcal{B} = 2^N$ of Naumova (2007). Generalized nucleolus is not single–point and its intersection with nonempty generalized kernel may be the empty set. Conditions on $\mathcal{A}$ which ensure that the intersections of the generalized nucleolus with two types of generalized bargaining sets are nonempty sets, are obtained. The generalized nucleolus always intersects the first type of the generalized kernel only if $\mathcal{A}$ is contained in a partition of the set of players.

Keywords: cooperative games; nucleolus; kernel; bargaining set.

1. Introduction

The theory of the bargaining set, the kernel, and the nucleolus for cooperative TU-games was born in papers (Aumann, Maschler, 1964, Davis, Maschler, 1963 (1), Davis, Maschler, 1967 (2), Maschler, Peleg, 1966, Schmeidler, 1969). First the proofs of existence theorems for the kernel and the bargaining set $\mathcal{M}_1^i$ used fixed point theorems, Schmeidler, 1969 defined the nucleolus which always exists and belongs to these sets. In these papers for each imputation x of TU–cooperative game, an objection of a player i against a player j at x and a counter–objection to this objection were defined. An imputation x^0 belongs to the bargaining set $\mathcal{M}_1^i$ if for each players i, j for each objection of i against j at x^0 there exists a counter–objection. At the same time some objections and counter–objections between coalitions were defined and it was shown that the existence theorem is not fulfilled if objections and counter–objections are permited between all pairs of disjoint coalitions.

This paper considers the case when objections and counter–objections are permited only between the members of a family of coalitions $\mathcal{A}$. Moreover, objections and counter–objections can use only the members of a family of coalitions $\mathcal{B} \supset \mathcal{A}$. Two versions of objections and two versions of counter–objections generalize the definitions for singletons. These definitions provide four generalized bargaining sets. Naumova, 1976, Naumova, 1978, Naumova, 2007 considered two of these bargaining sets for $\mathcal{B} = 2^N$.

We consider a class of n–person TU–games with nonnegative values of characteristic functions. For each of 4 types of generalized bargaining sets, necessary and sufficient conditions on $\mathcal{A}$ and $\mathcal{B}$ for existence these bargaining sets at each game of the considered class are obtained. Sufficiency results are proved as in (Naumova, 2007), necessity results are new.

Two types of generalized kernels are possible. In this paper only one of them is considered. This generalized kernel is contained in the largest bargaining set and its narrowing is contained in two generalized bargaining sets. The conditions that ensure their existence generalize the result for $\mathcal{B} = 2^N$ of Naumova, 2007.

Generalized nucleolus must use in its definition only elements of $\mathcal{B}$ which are suitable for objections or for counter–objections. It is not single–point and its intersection with nonempty generalized kernel may be the empty set. Conditions on $\mathcal{A}$ which ensure that the intersection of the generalized nucleolus with two types of generalized bargaining sets are nonempty sets, are obtained. The generalized nucleolus always intersects the generalized kernel only in trivial case (when "essential" elements of $\mathcal{A}$ are contained in a partition of the set of players).

2. Definitions

For simplicity, this paper considers not all coalition structures but only the case generated by the grand coalition.

Let Γ^0 be the set of cooperative TU–games (N, v) such that $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq 0$ for all $S \subset N$. (Such games are 0-normalizations of games (N, v) with $\sum_{i \in S} v(\{i\}) \leq v(S)$ for all $S \subset N$.) Let V_N^0 be the set of v such that $(N, v) \in \Gamma^0$.

Denote $x(S) = \sum_{i \in S} x_i$.

For $(N, v) \in \Gamma^0$, an *imputation* is a vector $x = \{x_i\}_{i \in N}$ such that $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$.

Consider two versions of generalized objections.

Let $(N, v) \in \Gamma^0$, $K, L \subset N$, x be an imputation for (N, v).

A *strong $\mathcal{B}$–objection of K against L at x* is a pair (C, y_C), such that $C \in \mathcal{B}$, $K \subset C$, $L \cap C = \emptyset$, $y_C = \{y_i\}_{i \in C}$, $y(C) = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$.

A coalition $C \in \mathcal{B}$ is *suitable for strong $\mathcal{B}$–objection of K against L* if $K \subset C$, $L \cap C = \emptyset$.

Let $K \cap L = \emptyset$ and $x(L) > 0$. A *weak $\mathcal{B}$–objection of K against L at x* is a pair (C, y_C), such that $C \in \mathcal{B}$, $K \subset C$, $L \not\subset C$, $y_C = \{y_i\}_{i \in C}$, $y(C) = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$.

A coalition $C \in \mathcal{B}$ is *suitable for weak $\mathcal{B}$– objection of K against L* if $K \subset C$, $L \not\subset C$.

Consider two versions of generalized counter–objections.

A *weak $\mathcal{B}$–counter–objection to strong or weak $\mathcal{B}$–objection (C, y_C) of K against L at x* is a pair (D, z_D) such that $D \in \mathcal{B}$, $L \subset D$, $K \not\subset D$, $z(D) = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

A coalition $D \in \mathcal{B}$ is *suitable for weak $\mathcal{B}$– counter–objection to objection of K against L* if $L \subset D$, $K \not\subset D$.

A *strong $\mathcal{B}$–counter-objection to strong or weak $\mathcal{B}$–objection* (C, y_C) of K against$L$ at x is a pair (D, z_D) such that $D \in \mathcal{B}$, $L \subset D$, $K \cap D = \emptyset$, $z(D) = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$.

A coalition $D \in \mathcal{B}$ is *suitable for strong $\mathcal{B}$–counter–objection to objection of K against L* if $L \subset D$, $K \cap D = \emptyset$.

Now we define 4 types of generalized bargaining sets.

