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Network Game with Production and Knowledge Externalities*

Vladimir D. Matveenko and Alexei V. Korolev

National Research University Higher School of Economics,
Soyuza Pechatnikov Street, 16, St. Petersburg 190121, Russia
E-mail: vmatveenko@hse.ru
E-mail: danitschi@gmail.com

Abstract We consider a game equilibrium in a network in each node of which an economy is described by the simple two-period model of endogenous growth with production and knowledge externalities. Each node of the network obtains an externality produced by the sum of knowledge in neighbor nodes. Uniqueness of the inner equilibrium is proved. Three ways of behavior of each agent are distinguished: active, passive, hyperactive. Behavior of agents in dependence on received externalities is studied. It is shown that the equilibrium depends on the network structure. We study the role of passive agents; in particular, possibilities of connection of components of active agents through components of passive agents. A notion of type of node is introduced and classification of networks based on this notion is provided. It is shown that the inner equilibrium depends not on the size of network but on its structure in terms of the types of nodes, and in similar networks of different size agents of the same type behave in similar way.

Keywords: network, structure of network, network game, Nash equilibrium, externality, network formation.

1. Introduction

Behavior of agents/actors [1] in a network structure is defined in much by actions of other agents neighboring in a networks, or by information received from them. Multi-agent networks is a natural object for studying interrelations in social and economic systems. Network economics and network games theory consider questions of network formation, spreading (diffusion) of information in networks, positive and negative externalities, complementarity and substitutability of activities (see reviews (Jackson, 2008, Galeotti et al., 2010, Jackson and Zenou, 2014).

In the modern world mutual dependence includes, first of all, exchange of information as well as other multiple externalities. *Externalities*, i.e. influence of other agents, which does not go through the price mechanism, possess properties of public goods and are not fully paid. In particular, so called "jacobian" positive externalities (Jacobs, 1969) are directly related to complementarity of agents' activities. Positive externalities, and among them externalities of knowledge and human capital, spring up both in processes of production (Romer, 1986, Lucas, 1988) and consumption (Azariadis et al., 2013).

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[1] The term "agent" is used in economics, while the term "actor" is used in management, sociology and politology. We speak further about "agents" despite results of our work may have applications in analysis of economic as well as social and political relations.

In case of *complementarity* (and, correspondingly, supermodularity) a marginal effect of the agent's effort depends positively on efforts of other agents (see for instance Bulow et al., 1985, Milgrom and Roberts, 1990, Milgrom and Roberts, 1994, Topkis, 1998, Martemyanov and Matveenko, 2014). The agent is interested in increase of her efforts if her neighbors in the network create enough externalities. The agent will make the more efforts the more efforts are made by other agents. Vice versa, in case of *substitutability* (submodularity), if other agents increase their efforts then the efforts of the agent can become unessential, and she may rely on other agents (Grossman and Maggi, 2000, Jackson and Zenou, 2014); thus so called free-ryder problem arises (Bramoullé and Kranton, 2007).

In game theory a branch related to analysis of the role of positive externalities in networks has appeared, but attention there is devoted not to production externalities but mostly to consumption externalities connected with distribution of efforts.

In the present paper we continue the line of research of Nash equilibria in networks in presence of positive externalities, but our work contains several principally new elements in comparison to previous research.

Firstly, we study production but not consumption externalities; efforts in our model have meaning of investments, in particular, investments into creation of knowledge. The presence of production block allows us to interpret concepts of complementarity (supermodularity) and substitutability (submodularity) as, correspondingly, absence and presence of productivity. We carry out comparative analysis of these concepts within the same model.

Secondly, our model, for the first time in the network games literature uses the notion of the "jacobian" production externality in definition of the concept of equilibrium. The essence of this notion is that any agent makes her decision staying in a particular environment which depends on actions by the agent herself and by her neighbors. When making her decision, the agent considers the the state of the environment as exogenous; this means that the agent does not take into account possibility that her actions can directly influence the state of the environment.

As a simplest example, imagine a game equilibrium in a collective of smokers and non-smokers. A smoker, when making in equilibrium a decision to continue or to give up smoking, makes it staying in an environment relating to her smoking.

The third novation of our work is the use of dynamic approach. Essentially, our model is a generalization of the simple two-period model of endogenous growth and knowledge externalities due to Romer, 1986.

We show that equilibria depend on the network structure, and explain presence of three ways of behavior of agents: passive, active and hyperactive.

We introduce a notion of type of node and propose an algorithm of subdividing the set of nodes into types. We provide a classification of networks on base of the types of nodes and show the role of this classification in characterizing equilibria in classes of networks which possess different sizes but similar structure of types of nodes.

The paper is organized in the following way. In Section 2 the model is described. The uniqueness of the inner equilibrium is proved, if it exists. A theorem is proved, which serves further as a basic tool for comparison of utilities. In Section 3 behavior of the agent in dependence on received pure externality is analized. Section 4 is devoted to pure corner equilibria. In Section 5 equilibria in equidegree networks are studied. In Section 6 possibilities of adjunction of a node with passive agent to an equidegree network with active agents are considered. In Section 7 possibilities of connection of equidegree components of active agents through nodes with passive agents are studied. In Section 8 a notion of type of node is introduced and an algorithm of subdivision is described. In Sections 9 and 10, correspondingly, inner and corner equilibria for networks with two types of nodes are studied. Section 11 concludes.

2. The model

We consider a network with n nodes $i = 1, 2, \ldots, n$. Let $\boldsymbol{M}$ be the incidence matrix: elements M_{ij} and M_{ji} of this matrix are equal 1 if nodes i and j are connected by a link and equal 0 in the opposite case. We set $M_{ii} = 0$ for all i.

In each of the nodes there is an agents, whose preferences at two periods of time, 1 and 2, are described by a twice continuously differentiable, increasing in each argument utility function $U(c_1^i, c_2^i)$, where c_1^i, c_2^i is consumption of the final good in node i in periods 1 and 2.

In period 1 each agent is endowed by volume e of final good. This quantity may be used for consumption in period 1, and for investment into knowledge: $e = c_1^i + k_i$. There is a research technology which produces knowledge one to one from the invested good.

