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Cooperation in Transportation Game*

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Abstract We consider a game-theoretic model of competition and cooperation of transport companies on a graph. First, a non-cooperative n-person game which is related to the queueing system $M/M/n$ is considered. There are n competing transport companies which serve the stream of passengers with exponential distribution of time with parameters $\mu^{(i)}$, $i = 1, 2, ..., n$ respectively on the graph of routes. The stream of passengers from a stop k to another stop t forms the Poisson process with intensity λ_{kt} . The transport companies announce the prices for the service on each route and the passengers choose the service with minimal costs. The incoming stream λ_{kt} is divided into n Poisson flows with intensities $\lambda_{kt}^{(i)}$, $i = 1, 2, ..., n$. The problem of pricing for each player in the competition and cooperation is solved.

Keywords: Duopoly, equilibrium prices, queueing system.

1. Introduction

We consider a game-theoretic model of competition and cooperation of transport companies on a graph. First, a non-cooperative n-person game which is related to the queueing system $M/M/n$ is considered. There are n competing transport companies which serve the stream of passengers with exponential distribution of time with parameters $\mu^{(i)}$, $i = 1, 2, ..., n$ respectively on the graph of linear route. Each transport company carry passengers from stop 1 to another stop t, $t = 2, ..., m$. Thus the linear route consists of $m-1$ pathes, i. e. $r_{12} = v_1v_2$, $r_{13} = v_1v_3$,..., $r_{1m} = v_1v_m$. The stream of passengers from a stop 1 to another stop t forms the Poisson process with intensity λ_{1t} . The transport companies announce the prices for the service on each path and the passengers choose the service with minimal costs. This approach was used in the Hotelling's duopoly (Hotelling, 1929), (D'Aspremont et al., 1979), (Mazalova, 2013) to determine the equilibrium price in the market. But the costs of each customer are calculated as the price for the service and expected time in queue. The incoming stream is divided into n Poisson flows with intensities $\lambda_{1t}^{(i)}$, $i = 1, 2, ..., n$, where $\sum_{i=1}^{n} \lambda_{1t}^{(i)} = \lambda_{1t}$.

Paragraph 2 is devoted to the competition of n players on the graph of linear route (see Melnik, 2014). The problem of pricing for each player is solved. Such articles as (Altman, Shimkin, 1980), (Levhari, Luski, 1978), (Hassin, Haviv, 2003), and (Chen et al., 2005), (Luski, 1976) are devoted to the similar game-theoretic problems of queuing processes.

In paragraph 3 cooperation is considered. In this case the additional player is introduced. This player serves the stream of passengers with exponential distribution of time with parameter μ_0 and has fixed prices for the service on each path. When

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coalition S is formed, s players from this coalition play game as single player with the intensity of service $\mu^{(s)} = \sum_{i \in S} \mu^{(i)}$ and other $n - s$ players form the equilibrium in this game. So, the characteristic function is defined as a payoff of the coalition in Nash equilibrium in competition among $n - s + 1$ players. Shapley value is used as a solution of this game.

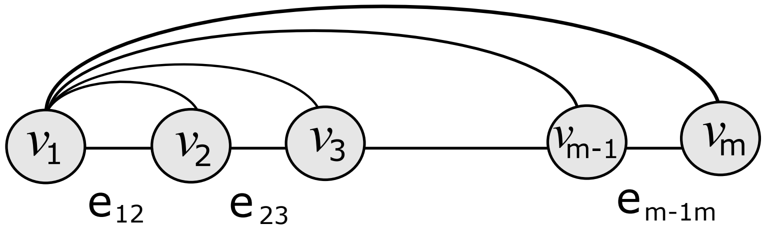

Fig. 1: Linear route

2. Game-theoretic model of pricing

Consider a noncooperative non-zero-sum n-player game associated with the operation of the queueing system $M/M/n$ on a graph of linear route (Fig. 1). n transport companies serve passengers on a graph $G =< V, E >$, V designates a vertex set and E indicates an edge set.

Suppose that all vertices are numbered: $V = \{v_1, ..., v_m\}$. Each player i has one route, from v_1 to all other vertices v_j, $j = 2, ..., m$. Player i serves the input flow of passengers with the exponential distribution of the service time described by the parameter $\mu^{(i)}$, $i = 1, 2, ..., n$. Player i assigns prices for its service $c_{1j}^{(i)}$ on all pathes r_{1j}, $j = 2, ..., m$.