Let $\mathcal{A}$ be a set of subsets of N. An imputation x of (N, v) belongs to the *strong–weak bargaining set* $\mathcal{M}^{sw}_{\mathcal{A},\mathcal{B}}(N, v)$ if for all $K, L \in \mathcal{A}$, for each strong $\mathcal{B}$–objection of K against L at x, there exists a weak $\mathcal{B}$–counter-objection.

In (Naumova, 2007) the strong–weak bargaining set $\mathcal{M}^{sw}_{\mathcal{A},2^N}(N, v)$ is called the bargaining set $\mathcal{M}^i_{\mathcal{A}}(N, v)$.

An imputation x of (N, v) belongs to the *weak–weak bargaining set* $\mathcal{M}^{ww}_{\mathcal{A},\mathcal{B}}(N, v)$ if for all $K, L \in \mathcal{A}$ for each weak $\mathcal{B}$–objection of K against L at x, there exists a weak $\mathcal{B}$–counter-objection.

An imputation x of (N, v) belongs to the *strong–strong bargaining set* $\mathcal{M}^{ss}_{\mathcal{A},\mathcal{B}}(N, v)$ if for all $K, L \in \mathcal{A}$, for each strong $\mathcal{B}$–objection of K against L at x, there exists a strong $\mathcal{B}$–counter-objection.

In (Naumova, 2007) the strong–strong bargaining set $\mathcal{M}^{ss}_{\mathcal{A},2^N}(N, v)$ is called the strong bargaining set $\bar{\mathcal{M}}^i_{\mathcal{A}}(N, v)$.

An imputation x of (N, v) belongs to the *weak–strong bargaining set* $\mathcal{M}^{ws}_{\mathcal{A},\mathcal{B}}(N, v)$ if for all $K, L \in \mathcal{A}$, for each weak $\mathcal{B}$–objection of K against L at x, there exists a strong $\mathcal{B}$–counter-objection.

For each of these 4 bargaining sets, a permited objection is *justified* if it has no permited counter–objection.

Note that

$$\mathcal{M}^{ws}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{ww}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{sw}_{\mathcal{AB}}(N, v)$$

and

$$\mathcal{M}^{ws}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{ss}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{sw}_{\mathcal{AB}}(N, v).$$

If $\mathcal{A}$ is the set of all singletons and $\mathcal{B}$ is the set of all subsets of N, then $\mathcal{M}^{ws}_{\mathcal{AB}}(N, v) = \mathcal{M}^{ww}_{\mathcal{AB}}(N, v) = \mathcal{M}^{ss}_{\mathcal{AB}}(N, v) = \mathcal{M}^{sw}_{\mathcal{AB}}(N, v) = \mathcal{M}^i_1(N, v)$.

For families of coalitions $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$, consider the following generalizations of the kernel.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathcal{B}$–outweighs L at x* if $K \cap L = \emptyset$, $x(L) > v(L)$, and $s^{\mathcal{B}}_{K,L}(x) > s^{\mathcal{B}}_{L,K}(x)$, where

$$s^{\mathcal{B}}_{P,Q}(x) = \max\{v(S) - x(S) : S \in \mathcal{B}, P \subset S, Q \not\subset S\}.$$

The set $\mathcal{K}_{\mathcal{AB}}(N, v)$ is the set of all imputations x of (N, v) such that no $K \in \mathcal{A}$ can $\mathcal{B}$–outweigh any $L \in \mathcal{A}$ at x.

In (Naumova, 2007) the set $\mathcal{K}_{\mathcal{A}2^N}(N, v)$ is denoted by $\mathcal{K}_{\mathcal{A}}(N, v)$.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathcal{B}$–weakly outweighs L at x* if $K \cap L = \emptyset$, $x(L) > 0$, and $s^{\mathcal{B}}_{K,L}(x) > s^{\mathcal{B}}_{L,K}(x)$.

The set $\mathcal{K}^0_{\mathcal{AB}}(N, v)$ is the set of all imputations x of (N, v) such that no $K \in \mathcal{A}$ can $\mathcal{B}$–weakly outweigh any $L \in \mathcal{A}$ at x.

Then $\mathcal{K}^0_{\mathcal{AB}}(N, v) \subset \mathcal{K}_{\mathcal{AB}}(N, v)$ because $v(L) \geq 0$ for all $L \in \mathcal{A}$.

If $\mathcal{A}$ is the set of all singletons and $\mathcal{B}$ is the set of all subsets of N, then $\mathcal{K}_{\mathcal{AB}}$ and $\mathcal{K}^0_{\mathcal{AB}}$ coincide with the kernel.

Another generalization of the kernel is possible.

Let $K, L \subset N$ and x be an imputation of (N, v). *K $\mathcal{B}$–prevails L at x* if $K \cap L = \emptyset$, $x(L) > v(L)$, and $t^{\mathcal{B}}_{K,L}(x) > t^{\mathcal{B}}_{L,K}(x)$, where

$$t^{\mathcal{B}}_{P,Q}(x) = \max\{v(S) - x(S) : S \in \mathcal{B}, P \subset S, Q \cap S = \emptyset\}.$$

The set $\bar{\mathcal{K}}_{\mathcal{AB}}(N,v)$ is the set of all imputations x of (N,v) such that no $K \in \mathcal{A}$ can $\mathcal{B}$–prevail any $L \in \mathcal{A}$ at x.

Now we define a generalization of the nucleolus.

Let $\mathcal{B}$ be a set of subsets of N, for each imputation y of (N,v), let $\theta^{\mathcal{B}}(N,v,y) = \{v(S) - y(S)\}_{S \in \mathcal{B}}$ with decreasing coordinates.

For $(N,v) \in \Gamma^0$, the *$\mathcal{B}$–nucleolus of* (N,v) $Nucl_{\mathcal{B}}(N,v)$ is the set of all imputations x of (N,v) such that $\theta^{\mathcal{B}}(N,v,x) \not>_{lex} \theta^{\mathcal{B}}(N,v,y)$ for all imputations y of (N,v).