For an agent (index i is omitted now for notational simplicity), let k be her investment into knowledge, $\tilde{K}$ is *externality* which is the sum of investments of her close neighbors, and $K = k + \tilde{K}$ is her *environment*. Thus, the environment is the sum of investments in the neighboring nodes and in the node itself. The vector $\boldsymbol{K} = (K_1, K_2, \ldots, K_n)^T$ of environments of the agents can be calculated by use of the incidence matrix in the following way:

$$\boldsymbol{K} = (\boldsymbol{M} + \boldsymbol{I})\boldsymbol{k},$$

where $\boldsymbol{I}$ is the unit matrix of order n, $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)^T$, T is the sign of transposition.

The knowledge is used in production of final good for consumption in period 2. Production of good in the node is described by a production function $F(k, K)$ depending on the state of knowledge (investment), k, and the environment, K. The production function $F(k, K)$ is assumed to increase in each of its arguments and to be concave (may be not strictly) in k for each environment K.

The concept of externality (Romer, 1986, Lucas, 1988), means that at the moment of decision making the agent takes the environment K as exogenously given, i.e. does not account for a possibility of its change in result of her choice of investment k.

Correspondingly, given K the agent solves the following optimization problem $P(K)$:

$$U(c_1, c_2) \xrightarrow[c_1,\ c_2,\ k]{} \max$$
$$\begin{cases} c_1 \le e - k, \\ c_2 \le F(k, K), \\ c_1 \ge 0,\ c_2 \ge 0,\ k \ge 0. \end{cases}$$

The first two constraints of problem $P(K)$ at the optimum point are, evidently, satisfied as equalities. Substituting these constraints into the objective function, one can define new function (indirect utility function):

$$V(k, K) = U(e - k, F(k, K)).$$

Solution of problem $P(K)$ is one-to-one defined by k which maximizes the function $V(k, K)$ under constraint $k \in [0, e]$ given environment K. Function V is, evidently, strictly concave in k and, consequently, has a unique stationary point k^s, to the left of which it increases in k, and to the right – decreases. The stationary point k^s satisfies the equation

$$D_1 V(k, K) = 0, \tag{1}$$

where D_1 denotes derivative with respect to the first argument. If $k^s \in (0, e)$ then the optimal solution of problem $P(K)$ is $k = k_s$; this solution will be referred as *inner solution.* If $k^s < 0$ then the optimal solution is $k = 0$; and if $k^s > e$, then the solution is $k = e$. In these cases the solution will be referred as *corner solution.*

Let us consider a game in which the players are the agents $i = 1, 2, \ldots, n$. Feasible strategies of each player i are her investments $k_i \in [0, e]$. The payoff of the player is her utility $V(k_i, K_i)$. If profile $(k_1, k_2, \ldots, k_n)$ defines a consistent set of environments and optimal solutions of the players, this profile is referred as *Nash equilibrium with externalities.* If all k_i are inner solutions then the equilibrium $(k_1, k_2, \ldots, k_n)$ will be referred as *inner equilibrium.* In the opposite case it will be referred as *corner equilibrium.* It is clear that the inner Nash equilibrium with externalities (if it exists under given values of parameters) is defined by the system of equations

$$D_1 V(k_i, K_i) = 0, i = 1, 2, \ldots, n. \tag{2}$$

We will choose a particular form of the utility function and production function which allows to study the structure of equilibria in dependence on parameters. Let the utility function have the quadratic form:

$$U(c_1, c_2) = c_1(e - ac_1) + bc_2, \tag{3}$$

where $0 < a < 1/2,\ b > 0$. Here a is a saturation coefficient. Let the production function have the form

$$F(k, K) = BkK,$$

where $B > 0$. Notice that, by the meaning of parameters b and B, their increase promotes investments of agents. We will use notation $A = bB$. It will be assumed that

$$a < A. \tag{4}$$

Remark 2.1. Under our assumptions, the utility function defined by (3), evidently, strictly increases in both arguments and is concave. We could use instead a strictly concave function by applying the following concave transformation:

$$U(c_1, c_2) = \frac{[c_1(e - ac_1) + bc_2]^{1-\sigma}}{1 - \sigma},$$

where $0 < \sigma < 1$, σ is a coefficient of relative risk aversion. The points of maximum for both functions do coincide; thus the problem $P(K)$ in our case also has a unique solution which is guaranteed by the following lemma.

Lemma 2.1. *The indirect utility function $V(k_i, K_i)$ for the i-th node, considered, given environment K_i, as a function of k_i on the whole real axis, has a unique strict global maximum. The system of equations (2) takes the form:*

$$(A-2a)\boldsymbol{k} + A\boldsymbol{M}\boldsymbol{k} = \overline{\boldsymbol{e}}, \tag{5}$$

where

$$\boldsymbol{e} = \begin{pmatrix} e(1-2a) \\ e(1-2a) \\ \cdots \\ e(1-2a) \end{pmatrix}.$$

Proof.

$$\begin{aligned} V(k_i, K_i) &= (e-k_i)(e-a(e-k_i)) + Ak_iK_i \\ &= e^2(1-a) - k_ie(1-2a) - ak_i^2 + Ak_iK_i, \\ D_1V(k_i, K_i) &= e(2a-1) - 2ak_i + AK_i, \end{aligned} \tag{6}$$

thus, the system of equations (2) takes the form (5). The second derivative of the function $V(k_i, K_i)$ with respect to the first argument in any point is $-2a < 0$. □

Theorem 2.1. *If $A \neq 2a$* [2]*, then the system of equations (5) has a unique solution.*

Proof. The matrix of system (5) is

$$\boldsymbol{T} = \begin{pmatrix} A-2a & a_{12} & \dots & a_{1n} \\ a_{21} & A-2a & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & A-2a \end{pmatrix},$$

where $a_{ij} = AM_{ij}$ under $i \neq j$. By dividing the elements of the matrix $\boldsymbol{T}$ by A, we receive the matrix

$$\tilde{\boldsymbol{T}} = \begin{pmatrix} \alpha & M_{12} & \dots & M_{1n} \\ M_{21} & \alpha & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & \dots & \alpha \end{pmatrix},$$

where, because of (4), the diagonal elements satisfy condition $0 < |\alpha| < 1$. To prove the theorem it is sufficient to check the non-singularity of the matrix $\tilde{\boldsymbol{T}}$.