Assume that passengers minimize their costs (ticket price plus expected service time) and choose the lowest-cost service. Consequently, the input flow λ_{1j}, $j = 2, ..., m$ is partitioned into n Poisson processes with intensities $\lambda_{1j}^{(i)}$, $i = 1, ..., n$, where $\sum_{i=1}^{n} \lambda_{1j}^{(i)} = \lambda_{1j}$, $j = 2, ..., m$.

The passengers costs incurred by choosing service i on path r_{1j} equal

$$c_{1j}^{(i)} + \sum_{k=1}^{j-1} \frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}}, \quad i = 1, ..., n, \quad j = 2, ..., m,$$

where

$$\frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}}, \quad i = 1, ..., n, \quad k = 1, ..., m - 1,$$

are the delays on the edge e_{kk+1}. Assume, that if the price for the service $c_{1j}^{(i)}$ of transport company i is too high, the passenger flow $\lambda_{1j}^{(i)} = 0$. The balance equations are

$$c_{1j}^{(1)} + \sum_{k=1}^{j-1} \frac{1}{\mu^{(1)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(1)}} - c_{1j}^{(i)} - \sum_{k=1}^{j-1} \frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}} = 0,$$

$$\text{for} \quad i = 2, ..., n, \quad j = 2, ..., m,$$

$$\sum_{i=1}^{n} \lambda_{1j}^{(i)} = \lambda_{1j}, \quad j = 2, ..., m, \tag{1}$$

and the payoff function

$$H_i(c_{12}^{(1)}, ..., c_{12}^{(n)}, ..., c_{1n}^{(1)}, ..., c_{1n}^{(n)}) = \sum_{j=2}^{m} c_{1j}^{(i)} \lambda_{1j}^{(i)}.$$

To find the best reply of the first transport company, we use the Lagrange method (Taha, 2011). So, we fix $c_{1j}^{(i)}$, $j = 2, ..., m$, $i = 2, ..., n$ and find the maximum of the payoff function H_1 under the constraints (1).

$$L_1 = \sum_{j=2}^{m} c_{1j}^{(1)} \lambda_{1j}^{(1)} + \sum_{j=2}^{m} \gamma_j \left(\sum_{i=1}^{n} \lambda_{1j}^{(i)} - \lambda_{1j} \right) +$$

$$+ \sum_{i=2}^{n} \sum_{j=2}^{m} k_j^i \left(c_{1j}^{(1)} + \sum_{k=1}^{j-1} \frac{1}{\mu^{(1)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(1)}} - c_{1j}^{(i)} - \sum_{k=1}^{j-1} \frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}} \right).$$

So,

$$\frac{\partial L_1}{\partial c_{1s}^{(1)}} = \lambda_{1s}^{(1)} + \sum_{i=2}^{n} k_s^i = 0, \quad s = 2, ..., m,$$

$$\frac{\partial L_1}{\partial \lambda_{1s}^{(1)}} = c_{1s}^{(1)} + \sum_{l=0}^{s-2} \sum_{i=2}^{n} \sum_{j=2+l}^{m} \frac{k_j^i}{(\mu^{(1)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(1)})^2} + \gamma_s, \quad s = 2, ..., m,$$

$$\frac{\partial L_1}{\partial \lambda_{1s}^{(k)}} = - \sum_{l=0}^{s-2} \sum_{i=2}^{m} \sum_{j=2+l}^{n} \frac{k_j^i}{(\mu^{(k)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(k)})^2} + \gamma_s, \quad s = 2, ..., m, \quad k = 2, ..., n.$$