If $\mathcal{B} = 2^N$ then $\mathcal{B}$–nucleolus is the nucleolus defined in (Schmeidler, 1969).

3. Existence condition for $\mathcal{K}_{\mathcal{AB}}(N,v)$

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Consider the following property of $\mathcal{A}$.

C0) If a single node is taken out from each component of $G(\mathcal{A})$, then the union of the remaining elements of $\mathcal{A}$ does not contain N.

Theorem 1. *Let $\mathcal{A}$ be a set of subsets of N. If $\mathcal{A}$ satisfies condition C0 then $\mathcal{K}^0_{\mathcal{AB}}(N,v) \neq \emptyset$ and $\mathcal{K}^0_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V^0_N$.*

If $\{i\}, S \in \mathcal{A}$ implies $\{i\} \cup S \in \mathcal{B}$ and $\mathcal{K}_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V^0_N$, then $\mathcal{A}$ satisfies condition C0.

The proof of this theorem coincides with the proof of Theorem 2 in (Naumova, 2007).

Remark 1. If $\mathcal{B} = \mathcal{A}$ then C0 is not necessary. The following example demonstrates this fact.

Example 1. $N = \{1,2,3\}$, $\mathcal{A} = \{\{1\},\{2\},\{1,3\}\}$. Then $\mathcal{K}_{\mathcal{AA}}(N,v) \neq \emptyset$ for all $v \in V^0_N$.

Proof. For (N,v), consider 2 cases.

Case 1. $v(\{1,3\}) \geq v(N)$. Take $x = (0,0,v(N))$. If $x \notin \mathcal{K}_{\mathcal{AA}}(N,v)$, then $\{2\}$ overweights $\{1,3\}$ at x. But $s^{\mathcal{A}}_{\{1,3\},\{2\}}(x) = v(\{1,3\}) - v(N) \geq 0$, $s^{\mathcal{A}}_{\{2\},\{1,3\}}(x) = 0 - x_2 = 0$.

Case 2. $v(\{1,3\}) < v(N)$. Take $y \in R^3$ such that $y_1 = y_2 = (v(N) - v(\{1,3\}))/2$, $y_3 = v(\{1,3\})$. Then

$s^{\mathcal{A}}_{\{1,3\},\{2\}}(y) = v(\{1,3\}) - y_1 - y_3 = -y_1$,

$s^{\mathcal{A}}_{\{2\},\{1,3\}}(y) = 0 - y_2 = -y_2 = -y_1$,

$s^{\mathcal{A}}_{\{2\},\{1\}}(y) = 0 - y_2$,

$s^{\mathcal{A}}_{\{1\},\{2\}}(y) = \max\{-y_1, v(\{1,3\}) - y_1 - y_3\} = -y_1 = -y_2$.

Thus, $y \in \mathcal{K}_{\mathcal{AA}}(N,v)$. □

4. Existence conditions for generalized bargaining sets

Lemma 1. *Let (C, y_C) be a weak or strong $\mathcal{B}$–objection of K against L at x. Let $D \subset N$, D be suitable for weak or strong $\mathcal{B}$–counter–objection to objection of K against L. Then there exists a counter-objection (D, z_D) to this objection if and only if*

$$v(D) - x(D) \geq (y - x)(C \cap D).$$

Corollary 1. *If (C, y_C) is a justified permited $\mathcal{B}$–objection of K against L at x, then $v(C) - x(C) > v(D) - x(D)$ for all D which are suitable for permited $\mathcal{B}$–counter–objection.*

The proof of this Lemma coincides with the proof of Lemma 1 in (Naumova, 2007).

Theorem 2. *Let $\mathcal{A} \subset \mathcal{B}$, $(N, v) \in \Gamma^0$. Then*
$\mathcal{K}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{sw}_{\mathcal{AB}}(N, v)$,
$\mathcal{K}^0_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{ww}_{\mathcal{AB}}(N, v)$,
$\bar{\mathcal{K}}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{ss}_{\mathcal{AB}}(N, v)$.

Proof. Let $x \in \mathcal{K}_{\mathcal{AB}}(N, v)$ and (C, y_C) be a strong $\mathcal{B}$–objection of K against L at x. If $x(L) \leq v(L)$ then by Lemma 1, there exists a weak $\mathcal{B}$–counter–objection (L, z_L) to this objection as $C \cap L = \emptyset$. If $s^{\mathcal{B}}_{K,L}(x) \leq s^{\mathcal{B}}_{L,K}(x)$ then there exists $D \in \mathcal{B}$ such that D is suitable for counter–objection to this objection and $v(D) - x(D) \geq v(C) - x(C)$. By Corollary to Lemma 1, (C, y_C) is not justified.

Similarly, we prove that $\bar{\mathcal{K}}_{\mathcal{AB}}(N, v) \subset \mathcal{M}^{ss}_{\mathcal{AB}}(N, v)$.

Let $x \in \mathcal{K}^0_{\mathcal{AB}}(N, v)$ and (C, y_C) be a weak $\mathcal{B}$–objection of K against L at x. Then $x(L) > 0$, hence $s^{\mathcal{B}}_{K,L}(x) \leq s^{\mathcal{B}}_{L,K}(x)$ and, as in the case $x \in \mathcal{K}_{\mathcal{AB}}(N, v)$, (C, y_C) is not justified. □

Now we describe the results of the author in (Naumova, 2007) that will be used later.

Let $N = \{1, \ldots, n\}$, $X \subset R^n$, $\mathcal{A}$ be a collection of subsets of N, $\{\succ_x\}_{x \in X}$ be a collection of binary relations. Then $x^0 \in X$ is an *equilibrium vector on* $\mathcal{A}$ if $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$.