The determinant of the matrix is

$$\alpha^n + a_2\alpha^{n-2} + a_3\alpha^{n-3} + \dots + a_{n-1}\alpha + a_n = 0, \tag{7}$$

where all coefficients $a_2, a_3, \dots, a_n$ are integer. Let m be the highest degree of the variable α under which the coefficient of the polynomial (7), a_{n-m}, differs from zero. If $a_n \neq 0$ then $m = 0$. In the opposite case we reduce the polynomial to obtain

$$\alpha^{n-m} + a_2\alpha^{n-m-2} + a_3\alpha^{n-m-3} + \dots + a_{n-m-1}\alpha + a_{n-m}. \tag{8}$$

[2] If $A = 2a$ then, as is seen from (5), there is no equilibrium value of investment of the agent under any strategies of her neighbors. In this case, the equilibrium investment of the agent exists only if $\tilde{K} = e(1-2a)$. We do not consider such artifact as far as the incidence matrix is not obliged to be non-singular.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-m}$ be roots of the polynomial (8). Then

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-m} = 0, \tag{9}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-m-1}\alpha_{n-m} = a_2, \tag{10}$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots + \alpha_{n-m-2}\alpha_{n-m-1}\alpha_{n-m} = -a_3, \tag{11}$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_3\alpha_5 + \cdots + \alpha_{n-m-3}\alpha_{n-m-2}\alpha_{n-m-1}\alpha_{n-m} = a_4, \tag{12}$$

$$\cdots$$

$$\alpha_1\alpha_2 \ldots \alpha_{n-m} = (-1)^{n-m} a_n. \tag{13}$$

Let us show that no one of the roots can satisfy condition

$$0 < |\alpha_i| < 1.$$

Assume the opposite: let, e.g., $0 < |\alpha_1| < 1$. Then (9) implies that there is another root, e.g. α_2, which is not integer. Then the product $\alpha_1\alpha_2$ is also noninteger, and it follows from (10) that there is another noninteger product, e.g. $\alpha_2\alpha_3$. In any case, we have a noninteger product of three or even four roots. Then it follows from (11) or (12) that there is one more noninteger product of three or even four roots. Continuing this process further, we see that the product of all roots $\alpha_1\alpha_2 \ldots \alpha_{n-k}$ can not be integer; this contradicts to (13). This absurd proves that no one of the roots of the polynomial (7) can satisfy condition $0 < |\alpha| < 1$. Hence, for any feasible values of parameters of the model, the matrix $\tilde{\boldsymbol{T}}$ is nonsingular. □

Corollary 2.1. *If, for given values of parameters, inner equilibrium exists, then it is unique.*

It follows from (5), that the stationary solution k_i^s for agent i is defined by

$$k_i^s = \frac{e(2a-1) + A\tilde{K}_i}{2a - A}, \tag{14}$$

where $\tilde{K}_i$ is the pure externality received by the agent. In the inner equilibrium: $k_i^* = k_i^s$; $i = 1, ..., n$.

Remark 2.2. If increase of neighbors' investments leads to increase of investments by the agent herself, then one says that the *strategic complementarity* takes place. If increase of neighbors' investments leads to decrease of the agent's investments, then one says that the *strategic substitutability* takes place. From formula (14) it becomes clear, that if $A < 2a$, then the strategic complementarity takes place, and if $A > 2a$, then the strategic substitutability takes place. In our model with production these inequalities show, is productivity relatively low or high.

Definition 2.1. If $A > 2a$, we say that the *productivity is present.* In contrary case, if $A < 2a$, we say, that the *productivity is absent.*

Remark 2.3. Since it is assumed that $a < 1/2$, the inequality $A > 1$ implies presence of productivity; and, correspondingly absence of productivity implies $A \leq 1$.

Remark 2.4. It is immediately seen from Remark 2.2 that under absence of productivity ($A < 2a$) strategic complementarity takes place, and under presence of productivity ($A > 2a$) strategic substitutability takes place.

We will prove a general theorem, that will serve an instrument for utilities comparison.

Theorem 2.2. *Let W^* and W^{**} be two networks in inner equilibria; i is a node of W^*, and j is a node of W^{**}; k_i^*, K_i^*, U_i^* are, correspondingly, optimal investment, environment, and utility of agent i; k_j^{**}, K_j^{**}, U_j^{**} are corresponding values for agent j. Then:*

1) if $K_i^* < K_j^{**}$, then $U_i^* < U_j^{**}$;

2) if $K_i^* \leq K_j^{**}$, then $U_i^* \leq U_j^{**}$;

3) if $K_i^* = K_j^{**}$, then $U_i^* = U_j^{**}$.

If k_i^* is not an inner, but a corner solution, $k_i^* = 0$, and $k_j^{**} > 0$, then

$$U_i^* = U(e, 0) < U_j^{**}.$$

Proof. Let be $K_i^* < K_j^{**}$ $\quad (K_i^* \leq K_j^{**})$. Since function $V(k_j, K_j^{**})$ reaches its maximum at point k_j^{**}, we have $V(k_i^*, K_j^{**}) \leq V(k_j^{**}, K_j^{**})$. Because of $\partial V(k, K)/\partial K > 0$ for any $k \neq 0$ and K, we obtain that $V(k_i^*, K_i^*) < V(k_i^*, K_j^{**})$ (respectively, $V(k_i^*, K_i^*) \leq V(k_i^*, K_j^{**})$). Thus, $V(k_i^*, K_i^*) < V(k_i^*, K_j^{**}) \leq V(k_j^{**}, K_j^{**})$ (respectively, $V(k_i^*, K_i^*) \leq V(k_i^*, K_j^{**}) \leq V(k_j^{**}, K_j^{**})$). Hence, $U_i^* < U_j^{**}$ (respectively, $U_i^* \leq U_j^{**}$).

Combining previous results, we see, that if $K_i^* = K_j^{**}$, then $U_i^* = U_j^{**}$.

The last statement of the theorem is quite obvious:

$$U_j^{**} = V(k_j^{**}, K_j^{**}) > V(0, K_j^{**}) = V(0, K_i^*) = U_i^*,$$

since, as far as $k = 0$, function V does not depend on K. □

3. The agent's behavior

Let us introduce the following terminology.

Definition 3.1. If the agent makes zero investments into knowledge, $k = 0$, we say that the agent is *passive*. If she makes investments $0 < k < e$, she is *active*. If the agent makes maximal possible investments, e (and, correspondingly, does not consume at period 1), we say that she is *hyperactive*.

The following lemma describes necessary and sufficient conditions of different ways of behavior by the agent in dependence on the size of externality $\tilde{K}$ which she receives. Index i is omitted.