Similarly, we can construct the Lagrange function for other players. So the equilibrium is found from the following system

$$c_{1s}^{(i)} = \sum_{l=0}^{s-2} \sum_{j=2+l}^{m} \lambda_{1j}^{(i)} \left(\frac{1}{(\mu^{(i)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(i)})^2)} + \frac{1}{\sum\limits_{k=2, k \neq i}^{n} (\mu^{(k)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(k)})^2} \right),$$

$$\text{for} \quad i = 2, ..., n, \quad s = 2, ..., m,$$

$$c_{1j}^{(1)} + \sum_{k=1}^{j-1} \frac{1}{\mu^{(1)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(1)}} - -c_{1j}^{(i)} - \sum_{k=1}^{j-1} \frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}} = 0 \qquad (2)$$

$$\text{for} \quad i = 2, ..., n, \quad j = 2, ..., m,$$

$$\lambda_{1s} = \sum_{i=1}^{m} \lambda_{1s}^{(i)}, \quad s = 2, 3, ..., n.$$

3. Coalition formation

Assume that there is another player in our model – public transport, who has the fixed service price $c_{1j}^{(0)}$, $j = 2, ..., m$ and fixed intensity of service $\mu^{(0)}$. This player serves the passenger flow on the same linear route. When the passenger comes to the bus stop, he chooses the service of n transport companies or public transport according to the minimal sum of the service price and the expected waiting time.

The equilibrium in competition of n transport companies can be found from the similar to the (2) system

$$c_{1s}^{(i)} = \sum_{l=0}^{s-2} \sum_{j=2+l}^{m} \lambda_{1j}^{(i)} \left(\frac{1}{(\mu^{(i)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(i)})^2)} + \frac{1}{\sum\limits_{k=2, k\neq i}^{n} (\mu^{(k)} - \sum\limits_{h=l+2}^{m} \lambda_{1h}^{(k)})^2} \right),$$

$$\text{for} \quad i = 1, ..., n, \quad s = 2, ..., m,$$

$$c_{1j}^{(0)} + \sum_{k=1}^{j-1} \frac{1}{\mu^{(0)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(0)}} - -c_{1j}^{(i)} - \sum_{k=1}^{j-1} \frac{1}{\mu^{(i)} - \sum\limits_{s=k+1}^{m} \lambda_{1s}^{(i)}} = 0 \qquad (3)$$

$$\text{for} \quad i = 1, ..., n, \quad j = 2, ..., m,$$

$$\lambda_{1s} = \sum_{i=0}^{m} \lambda_{1s}^{(i)}, \quad s = 2, 3, ..., n.$$

Suppose that s players want to cooperate. Then they serve the passenger flow with the parameter $\mu^{(s)} = \sum\limits_{i\in S} \mu^{(i)}$. The coalition S announce the price for the service $c_{1j}^{(s)}$, $j = 2, ..., m$ on all pathes and the passengers, like before, choose the service.

To determine the characteristic function of a cooperative game, it is necessary to determine the values $v(S)$ of this function for each coalition S. It can be done in two ways. First, traditional, when the remaining players are united in a coalition and play against the S. In this case, the own payoff of coalition $N \setminus S$ is not important for this coalition, its aim is to minimize the payoff of coalition S. Then, coalition $N \setminus S$, as its strategy may use $c_{N\setminus S} = 0$, thus its flow $\lambda_{N\setminus S}$ will increase, but the coalition will receive a payoff equal to zero. The payoff of coalition S in this case is reduced. We use a different approach, in which the characteristic function is constructed as follows. Assume that s transport companies decide to form the coalition S. The coalition S plays as a one player, and all other $n - s$ transport companies are in the equilibrium with it, i. e. equilibrium prices are used as a strategies. This prices are the Nash equilibrium in transportation game of $n-s+1$ players and can be found from the system (3), when the number of players is $n-s+1$. Then the value of the characteristic function is a payoff of the player or coalition in the equilibrium situation.

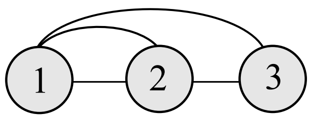

Fig. 2: Example

4. Numeric examples

Consider the following transportation game. Three transport companies and public transport are competing on the linear route with the three stops (see Fig. 2). The passengers come at first stop, choose the service and travel to the stop they need, i. e. the stop 2 or 3. The balance equation are

$$c_{12}^{(0)} + \frac{1}{\mu^{(0)} - \lambda_{12}^{(0)} - \lambda_{13}^{(0)}} - c_{12}^{(i)} - \frac{1}{\mu^{(i)} - \lambda_{12}^{(i)} - \lambda_{13}^{(i)}} = 0, \quad i = 1,\ 2,\ 3,$$

$$c_{13}^{(0)} + \frac{1}{\mu^{(0)} - \lambda_{12}^{(0)} - \lambda_{13}^{(0)}} + \frac{1}{\mu^{(0)} - \lambda_{13}^{(0)}} -$$