For $b > 0$, $K \in \mathcal{A}$ denote

$$X(b) = \{x \in R^n : x_i \geq 0, \ \ x(N) = b\},$$

$$F^K(b) = \{x \in X(b) : L \not\succ_x K \quad \text{for all} \quad L \in \mathcal{A}\}.$$

Then x is an equilibrium vector on $\mathcal{A}$ iff $x \in \bigcap_{K \in \mathcal{A}} F^K(b)$.

The following theorem is Theorem 1 in (Naumova, 2007).

Theorem 3. *Let a family of binary relations $\{\succ_x\}_{x \in X(b)}$ on $\mathcal{A}$ satisfy the conditions:*
1) for all $K \in \mathcal{A}$, the set $F^K(b)$ is closed;
2) if $x_i = 0$ for all $i \in K$, then $x \in F^K(b)$;
3) for each $x \in X(b)$, the set of coalitions $\{L \in \mathcal{A} : K \succ_x L \quad \text{for some} \quad K \in \mathcal{A}\}$ does not cover N.

Then there exists an equilibrum vector $x^0 \in X(b)$ on $\mathcal{A}$.

For all 4 types of generalized bargaining sets, sufficient conditions of their existence will be obtained using this theorem.

Definition 1. Let $\mathcal{A}$ be a family of subsets of N and $\mathcal{B} \supset \mathcal{A}$. A directed graph Gr is called *$\mathcal{AB}$–strong–weak–admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map f defined on the set of edges of Gr, that takes each oriented edge (K, L) to a pair $f(K, L) = (Q, r)$ ($Q \in \mathcal{B}$, $r \in R^1$, $r = r(Q)$) and satisfies the following 3 conditions:

C1. If $f(K, L) = (Q, r(Q))$, then $K \subset Q$, $Q \cap L = \emptyset$, $|Q| > 1$.

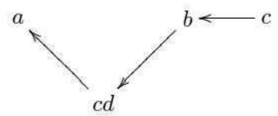
C2. If $f(K, L) = (Q, r)$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \not\subset S$, then $Q \cap S \neq \emptyset$.

C3. If $f(K, L) = (Q, r(Q))$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \not\subset S$, then $r(Q) > r(S)$.

Condition C1 means that Q is suitable for strong $\mathcal{B}$–objection of K against L. Condition C2 means that if S is suitable for weak $\mathcal{B}$–counter–objection to objection of K against L, then $Q \cap S \neq \emptyset$. Condition C3 means that if S is suitable for weak $\mathcal{B}$–counter–objection to objection of K against L. then $r(Q) > r(S)$.

Example 2. Let $\mathcal{A}_1 = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. Let $\mathcal{B}$ consist of all unions of members of $\mathcal{A}$. Let Gr_1 be a digraph, where $\mathcal{A}_1$ is the set of vertices and $\{(K, M), (M, L)\}$ is the set of edges. Then Gr_1 is not $\mathcal{A}_1\mathcal{B}$–strong–weak–admissible. Indeed, if Gr_1 is $\mathcal{A}_1\mathcal{B}$–strong–weak–admissible and f is the corresponding map, then, by C1, $f(M, L) = (M, r)$, $f(K, M) = (Q, t)$, $Q \cap M = \emptyset$, but this contradicts C2.

Example 3. Let $N = \{a, b, c, d\}$, $\mathcal{A} = \{\{a\}, \{b\}, \{c\}, \{c, d\}\}$, $\mathcal{B}$ contain all unions of elements of $\mathcal{A}$. The following digraph is $\mathcal{AB}$–strong–weak–admissible.



Indeed, take
$f(\{c\}, \{b\}) = (\{a, c\}, 2)$,
$f(\{b\}, \{c, d\}) = (\{a, b\}, 1)$,
$f(\{c, d\}, \{a\}) = (\{b, c, d\}, 3)$.

Theorem 4. *Let $\mathcal{A}$ be a set of subsets of N and $\mathcal{B} \supset \mathcal{A}$. Then $\mathcal{M}^{sw}_{\mathcal{AB}}(N, v) \neq \emptyset$ for all $v \in V^0_N$ if and only if for each $\mathcal{AB}$–strong–weak–admissible graph Gr the set of the ends of its edges does not cover N.*

Proof. Let for each $\mathcal{AB}$–strong–weak–admissible graph Gr the set of the ends of its edges does not cover N. For each imputation x define the following binary relation $\succ_x$ on $\mathcal{A}$. $K \succ_x L$ iff K has a justified strong $\mathcal{B}$–objection against L at x. We check that this relation satisfies all conditions of Theorem 3. Condition (1) was checked in (Naumova, 2007), condition (2) follows from $v \in V^0_N$.

Let us check condition (3). For x, define $\mathcal{AB}$–strong–weak–admissible graph as follows. (K, L) is the edge iff $K \succ_x L$ and $f(K, L) = (Q, v(Q) - x(Q))$ for some justified strong $\mathcal{B}$–objection of K against L. Then the map f satisfies conditions C1, C2, C3 in the definition of strong–weak–admissible graph. Indeed, C1 follows from the definition of strong objection, C3 follows from Corollary to Lemma 1, C2 follows from Lemma 1. Then condition (3) follows from our supposition.

Let there exist a $\mathcal{AB}$–strong–weak–admissible graph Gr such that the set of the ends of its edges covers N. We construct $v \in V_N^0$ such that $\mathcal{M}^{sw}_{\mathcal{AB}}(N,v) = \emptyset$ as follows. $v(N) = 1$, $v(T) = 0$ if there is no edge (K,L) such that $f(K,L) = (T, r(T))$. Otherwise, we define first $v(Q)$ for Q with minimal $r(Q)$. For all such Q take $v(Q) > 1$.