Lemma 3.1. *The necessary and sufficient conditions for various types of behavior by the agent are the following.*

1) Under absence of productivity:

the agent is passive if

$$\tilde{K} \leq \frac{e(1 - 2a)}{A};$$

the agent is active if

$$\frac{e(1 - 2a)}{A} < \tilde{K} < \frac{e(1 - A)}{A};$$

the agent is hyperactive if

$$\tilde{K} \geq \frac{e(1-A)}{A}.$$

2) Under presence of productivity:
the agent is passive if

$$\tilde{K} \geq \frac{e(1-2a)}{A};$$

the agent is active if

$$\frac{e(1-A)}{A} < \tilde{K} < \frac{e(1-2a)}{A};$$

the agent is hyperactive if

$$\tilde{K} \leq \frac{e(1-A)}{A}.$$

Proof. It follows from (6) that

$$k^s = \frac{e(2a-1) + A\tilde{K}}{2a - A}. \tag{15}$$

If $k^s \leq 0$ then the agent makes no investments, $k = 0$. If $0 < k^s < e$, the solution is in the inner point $k = k^s$. If $k^s \geq e$, the agent makes maximal possible investment, $k = e$. Writing these conditions in details, we receive the inequalities listed in the formulation of the lemma. □

Proposition 3.1. *Under presence of productivity and $A > 1/2$, each agent, who has a hyperactive neighbor, is not hyperactive; moreover, if $A + 2a \geq 1$ (which implies $A > 1/2$), she is even passive.*

Proof. If agent i has a hyperactive neighbor, then i obtains externality $\tilde{K} \geq e$. Hence, if $A > 1/2$ then $\tilde{K} \geq e > e(1-A)/A$, and, by Lemma 3.1, agent i is not hyperactive in equilibrium. Moreover, let $A + 2a \geq 1$, i.e. $(1-2a)/A \leq 1$. Then $\tilde{K} \geq e \geq e(1-2a)/A$ and, by Lemma 3.1, the agent is passive. □

Proposition 3.2. *Under absence of productivity, if $A \geq 1/2$, then each agent, who has a hyperactive neighbor, is hyperactive.*

Proof. If the agent has a hyperactive neighbor then, similarly to the proof of Proposition 3.1, $\tilde{K} \geq e$. Hence, if $A \geq 1/2$, then $\tilde{K} \geq e \geq e(1-A)/A$. By Lemma 3.1, the agent is hyperactive. □

Proposition 3.3. *Under presence of productivity, an agent, who stays in an isolated node, or for whom all neighbors are passive, is active if $A > 1$, and hyperactive if $A \leq 1$.*

Proof. Since $\tilde{K} = 0$, Lemma 3.1 implies that the agent is active under $1 - A < 0$; and hyperactive under $1 - A \geq 0$. □

Proposition 3.4. *Under absence of productivity, the agent, who stays in an isolated node, or for whom all neighbors are passive, is passive.*

Proof. Since $\tilde{K} = 0$, by Lemma 3.1, the agent is passive. □

4. Pure corner equilibria

Definition 4.1. *Pure corner equilibrium* is such equilibrium, in which knowledge in each node is equal either to 0 or to e, i.e. each agent is either passive or hyperactive.

Proposition 4.1. *Under absence of productivity, the situation when all agents are passive is equilibrium.*

Proof. For each node i, if $\tilde{K}_i = 0$, then, by Lemma 3.1, the agent is passive. □

Proposition 4.2. *For a connected network with more than one node, let μ be the smallest degree (number of neighbors) and M the biggest degree. Under absence of productivity, the situation when all agents are hyperactive is equilibrium iff $A \geq 1/(\mu+1)$. Under presence of productivity, the situation when all agents are hyperactive is equilibrium iff $A \leq 1/(M+1)$.*

Proof. By Lemma 3.1, the agent who stays in a node with the smallest degree, is hyperactive iff $\mu e = \tilde{K} \geq e(1-A)/A$; this is equivalent to condition $A \geq 1/(\mu+1)$. If the latter condition is fulfilled then the agents in all other nodes are all the more hyperactive.

Similarly, the agent who stays in a node with the biggest degree, is hyperactive under presence of productivity iff $Me = \tilde{K} \leq e(1-A)/A$; the latter is equivalent to condition $A \leq 1/(M+1)$. Under this condition, the agents in all other nodes are all the more hyperactive. □

Definition 4.2. A network is referred as *equidegree network* if each node has the same degree m, where $m \geq 1$.

Corollary 4.1. *In equidegree network, if $A \geq 1/(1+m)$ under absence of productivity, or if $A \leq 1/(m+1)$ under presence of productivity, then the situation when all agents are hyperactive is equilibrium.*

Proof. In equidegree network: $\mu = M = m$. □

Corollary 4.2. *In full network with n nodes, equilibrium in which all agents are hyperactive is possible under absence of productivity iff $A \geq 1/n$, and under presence of productivity iff $A \leq 1/n$.*

Proof. In full network: $\mu = M = n-1$. □

Propositions 3.1 and 3.2 imply the following fact.

Corollary 4.3. *In connected network under absence of productivity, if $A \geq 1/2$ then a situation when all agents are hyperactive is equilibrium; moreover, it is a unique possible equilibrium in which at least one agent is hyperactive. If in a network there is at least one link then, under presence of productivity, if $A > 1/2$ then the situation in which all agents are hyperactive is not equilibrium.*

Proposition 4.3. *Under absence of productivity, in each network there is equilibrium in which all agents are passive. Under presence of productivity, such equilibrium is impossible.*

Proof. This follows directly from Propositions 3.3 and 3.4. □

Theorem 4.1. *Under presence of productivity, let r be a natural number such that $r \le n$, $Ar \le 1$, $Ar + 2a \ge 1$. In full network, C_n^r pure corner equilibria are possible, in each of which r nodes are hyperactive and other $n - r$ nodes are passive.*

Proof. A node, for which not more than $r - 1$ neighbors are hyperactive and other neighbors are passive, receives externality $\tilde{K} = (r-1)e$. By Lemma 3.1, such node is hyperactive iff $Ar \le 1$. Similarly, a node, for which more than $r - 1$ neighbors are hyperactive and other neighbors are passive, is itself passive iff $r \ge (1-2a)/A$, which is equivalent to $Ar + 2a \ge 1$. □

Remark 4.1. Besides pure corner equilibria which are listed in Corollary 4.2, Proposition 4.3 and Theorem 4.1, there may exist corner equilibria and a unique inner equilibrium.