$$-c_{12}^{(i)} - \frac{1}{\mu^{(i)} - \lambda_{12}^{(i)} - \lambda_{13}^{(i)}} - \frac{1}{\mu^{(i)} - \lambda_{13}^{(i)}} = 0, \quad i = 1,\ 2,\ 3,$$

$$\sum_{i=0}^{3} \lambda_{12}^{i} = \lambda_{12}, \tag{4}$$

$$\sum_{i=0}^{3} \lambda_{13}^{i} = \lambda_{13}.$$

So the equilibrium prices for each transport company in competition can be found from (4) and the following system

$$c_{12}^{(i)} = (\lambda_{12}^{(i)} + \lambda_{13}^{(i)}) \left(\frac{1}{(\mu^{(i)} - \lambda_{12}^{(i)} - \lambda_{13}^{(i)})^2} + \frac{1}{\sum\limits_{j=0,\ j\neq i}^{3} (\mu^{(j)} - \lambda_{12}^{(j)} - \lambda_{13}^{(j)})^2} \right),$$

$$c_{13}^{(i)} = (\lambda_{12}^{(i)} + \lambda_{13}^{(i)}) \left(\frac{1}{(\mu^{(i)} - \lambda_{12}^{(i)} - \lambda_{13}^{(i)})^2} + \frac{1}{\sum\limits_{j=0,\ j\neq i}^{3} (\mu^{(j)} - \lambda_{12}^{(j)} - \lambda_{13}^{(j)})^2} \right) +$$

$$+(\lambda_{13}^{(i)}) \left(\frac{1}{(\mu^{(i)} - \lambda_{13}^{(i)})^2} + \frac{1}{\sum\limits_{j=0,\ j\neq i}^{3} (\mu^{(j)} - \lambda_{13}^{(j)})^2} \right), \quad i = 1,\ 2,\ 3.$$

The equilibrium prices and the streams of passengers are in Table 1.

Table 1: Equilibrium in transportation game with $\lambda_{12} = 5$, $\lambda_{13} = 10$

μ_i	$c_{12}^{(i)}$	$c_{13}^{(i)}$	$\lambda_{12}^{(i)}$	$\lambda_{13}^{(i)}$
$\mu_1 = 17$	0,04	0,06	1,59	2,77
$\mu_2 = 16$	0,033	0,053	1,54	2,28
$\mu_3 = 16$	0,033	0,053	1,54	2,28
$\mu_0 = 15$	0,033	0,041	0,32	2,67
$\mu_{12} = 33$	0,055	0,089	2,84	8,62
$\mu_3 = 16$	0,024	0,031	1,88	1,26
$\mu_0 = 15$	0,033	0,041	0,27	0,11
$\mu_{13} = 33$	0,055	0,089	2,84	8,62
$\mu_2 = 16$	0,024	0,031	1,88	1,26
$\mu_0 = 15$	0,033	0,041	0,27	0,11
$\mu_{23} = 32$	0,053	0,086	2,8	8,26
$\mu_1 = 17$	0,026	0,036	1,95	1,7
$\mu_0 = 15$	0,033	0,041	0,26	0,04
$\mu_{123} = 49$	0,102	0,154	0,89	9,58
$\mu_0 = 15$	0,033	0,041	4,11	0,42

Using the results from Table 1 we can construct the characteristic function, which is the payoff of a player or coalition in the Nash equilibrium situation. The values of characteristic function are in Table 2.

Table 2: Characteristic function

$v(\{1\})$	$0,215$
$v(\{2\})$	$0,168$
$v(\{3\})$	$0,168$
$v(\{12\})$	$0,925$
$v(\{23\})$	$0,859$
$v(\{13\})$	$0,925$
$v(\{123\})$	$1,56$

In cooperation players can get a total payoff equal to $1,56$. As a rule the division we use the Shapley value, which equals

$$\phi_1(v) = \frac{1}{3}v(1) + \frac{1}{6}(v(12) - v(2)) + \frac{1}{6}(v(13) - v(3)) + \frac{1}{3}(v(123) - v(23)) = 0,558,$$

$$\phi_2(v) = \frac{1}{3}v(2) + \frac{1}{6}(v(12) - v(1)) + \frac{1}{6}(v(23) - v(3)) + \frac{1}{3}(v(123) - v(13)) = 0,501,$$

$$\phi_3(v) = \frac{1}{3}v(3) + \frac{1}{6}(v(13) - v(1)) + \frac{1}{6}(v(23) - v(2)) + \frac{1}{3}(v(123) - v(12)) = 0,501,$$

and the price for the service is the same for all transport companies and is equal to $c_{12}^{123} = 0,102$, $c_{13}^{123} = 0,154$. Thus, transport companies have benefit form a cooperation.