Suppose that $v(Q)$ is defined for all Q with $r(Q) < \bar{r}$. Let

$$\alpha(\bar{r}) = \max\{v(Q) : r(Q) < \bar{r}\}.$$

If $r(T) = \bar{r}$, then take $v(T) > n\alpha + n + 1$. By this way, all $v(T)$ will be defined inductively.

Suppose that $x \in \mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$. We prove that for each edge (K,L) of Gr, $x(L) = 0$. Let $f(K,L) = (Q, r(Q))$. Take $y_i = (v(Q) - x(Q))/|Q|$ for each $i \in Q$, then due to C1, (Q, y_Q) is a strong objection of K against L at x. Let S be suitable for weak counter–objection. If $v(S) = 0$ then $x(S) = 0$ and $x(L) = 0$. If $v(S) > 0$ then $S \cap Q \neq \emptyset$, $r(Q) > r(S)$,

$$(y - x)(S \cap Q) \geq (v(Q) - x(Q))/|Q| - x(S) > (v(Q) - 1)/n - 1 > \alpha(r(Q)) > v(S).$$

As $(y - x)(S \cap Q) > v(S)$, by Lemma 1, there is no counter–objection (S, z_S) to objection (Q, y_Q). Thus only the case $x(L) = 0$ is possible. Since the ends of the edges of Gr cover N, we get $x(N) = 0$ and this contradicts $x(N) = 1$. □

Remark 2. Note that the second part of the proof used v with $v(S) > v(N)$. In (Naumova, 2007)for $\mathcal{B} = 2^N$ and $|N| \leq 5$, this part of the theorem was proved using $v(S) \in \{0, 1\}$, but that proof is not suitable for arbitrary $\mathcal{B}$.

Similarly, for each of 3 remaining types of generalized bargaining sets, we define its type of $\mathcal{AB}$– admissible graphs and obtain the corresponding versions of existence theorems.

Definition 2. Let $\mathcal{A}$ be a family of subsets of N and $\mathcal{B} \supset \mathcal{A}$. A directed graph Gr is called *$\mathcal{AB}$–strong–strong–admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map g defined on the set of edges of Gr, that takes each oriented edge (K,L) to a pair $g(K,L) = (Q,r)$ ($Q \in \mathcal{B}$, $r \in R^1$, $r = r(Q)$) and satisfies the following 3 conditions:

C1. If $g(K,L) = (Q, r(Q))$, then $K \subset Q$, $Q \cap L = \emptyset$, $|Q| > 1$.

C2'. If $g(K,L) = (Q, r(Q))$, $g(R,P) = (S, r(S))$, $L \subset S$, $K \cap S = \emptyset$, then $Q \cap S \neq \emptyset$.

C3'. If $g(K,L) = (Q, r(Q))$, $g(R,P) = (S, r(S))$, $L \subset S$, $K \cap S = \emptyset$, then $r(Q) > r(S)$.

Theorem 5. *Let $\mathcal{A}$ be a set of subsets of N and $\mathcal{B} \supset \mathcal{A}$. Then $\mathcal{M}^{ss}_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V_N^0$ iff for each $\mathcal{AB}$–strong–strong–admissible graph Gr the set of the ends of its edges does not cover N.*

Definition 3. Let $\mathcal{A}$ be a family of subsets of N and $\mathcal{B} \supset \mathcal{A}$. A directed graph Gr is called *$\mathcal{AB}$–weak-weak–admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map f defined on the set of edges of Gr, that takes each oriented edge (K, L) to a pair $f(K, L) = (Q, r)$ $(Q \in \mathcal{B},\ r \in R^1,\ r = r(O))$ and satisfies the following 3 conditions:

C1'. If $f(K, L) = (Q, r(Q))$, then $K \subset Q$, $L \not\subset Q$, $|Q| > 1$.

C2. If $f(K, L) = (Q, r(Q))$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \not\subset S$, then $Q \cap S \neq \emptyset$.

C3. If $f(K, L) = (Q, r(Q))$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \not\subset S$, then $r(Q) > r(S)$.

Theorem 6. *Let $\mathcal{A}$ be a set of subsets of N and $\mathcal{B} \supset \mathcal{A}$. Then $\mathcal{M}^{ww}_{\mathcal{AB}}(N, v) \neq \emptyset$ for all $v \in V^0_N$ iff for each $\mathcal{AB}$–weak–weak–admissible graph Gr, the set of the ends of its edges does not cover N.*

Example 4. Let $\mathcal{A} = \{K, L, M\}$, where $L \cup M = N$, $L \cap M = \emptyset$, $K \subset L$, $K \neq L$ and $\mathcal{B}$ consist of all unions of members of $\mathcal{A}$. Then there exists $(N, v) \in \Gamma^0$ such that $\mathcal{M}^{ww}_{\mathcal{AB}}(N, v) = \emptyset$ and $\mathcal{M}^{sw}_{\mathcal{AB}}(N, v) \neq \emptyset$ for all $v \in V^0_N$. Indeed the graph with edges (K, M) and (M, L) is $\mathcal{AB}$–weak–weak–admissible since we can take $f(K, M) = (L, 1)$ and $f(M, L) = (K \cup M, 2)$. Moreover, in view of Example 2, there is no $\mathcal{AB}$–strong–weak–admissible graph with the same set of vertices such that the ends of its edges cover N.

Definition 4. Let $\mathcal{A}$ be a family of subsets of N and $\mathcal{B} \supset \mathcal{A}$. A directed graph Gr is called *$\mathcal{AB}$–weak-strong–admissible* if $\mathcal{A}$ is the set of its vertices and there exists a map f defined on the set of edges of Gr, that takes each oriented edge (K, L) to a pair $f(K, L) = (Q, r)$ $(Q \in \mathcal{B},\ r \in R^1,\ r = r(O))$ and satisfies the following 3 conditions:

C1'. If $f(K, L) = (Q, r(Q))$, then $K \subset Q$, $L \not\subset Q$, $|Q| > 1$.