Example 4.1. In full network with four nodes, under presence of productivity and under $2A \le 1$, $2A + 2a \ge 1$, there is equilibrium with two hyperactive and two passive agents. Because of symmetry, any two nodes can be hyperactive, and others passive; in all $C_4^2 = 6$ purely corner equilibria exist.

Under $A \le 1$, $A + 2a \ge 1$ in the same network there is an equilibrium with one hyperactive and three passive nodes; in all $C_4^1 = 4$ such equilibria exist.

Under $3A \le 1$, $3A + 2a \ge 1$ in the same network there is equilibrium with three hyperactive and one passive nodes. In all there are $C_4^3 = 4$ such equilibria.

In case when $3A \le 1$, $A + 2a \ge 1$ all these pure corner equilibria exist; $6+4+4 = 14$ in all.

Besides these 14 pure corner equilibria, in this network there are also nonpure corner equilibria and unique inner equilibrium.

5. Equilibria in equidegree networks

For equidegree network, an equilibrium in which all agents have the same level of knowledge (i.e. make the same investments), will be referred as *symmetric*. For a symmetric equilibrium, Equ. (15) (under $\tilde{K} = mk$) implies

$$k^s = \frac{e(1-2a)}{A(m+1) - 2a}. \tag{16}$$

If $A > 1/(m+1)$ then $k = k^s$, i.e. the equilibrium is inner: the agents are active. If $2a/(m+1) < A \le 1/(m+1)$ then $k = e$, i.e. the equilibrium is corner: the agents are hyperactive.

Remark 5.1. Thus, the condition of existence of inner equilibrium in equidegree network is $A > 1/(m+1)$ (remind also permanent constraints $a < 1/2$, $a < A$).

Examples of equidegree networks

1. Cycle. In this case $m = 2$. According to (16), investment by an agent does not depend on the size of cycle.

2. Full network. In this case $m = n - 1$, where n is the number of nodes in the network. This case is similar to Romer, 1986. According to (16), knowledge in nodes declines with increase in the size of network. The sum of knowledge $nk^s = e(1-2a)/(A - 2a/n)$ also declines and converges to $e(1-2a)/A$.

3. Chain with two nodes. It is the case of $m = 1$ (and also a case of full network).

4. Networks in which each node has degree $m = 3$ (see Fig. 1).

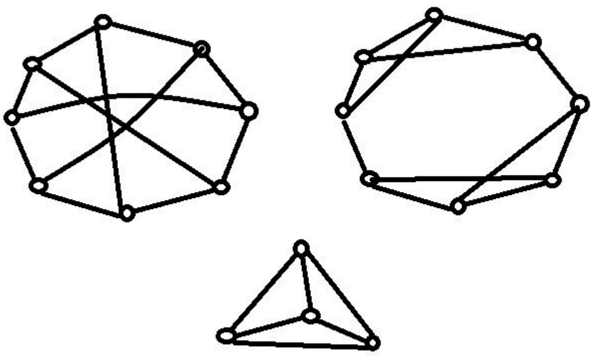

Fig. 1: Examples of equidegree networks with degree $m = 3$.

Remark 5.2. Equation (16) is also true for isolated node, under $m = 0$. It is seen from (16) that in an isolated node:
1) under $A < 2a$ the agent is passive;
2) under $2a < A \leq 1$ the agent is hyperactive;
3) under $A > 1$ the agent is active and

$$k = \frac{e(1-2a)}{A-2a}.$$

Generally, (16) allows to study dependence of knowledge, consumption and utility on degree of nodes, m, of equidegree network. Knowledge k decreases with respect to degree. Consumptions at the first and the second periods of time are, correspondingly,

$$c_1 = e - k = e\frac{(m+1)A-1}{(m+1)A-2a},$$

$$c_2 = F(k,(1+m)k) = B(1+m)k^2 = \frac{B(m+1)e^2(1-2a)^2}{[(m+1)A-2a]^2}.$$

One can check that c_1 increases and c_2 decreases with respect to degree.

Proposition 5.1. *For inner equilibria in equidegree networks, utility decreases with respect to degree and converges to $U(e,0)$.*

Proof. The utility function turns into

$$U = e^2\frac{[(m+1)A-1][(m+1)A(1-a)-a]+A(m+1)(1-2a)^2}{[(m+1)A-2a]^2}.$$

Differentiating U with respect to $x = 1+m$ (as if x is continuous) we obtain

$$\frac{dU}{dx} = -\frac{2aA(Ax-2a)(2a-1)^2}{(x^2A^2-4aAx+4a^2)^2}.$$

Under $x \geq 2$, i.e. under $m \geq 1$, the inequality $Ax - 2a > 0$ is fulfilled, hence $dU/dx < 0$; thus, utility in each node decreases with respect to degree. Under $m \to +\infty$ we have $c_1 \to e$, $c_2 \to 0$ and, because of continuity, the limit of the utility is equal to $U(e, 0)$. □

This result corresponds to intuition: in big social and economic systems utility can be high because of diversity, but in a system consisting of similar agents, the world, probably, loses its utility under very high degrees of nodes if there is no diversity.

6. Adding a node with passive agent to an equidegree network

We have seen that, in definite areas of parameters, equilibrium in equidegree network can be rather simple: all agents are hyperactive, or all are active. At the same time, passive agents do not create externalities, i.e. do not influence the environments of other agents. This means that under some conditions, an equilibrium is possible which consists of components with either active or hyperactive agents and of groups of passive agents which connect these components.

Below in Section 7 we consider connection of equidegree networks through nodes with passive agents. As a preliminary, in this section we study a possibility of addition a node with passive agent to equidegree network.

Proposition 6.1. *Let a node with passive agent be connected by use of l links to an equidegree network with degree $m > 0$, which is initially in inner equilibrium. Necessary a sufficient condition of existence of such equilibrium, in which the adjoined agent remains passive and the active agents remain in the previous inner equilibrium, is the following:*

under absence of productivity,

$$l \leq m + 1 - \frac{2a}{A}; \tag{17}$$

under presence of productivity,

$$l \geq m + 1. \tag{18}$$

Proof. For the newly adjoined node the externality is equal to

$$\tilde{K} = \frac{le(1 - 2a)}{A(m + 1) - 2a}.$$

By Lemma 3.1, the adjoined agent can stay passive in equilibrium iff

$$A < 2a; \quad \frac{l(1 - 2a)e}{A(m + 1) - 2a} \leq \frac{(1 - 2a)e}{A} \tag{19}$$

or

$$A > 2a; \quad \frac{l(1 - 2a)e}{A(m + 1) - 2a} \geq \frac{(1 - 2a)e}{A} \tag{20}$$

Conditions (19) and (20) are equivalent, correspondingly, to (17) and (20). □

The meaning of (17) is that, under absence of productivity, the adjoined agent can stay passive only as long as she is not sufficiently connected with active agents. Staying passive, she does not influence the initial equilibrium of active agents. But if the number of her links with active agents becomes sufficiently big, the adjoined agent receives so big externality that she is not able to hold the indifferent behavior in equilibrium. When she starts making investments, absolutely different inner equilibrium appears in the network.