Let increase the number of stops. Consider now that three transport companies and public transport are competing on the linear route with the four stops. The equilibrium prices and the streams of passengers are in Table 3.

Table 3: Equilibrium in transportation game with $\lambda_{12} = 2$, $\lambda_{13} = 3$, $\lambda_{14} = 10$

μ_i	$c_{12}^{(i)}$	$c_{13}^{(i)}$		$c_{14}^{(i)}$	$\lambda_{12}^{(i)}$	$\lambda_{13}^{(i)}$ $\lambda_{14}^{(i)}$
$\mu_1 = 17$	0,035	0,063	0,082	0,59	0,85	2,76
$\mu_2 = 16$	0,032	0,057	0,073	0,57	0,83	2,28
$\mu_3 = 16$	0,032	0,057	0,073	0,57	0,83	2,28
$\mu_0 = 15$	0,027	0,045	0,053	0,27	0,79	2,68
$\mu_{12} = 33$	0,053	0,097	0,132	1	1,4	8,62
$\mu_3 = 16$	0,022	0,037	0,044	0,7	1,01	1,26
$\mu_0 = 15$	0,027	0,045	0,053	0,3	0,59	0,12
$\mu_{12} = 33$	0,053	0,097	0,132	1	1,4	8,62
$\mu_3 = 16$	0,022	0,037	0,044	0,7	1,01	1,26
$\mu_0 = 15$	0,027	0,045	0,053	0,3	0,59	0,12
$\mu_{23} = 32$	0,051	0,094	0,127	0,995	1,38	8,26
$\mu_1 = 17$	0,024	0,041	0,051	0,713	1,04	1,7
$\mu_0 = 15$	0,027	0,045	0,053	0,292	0,58	0,04
$\mu_{123} = 49$	0,16	0,24	0,29	0	0,34	9,58
$\mu_0 = 15$	0,027	0,045	0,053	0,292	0,58	0,04

Using the results from Table 3 we can construct the characteristic function, which is the payoff of a player or coalition in the Nash equilibrium situation. The values of characteristic function are in Table 4.

Table 4: Characteristic function

$v(\{1\})$	$0,302$
$v(\{2\})$	$0,234$
$v(\{3\})$	$0,234$
$v(\{12\})$	$1,324$
$v(\{23\})$	$1,227$
$v(\{13\})$	$1,324$
$v(\{123\})$	$2,832$

In cooperation players can get a total payoff equal to $2,832$. As a rule the division we use the Shapley value, which equals

$$\phi_1(v) = \frac{1}{3}v(1) + \frac{1}{6}(v(12) - v(2)) + \frac{1}{6}(v(13) - v(3)) + \frac{1}{3}(v(123) - v(23)) = 0,999,$$

$$\phi_2(v) = \frac{1}{3}v(2) + \frac{1}{6}(v(12) - v(1)) + \frac{1}{6}(v(23) - v(3)) + \frac{1}{3}(v(123) - v(13)) = 0,9165,$$

$$\phi_3(v) = \frac{1}{3}v(3) + \frac{1}{6}(v(13) - v(1)) + \frac{1}{6}(v(23) - v(2)) + \frac{1}{3}(v(123) - v(12)) = 0,9165.$$

5. Conclusion

So we solved the pricing problem in cooperative transport game. It follows from simulation results, that the higher the intensity of service of transport company is, the higher ticket price this transport company declares. It also follows from results (Tables 1, 3) that cooperation of transport companies (or the increasing of the intensity of service) attracts passengers to use the coalition service in almost all cases, except the case, when all players unite, where passengers prefer to use the service of the transport company, which has the lower service price at short distances, while at long distances passengers prefer to use the service of the transport company, which has greater intensity of service.

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