C2'. If $f(K, L) = (Q, r(Q))$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \cap S = \emptyset$, then $Q \cap S \neq \emptyset$.

C3'. If $f(K, L) = (Q, r(Q))$, $f(R, P) = (S, r(S))$, $L \subset S$, $K \cap S = \emptyset$, then $r(Q) > r(S)$.

Theorem 7. *Let $\mathcal{A}$ be a set of subsets of N and $\mathcal{B} \supset \mathcal{A}$. Then $\mathcal{M}^{ws}_{\mathcal{AB}}(N, v) \neq \emptyset$ for all $v \in V^0_N$ iff for each $\mathcal{AB}$–weak–strong–admissible graph Gr, the set of the ends of its edges does not cover N.*

5. Conditions for intersection of $\mathcal{B}$–nucleolus with generalized bargaining sets

Definition 5. $S \in \mathcal{A}$ is *unessential for* $\mathcal{A}$ if $S \cap T \neq \emptyset$ for all $T \in \mathcal{A}$ and $S \supset T_1 \cup T_2$ for all $T_1, T_2 \in \mathcal{A}$ with $T_1 \cap T_2 = \emptyset$.

If S is unessential for $\mathcal{A}$ then there are no objections neither of S against any $L \in \mathcal{A}$ nor against S and unions with S are not suitable for objections and counter–objections. Hence we can take off all unessential elements for $\mathcal{A}$ from $\mathcal{A}$.

Denote $\mathcal{A}^0 = \{S \in \mathcal{A} : S \quad \text{is not unessential for} \quad \mathcal{A}\}$.

Theorem 8. *Let $\mathcal{B}^0$ consist of the unions of elements of $\mathcal{A}^0$. Then $Nucl_{\mathcal{B}^0}(N, v) \cap \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N, v) \neq \emptyset$ for all $v \in V^0_N$ iff $\mathcal{A}^0$ is contained in a partition of N.*

Proof. Let $\mathcal{A}^0$ be contained in a partition of N, then the proof that $Nucl_{\mathcal{B}^0}(N,v) \subset \mathcal{M}^{ww}_{\mathcal{AB}}(N,v)$ is the same as for the case of singletons.

Let $Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v) \neq \emptyset$ for all $v \in V^0_N$.

Step 1. We prove that if there exist $S,P,Q \in \mathcal{A}$ such that $P \neq Q$, $P \cap Q \neq \emptyset$, $P \cap S = \emptyset$, $Q \cap S = \emptyset$ then there exists $v \in V^0_N$ with $Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v) = \emptyset$.

Let $Q \not\subset P$. Consider the following (N,v). Let $\epsilon > 0$.

$$v(T) = \begin{cases} 1 & \text{for} \quad T = N, \\ 1+\epsilon & \text{for} \quad T = P \cup S, \\ 0 & \text{otherwise.} \end{cases}$$

Let $y \in Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v)$. Note that $Q \not\subset P \cup S$ and $y(P\cup S) < v(P \cup S)$, therefore by Lemma 1, if $y(Q) > 0$ then S has a justified objection $(P \cup S, z_{P\cup S})$ against Q. Hence $y(Q) = 0$. As $y \in Nucl_{\mathcal{B}^0}(N,v)$, we have $y(P \cup S) = 1$. There exist $i_0 \in P \cap Q$, $j_0 \in P \cup S$ with $y_{j_0} \geq 1/n$. Let $0 < \delta < 1/(2n)$. Take y^δ such that

$$y^\delta_i = \begin{cases} y_{i_0} + \delta & \text{for} \quad i = i_0, \\ y_{j_0} - \delta & \text{for} \quad i = j_0, \\ y_i & \text{otherwise.} \end{cases}$$

Then $\theta^{\mathcal{B}^0}(N,v,y) >_{lex} \theta^{\mathcal{B}^0}(N,v,y^\delta)$. Indeed, $Q \in \mathcal{B}^0$, $v(Q) - y(Q) = 0 > v(Q) - y^\delta(Q) > -1/(2n)$ and if $v(T) - y(T) < v(T) - y^\delta(T)$ then $j_0 \in T$, $v(T) = 0$ and $v(T) - y^\delta(T) < -1/(2n)$, hence $v(Q) - y^\delta(Q) > v(T) - y^\delta(T)$. Thus $y \notin Nucl_{\mathcal{B}^0}(N,v)$.

Step 2. Let $S \cap T \neq \emptyset$ for all $S, T \in \mathcal{A}$. Then no objections are possible and $\mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v)$ is the set of all imputations. Each $S \in \mathcal{A}$ is unessential, hence $\mathcal{B}^0 = \emptyset$, $Nucl_{\mathcal{B}^0}(N,v)$ is the set of all imputations., and $Nucl_{\mathcal{B}^0}(N,v) = \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v)$.

Let $P \cap T = \emptyset$ for some $P, T \in \mathcal{A}$. If $\{Q \in \mathcal{A} : Q \cap P = \emptyset\} \cup \{P\} = \mathcal{A}^0$ then it was proved at Step 1 that $\mathcal{A}^0$ is contained in a partition of N.