Vice versa, under presence of productivity, the agent can preserve her indifferent behavior only until she has sufficiently big number of active neighbors.

Corollary 6.1. *In case of a cycle (equidegree network with degree $m = l$) of active agents, adjunction of a passive agent, in such way that each of the agents preserves her behavior in equilibrium, is impossible.*

Corollary 6.2. *Under absence of productivity, if an equidegree network with degree $m \geq 2$ is in inner equilibrium, then a node with passive agent can be adjoined by one link in such way that each of the agents preserves her behavior in equilibrium.*

Proposition 6.2. *A node with passive agent can be adjoined to an isolated node, in such way that each of the agents preserves her behavior in equilibrium, in the following three cases:*

1) $A < 2a$ (in this case the agent in the isolated node is passive);

2) $2a < A \leq 1, \quad A + 2a \geq 1$ (in this case the agent in the isolated node is hyperactive);

3) $A > 1$ (in this case the agent in the isolated node is active and $k = e(1 - 2a)/(A - 2a)$).

Proof. If $A < 2a$ then there are no externalities and nothing changes after adjunction. Thus, we have equilibrium of two passive agents.

If $2a < A \leq 1$ then the adjoined agent receives externality $\tilde{K} = e$ and, by Lemma 3.1, remains passive under $A \geq 1 - 2a$.

If $A > 1$ (this is a condition for the isolated agent to be active), then the adjoined agent receives externality $\tilde{K} = e(1 - 2a)/(A - 2a)$. By Lemma 3.1, the agent will remain passive if $A - 2a \leq A$, but this inequality is certainly true. □

Examples of adjunction of a passive agent to an equidegree network

Example 6.1. In the chain of 3 nodes 1–2–3, under $A > 1/2$, equilibrium with $k_1 = 0$, $k_2 = k_3 = (1 - 2a)e/[2(A - a)]$ is impossible, by virtue of Corollary 6.1.

Example 6.2. If initially there is a chain of two active agents, 2–3, and passive agent 1 establishes links to both of them, i.e. $l = 2$, $m = 1$, then, by Proposition 6.1, there is equilibrium, in which all three agents maintain their initial behavior.

Example 6.3. Let a passive agent establish $l = 4$ links with agents in equidegree network with degree $m = 3$, which is in inner equlilibrium. The initial equilibrium exists only if $A > 1/4$ (see Remark 5.1). If, moreover, productivity takes place, then, by Proposition 6.1, there exists equilibrium in which all agents maintain their initial behavior.

Remark 6.1. Under absence of productivity, similarly to adjunction of one node with passive agent, any connected network with passive agents can be added.

7. Connection of equidegree networks through nodes with passive agents

In this section we consider connection of two equidegree networks being initially in inner equilibrium. We wonder is it possible to construct a new network from such blocks, connecting them by components of passive agents in such way that in the new network there exists an equilibrium, in which all the agents behave in the same way as before unification.

Proposition 7.1. *Under $2a/(m-1) \leq A < 2a$ (what implies $m \geq 3$), two equidegree networks with the same degree m, being initially in inner equilibrium, can be connected through a node with passive agent in such way that all agents maintain their initial behavior in equilibrium. Under $m = 2$ (case of cycles) such equilibrium is impossible. Under $m = 1$ (case of active dyads, when $A > 1/2$) such equilibrium is possible under presence of productivity ($A > 2a$). Under $m = 0$ (case of active isolated nodes,when $A > 1$) such connection is always possible.*

Proof. The connecting passive node receives externality from two active nodes: $l = 2$. Under absence of productivity, condition (6.1) takes the form $2a(m-1) \leq A$. Condition (6.2) takes the form $2 \geq m+1$ and is fulfilled under $m = 1$. Let $m = 0$, $A > 1$ and, hence, $A > 2a$. Initially the inner equilibrium in two isolated nodes was $k = e(1-2a)/(A-2a)$, hence the connecting node receives externality

$$\tilde{K} = \frac{2e(1-2a)}{A-2a}.$$

By Lemma 3.1, this node remains passive under

$$\frac{2}{A-2a} > \frac{1}{A},$$

and the latter inequality is certainly fulfilled. □

Proposition 7.2. *Under $m \geq 2$ and absence of productivity, two equidegree networks with the same degree m, being initially in inner equilibrium, can be connected by a chain of two or more passive nodes in such way that behavior of the agents does not change in equilibrium. Such connection is impossible if $m = 1$. In case of $m = 0$ (isolated nodes) such connection is possible under $A > 1$ through a chain of two passive nodes, but is not possible through chains of three or more passive nodes.*

Proof. Statements for $m \geq 2$ and $m = 1$ follow from Proposition 6.1 and Corollary 6.1. Statement for $m = 0$ follows from Proposition 6.2. If two active agents are connected by a chain of three or more passive nodes, then, by Proposition 3.3, the agent, who has no active neighbors, could not stay passive in equilibrium under presence of productivity. □

Remark 7.1. Under the same conditions, there exists a "cycle", consisting of equidegree networks connected by chains of nodes with passive agents. Components of active agents in such cycle alternate with components of passive agents.

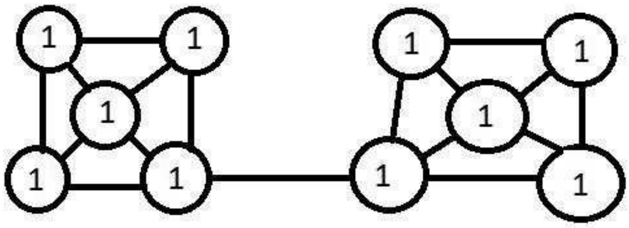

Fig. 2: Start of the algorithm: $s = 1$.

8. Types of nodes

Definition 8.1. Let the set of nodes $1, 2, \ldots, n$ be decomposed into disjoint classes in such way that any nodes belonging the same class have the same numbers of neighbors from each class. The classes will be referred as *types* of nodes. Type j is characterized by vector $\boldsymbol{l}(j) = (l_1(j), l_2(j), \ldots, l_k(j))$, where $l_i(j)$ is the number of neighbors in class i for each node of class j.

Let us describe an algorithm of subdivision of the set of nodes of network into types.

Let s be a current number of subsets of subdivision. Initially $s = 1$.

Iteration of the algorithm. Consider nodes of the first subset. If all of them have the same numbers of neighbors in each subset $1, 2, ..., s$, then the first subset is not changed. In the opposite case, we divide the first subset into new subsets in such way that all nodes of each new subset have the same vector of numbers of neighbors in subsets.

We proceed in precisely same way with the second, the third, ..., the s-th subset. If on the present iteration the number of subsets s have not changed, then the algorithm finishes its work. If s has increased, then the new iteration is executed.

The number of subsets s does not decrease in process of the algorithm. Since s is bounded from above by the number of nodes in the network, the algorithm converges.

It is clear that the algorithm divides the set of nodes into the minimal possible number of classes.

Example

Let us apply the algorithm to the network depicted in Fig. 2. Initially $s = 1$, all nodes are from the same one set (Fig. 2).

After the first iteration we obtain the division depicted in Fig. 3.

Then, on the first step of the second iteration, we receive the division depicted in Fig. 4.

On the second step of the second iteration we receive the division shown in Fig. 5.

On the third iteration nothing changes, and the algorithm stops. We have received a subdivision of the set of nodes of the network into four types which are characterized by the vectors of numbers of neighbors: $\boldsymbol{l}(1) = (1, 2, 1, 0)$, $\boldsymbol{l}(2) = (1, 0, 1, 1)$, $\boldsymbol{l}(3) = (1, 2, 0, 1)$, $\boldsymbol{l}(4) = (0, 2, 1, 0)$.

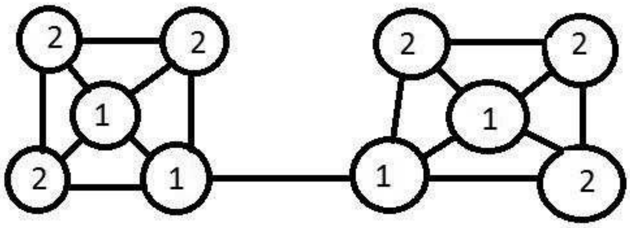

Fig. 3: Result of the first iteration: $s = 2$.

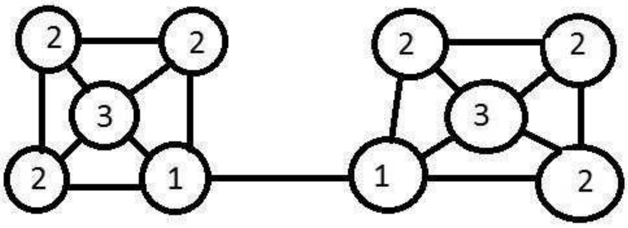

Fig. 4: The first step of the second iteration.

Definition 8.2. Let us call *symmetric* such equilibrium in which agents of the same type make the same investments.

Remark 8.1. Inner equilibrium is always symmetric. It follows from the uniqueness of solution of the system of equations (2.1) and symmetry of this system with respect to types.

Later on k_i will denote investment in any node of type i.

Remark 8.2. If two networks have the same number of types of nodes, S, and these types are characterized by the same set of vectors $\boldsymbol{l}(1)$, $\boldsymbol{l}(2)$, $\ldots$, $\boldsymbol{l}(S)$, then the inner equilibria in these networks do coincide, in the sense that agents in the nodes of the same type behave in the same way.

Fig. 6 provides an example of 3 networks which possess the same types of nodes characterized by vectors $\boldsymbol{l}(1) = (1, 2)$ and $\boldsymbol{l}(2) = (0, 2)$. Correspondingly, these networks have the same inner equilibria, despite these networks have different sizes.

9. Inner equilibria in networks with two types of nodes

Let a network have two types of nodes which are characterized by vectors $\boldsymbol{l}(1) = (s_1, s_2)$ and $\boldsymbol{l}(2) = (t_1, t_2)$. Here s_i is the number of links connecting a node of type 1 with nodes of type i; t_i is the number of links connecting a node of type 2 with nodes of type i; $i = 1, 2$. Then (5) implies the following system of equations:

$$\begin{cases} (A - 2a + s_1 A)k_1 + s_2 A k_2 = e(1 - 2a), \\ t_1 A k_1 + (A - 2a + t_2 A)k_2 = e(1 - 2a). \end{cases} \tag{21}$$

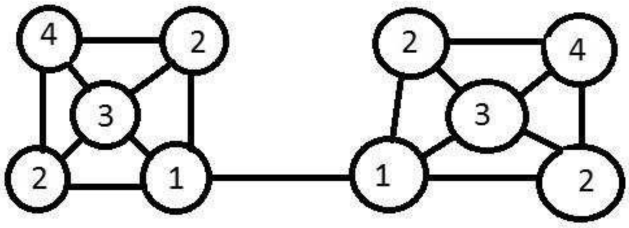

Fig. 5: The second step of the second iteration: $s = 4$.

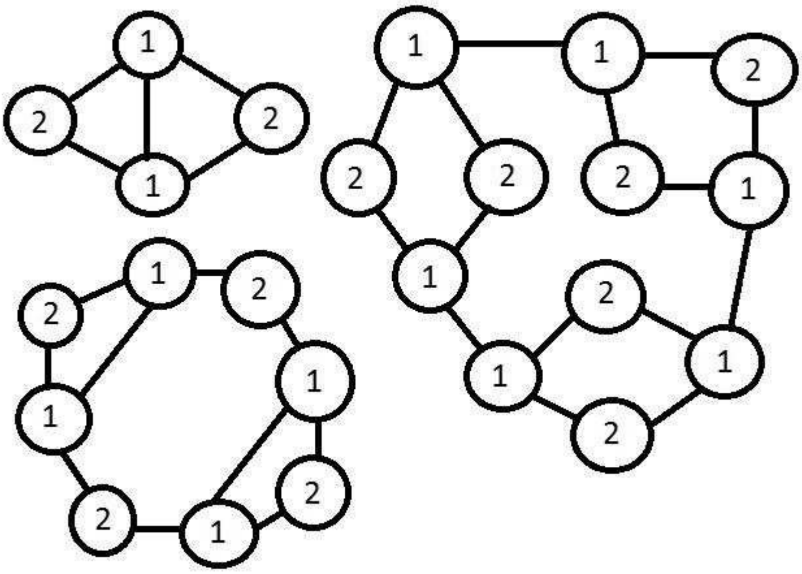

Fig. 6: Networks with "coinciding" inner equilibria.