Else there exist $T_1, T_2, S \in \mathcal{A}^0$ such that $T_1 \cap T_2 = \emptyset$, $S \cap T_i \neq \emptyset$ for $i = 1,2$, and $T_2 \not\subset S$. We prove that this case is impossible. Consider the following $v \in V^0_N$. Let $\epsilon > 0$.

$$v(Q) = \begin{cases} 1 & \text{for} \quad Q = N, \\ 1+\epsilon & \text{for} \quad Q = S \cup T_1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $x \in Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{M}^{ww}_{\mathcal{A}^0\mathcal{B}^0}(N,v)$. As $T_1 \cup S \not\supset T_2$ and $x(T_1 \cup S) < v(T_1 \cup S)$, there exists an objection of T_1 against T_2. As this objection is not justified, $x(T_2) = 0$. Now the proof is the same as at Step 1. As $x \in Nucl_{\mathcal{B}^0}(N,v)$, $x(T_1 \cup S) = 1$, there exist $i_0 \in S \cap T_2$, $j_0 \in T_1 \cup S$ with $x_{j_0} \geq 1/n$. Take $0 < \delta < 1/(2n)$ and x^δ as at Step 1. Then $\theta^{\mathcal{B}^0}(N,v,x) >_{lex} \theta^{\mathcal{B}^0}(N,v,x^\delta)$ and $x \notin Nucl_{\mathcal{B}^0}(N,v)$. □

Corollary 2. *Let $\mathcal{B}$ consist of the unions of elements of $\mathcal{A}$. Then $Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{K}^0_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V^0_N$ iff $\mathcal{A}^0$ is contained in a partition of N.*

Remark 3. Note that it does not follow from this theorem that $Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{K}_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V^0_N$ iff $\mathcal{A}^0$ is contained in a partition of N because only $\mathcal{K}^0_{\mathcal{AB}}(N,v)$ is contained in $\mathcal{M}^{ww}_{\mathcal{AB}}(N,v)$. However, this can be proved by exactly the same constructions as in Theorem 8

Theorem 9. *Let $\mathcal{B}^0$ consist of the unions of elements of $\mathcal{A}^0$. Then $Nucl_{\mathcal{B}^0}(N,v) \cap \mathcal{K}_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V_N^0$ iff $\mathcal{A}^0$ is contained in a partition of N.*

Now we describe conditions on $\mathcal{A}$ that ensure intersection of $\mathcal{B}$–nucleolus with the bargaining set $\mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$.

For $i \in N$, denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

Definition 6. A collection of coalitions $\mathcal{A}$ is *weakly mixed at* N if $\mathcal{A} = \cup_{i=1}^k \mathcal{B}^i$, where
1) each $\mathcal{B}^i$ is contained in a partition of N;
2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
3) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Weakly mixed collections of coalitions were introduced in (Naumova, 2012) for another problem associated with $\mathcal{A}$–nucleolus.

Remark 4. If $k \leq 2$ then condition 3 follows from conditions 1 and 2.

Example 5. Let $N = \{1,2,\ldots,5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1,2,3\},\{4,5\}\}$,
$\mathcal{B}^2 = \{\{1,4\},\{2,5\}\}$,
then $\mathcal{C}$ is weakly mixed at N.

Example 6. Let $N = \{1,2,\ldots,6\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1,2\},\{3,4\}\}$,
$\mathcal{B}^2 = \{\{1,3\},\{2,4\}\}$,
$\mathcal{B}^3 = \{\{1,4,5\},\{2,3,6\}\}$,
then $\mathcal{C}$ satisfies conditions C0, (1), and (2), but does not satisfy (3) (for $i = 1$ and $Q = \{1,2\}$), hence $\mathcal{C}$ is not weaky mixed at N.

Theorem 10. *Let $\mathcal{A}$ be a weakly mixed at N collection of coalitions, $\mathcal{B}$ consist of all unions of elements of $\mathcal{A}$. Then $Nucl_{\mathcal{B}}(N,v) \subset \mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$ for all $v \in V_N^0$.*

Proof. Let $x \in Nucl_{\mathcal{B}}(N,v)$. Suppose that for some v, $x \notin \mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$. Then there exist $S, Q \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and S has a justified strong objection (T, z_T) at x against Q. Then $T \cap Q = \emptyset$ and $x(Q) > v(Q)$. Take $i_0 \in Q$ such that $x_{i_0} > 0$. Since $\mathcal{A}$ is weakly mixed, there exists $j_0 \in N$ such that $\mathcal{A}_{j_0} \supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Let $\delta > 0$, $y \in R^n$, where

$$y_i = \begin{cases} x_i - \delta & \text{for} \quad i = i_0, \\ x_i + \delta & \text{for} \quad i = j_0, \\ x_i & \text{otherwise.} \end{cases}$$

Let $P \in \mathcal{B}$ and $y(P) < x(P)$. Then $i_0 \in P$ and $j_0 \notin P$. Since $j_0 \in S$, $P \not\supset S$.

There exists $P^0 \in \mathcal{A}$ such that $P^0 \subset P$, $i_0 \in P^0$ and $j_0 \notin P^0$. By the definition of j_0, only the case $P^0 = Q$ is possible, hence $P \supset Q$ and P can be used for weak counter–objection to any objection of S against Q.

Since (T, z_T) is a justified strong objection of S against Q at x, we have $i_0 \notin T$, $j_0 \in S \subset T$, and $v(T) - x(T) > v(P) - x(P)$.

If δ is small enought, $v(T) - y(T) > v(P) - y(P)$ for all $P \in \mathcal{B}$ such that $v(P) - y(P) > v(P) - x(P)$. Since $v(T) - y(T) < v(T) - x(T)$, hence $\theta^{\mathcal{B}}(N,v,x) >_{lex} \theta^{\mathcal{B}}(N,v,y)$ and $x \notin Nucl_{\mathcal{B}}(N,v)$. □

Theorem 11. *Let $\mathcal{A}$ do not contain singletons, do not contain S such that $S \cap T \neq \emptyset$ for all $T \in \mathcal{A}$ and $\mathcal{B}$ consist of all unions of elements of $\mathcal{A}$. If $Nucl_{\mathcal{B}}(N,v) \cap \mathcal{M}^{sw}_{\mathcal{AB}}(N,v) \neq \emptyset$ for all $v \in V^0_N$ then $\mathcal{A}$ is a weakly mixed at N collection of coalitions.*

Proof. Step 1. We prove that if there exist $S, P, Q \in \mathcal{A}$ such that $P \neq Q$, $P \cap Q \neq \emptyset$, $P \cap S = \emptyset$, $Q \cap S = \emptyset$ then there exists $v \in V^0_N$ with $Nucl_{\mathcal{B}}(N,v) \cap \mathcal{M}^{sw}_{\mathcal{AB}}(N,v) = \emptyset$. We can suppose that $P \not\supset Q$.

Take the following $v \in V^0_N$.

$$v(T) = \begin{cases} 1 & \text{for} \quad T = N, S, P, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in Nucl_{\mathcal{B}}(N,v) \cap \mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$. Let $x(Q) > 0$, then $x(S) < 1$ and there exists a strong objection of S against Q at x. Since $P \not\supset Q$, such strong objection is a justified strong objection. Thus $x(Q) = 0$.

As $x \in Nucl_{\mathcal{B}}(N,v)$, $x(P) = x(S) = 1/2$. Hence, as $x(Q) = 0$, there exists $i_0 \in P \setminus Q$ such that $x_{i_0} > 0$. Fix $j_0 \in P \cap Q$. Let $0 < \delta < x_(i_0)/2$. Take the following $y \in R^n$.

$$y_i = \begin{cases} x_{i_0} - \delta & \text{for} \quad i = i_0, \\ x_i + \delta & \text{for} \quad i = j_0, \\ x_i & \text{otherwise.} \end{cases}$$

Consider 2 cases.

Case 1. $i_0 \notin T$ for all $T \in \mathcal{A} \setminus \{P\}$. Then $y(T) \geq x(T)$ for all $T \in \mathcal{B}$ and $y(Q) > x(Q)$, hence $x \notin Nucl_{\mathcal{B}}(N,v)$.

Case 2. $i_0 \in T$ for some $T \in \mathcal{A} \setminus \{P\}$. Then $T \neq S$, $v(T) - x(T) \leq -xi_0$, $v(T) - y(T) \leq -x_{i_0} + \delta$, $v(Q) - x(Q) = 0$, $v(Q) - y(Q) = -\delta$. The condition $\delta < x_(i_0)/2$ implies $v(Q) - y(Q) > v(T) - y(T)$ for such T. Therefore, $\theta^{\mathcal{B}}(N,v,x) >_{lex} \theta^{\mathcal{B}}(N,v,y)$ and $x \notin Nucl_{\mathcal{B}}(N,v)$.

Step 2. Now we check the fulfilment of three conditions in the definition of weakly mixed at N collection of coalitions.

Let $\mathcal{B}^i$ be components of the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$. Then the fulfilment of condition (2) follows from the definition of $\mathcal{B}^i$ and the fulfilment of condition (1) was proved at Step 1.

Let us check the condition (3). Suppose that $\mathcal{A}$ does not satisfy this condition, i.e., there exist $i_0 \in \mathcal{A}$, $Q \in \mathcal{A}_{i_0}$, $S \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and for each $j \in N$, $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \setminus \{Q\} \cup \{S\}$. Take the following $v \in V^0_N$.

$$v(T) = \begin{cases} 1 & \text{for} \quad T = N, S, \\ 2 & \text{for} \quad T \in \mathcal{A}_{i_0} \setminus \{Q\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in Nucl_{\mathcal{B}}(N,v) \cap \mathcal{M}^{sw}_{\mathcal{AB}}(N,v)$. Suppose that $x(Q) > 0$, then $x(S) < 1$ and there exists a strong objection (S, y_S) of S against Q at x. Let (D, z_D) be a weak counter–objection to this objection. Then $D \supset Q$, $v(D) \geq x(D) \geq x(Q) > 0$ hence $D \neq Q$. There exists $D' \in \mathcal{A}$ such that $D \cap D' = \emptyset$. As $D \cap Q \neq \emptyset$, it follows by the proved at Step 1, that $D' \cap Q \neq \emptyset$, thus $D \not\supset Q$ and D is not suitable for counter–objection. This contradiction proves that $x(Q) = 0$.

There exists $j_0 \in N$ such that $x_{j_0} \geq 1/n$. Then $j_0 \notin Q$ and $j_0 \neq i_0$. Let $0 < \delta < 1/(2n)$. Take the following $x^\delta \in R^n$.

$$x_i^\delta = \begin{cases} x_{i_0} - \delta & \text{for} \quad i = j_0, \\ x_i + \delta & \text{for} \quad i = i_0, \\ x_i & \text{otherwise.} \end{cases}$$

Consider 2 cases.

Case 1. $j_0 \notin S$. Let $x^\delta(T) < x(T)$, then $j_0 \in T$, $i_0 \notin T$, hence $T \neq S$ and $v(T) = 0$ in this case. Then $v(T) - x(T) \leq -x_{j_0}$, $v(T) - x^\delta(T) \leq -x_{j_0} + \delta$, $v(Q) - x(Q) = 0$, $v(Q) - x^\delta(Q) = -\delta$. Since $\delta < 1/(2n)$, we have $v(Q) - x^\delta(Q) > v(T) - x^\delta(T)$ and $x \notin Nucl_{\mathcal{B}}(N, v)$.

Case 2. $j_0 \in S$. Then due to our supposition, there exists $P \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, hence $v(P) = 2$ and

$$v(P) - x(P) > v(P) - x^\delta(P) \geq 1.$$

If $v(T) - x(T) < v(T) - x^\delta(T)$ then $j_0 \in T$, $i_0 \notin T$ and either $T = S$ or $v(T) = 0$. For $T = S$, $v(T) - x^\delta(T) \leq 1 - x_{j_0} + \delta < 1$. For $v(T) = 0$, $v(T) - x^\delta(T) < 0$. Thus $v(P) - x^\delta(P) > v(T) - x^\delta(T)$ and $x \notin Nucl_{\mathcal{B}}(N, v)$. □

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