Its solution is the pair

$$k_1^s = \frac{e(1-2a)[A-2a+(t_2-s_2)A]}{(A-2a)^2+(s_1+t_2)(A-2a)A+(s_1t_2-t_1s_2)A^2}. \tag{22}$$

$$k_2^s = \frac{e(1-2a)[A-2a+(s_1-t_1)A]}{(A-2a)^2+(s_1+t_2)(A-2a)A+(s_1t_2-t_1s_2)A^2}. \tag{23}$$

If $0 < k_1^s < e$, $0 < k_2^s < e$, then the stationary values k_1^s, k_2^s define the inner equilibrium in the network.

Special cases of the network with two types of nodes

1. Chain of four nodes: 2–1–1–2. Types 1 and 2 are characterized by vectors $\boldsymbol{l}(1) = (1,1)$ and $\boldsymbol{l}(2) = (1,0)$. Formulas (22) – (23) take the form:

$$k_1 = \frac{2ae(1-2a)}{6aA-4a^2-A^2}, \tag{24}$$

$$k_2 = \frac{e(1-2a)(2a-A)}{6aA - 4a^2 - A^2}. \tag{25}$$

Inequalities $0 < k_i < e$, $i = 1, 2$ are fulfilled under absence of productivity ($A < 2a$).

2. A generalization of the previous case is a fan, i.e. a chain of two nodes, to each of which a bundle of ν hanging nodes is adjoined. The types are characterized by vectors $\boldsymbol{l}(1) = (1, \nu)$ and $\boldsymbol{l}(2) = (1, 0)$.

3. Star of order ν, i.e. a network, in which a central node of type 1 has ν peripheral neighbors of type 2. The types are characterized by vectors $\boldsymbol{l}(1) = (0, \nu)$ and $\boldsymbol{l}(2) = (1, 0)$. Equations (22) – (23) turn into

$$k_1 = \frac{e(1-2a)[(\nu-1)A + 2a]}{\nu A^2 - (A-2a)^2}, \tag{26}$$

$$k_2 = \frac{2ea(1-2a)}{\nu A^2 - (A-2a)^2}, \tag{27}$$

The pair k_1, k_2 defines inner equilibrium if $0 < k_i < e$, $i = 1, 2$, i.e. if

$$\begin{cases} \nu A^2 - (A-2a)^2 > 0, \\ \nu A^2 - (A-2a)^2 > (1-2a)[(\nu-1)A + 2a], \\ \nu A^2 - (A-2a)^2 > 2a(1-2a). \end{cases}$$

Evidently, this system of inequalities is equivalent to the second of them,

$$(\nu-1)A^2 + [2a(\nu+1) - (\nu-1)]A - 2a > 0. \tag{28}$$

This inequality is fulfilled iff

$$A > \frac{-2a(\nu+1) + \nu - 1 + \sqrt{[2a(\nu+1) - (\nu-1)]^2 + 8a(\nu-1)}}{2(\nu-1)}.$$

We see that under big ν inequality (28) is true if $A^2 + 2aA - A > 0$, which is equivalent to $A + 2a > 1$. It is also easily seen that the left-hand side of (28) increases in ν. Hence, if (28) is fulfilled for $\nu = 2$, it is fulfilled for all ν. This implies that if $A + 2a > 1$ and

$$A > \frac{-6a + 1 + \sqrt{36a^2 - 4a + 1}}{2},$$

then formulas (26)–(27) define inner equilibrium for all $\nu \geq 2$.

Proposition 9.1. *In a star, if the number of peripheral nodes, ν, increases, then knowledge and utility in the central node decrease under absence of productivity, but increase under presence of productivity. Knowledge and utility in each peripheral node always decrease.*

Proof. Derivative of k_1 in ν, if ν is considered as a continuous parameter, is

$$\frac{2Aae(1-2a)(A-2a)}{[(A-2a)^2 - \nu A^2]^2}.$$

Hence, knowledge in the central node decreases in ν if $A < 2a$, and increases if $A > 2a$. It is directly seen from (27) that k_2 decreases in ν.

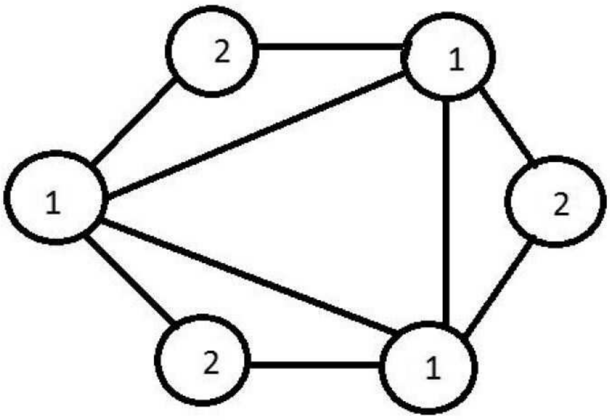

Fig. 7: A network with 2 types of nodes: the numbers mean types.

Environment for the central node is

$$K_1 = \frac{e(1-2a)[\nu(A+2a)-(A-2a)]}{\nu A^2-(A-2a)^2}.$$

Derivative of K_1 is

$$\frac{4a^2e(1-2a)(A-2a)}{[\nu A^2-(A-2a)^2]^2}.$$

Theorem 2.2 implies that utility in the central node decreases in ν if $A < 2a$, and increases if $A > 2a$. Environment for any peripheral node is

$$K_2 = \frac{e(1-2a)[(\nu-1)A+4a]}{\nu A^2-(A-2a)^2}.$$

Derivative of K_2 is

$$\frac{-4a^2Ae(1-2a)}{[\nu A^2-(A-2a)^2]^2} < 0.$$

Hence, by Theorem 2.2, utility in a peripheral node decreases in ν. □

Remark 9.1. When the order of the star, ν, increases, the sum of knowledge in the peripheral nodes decreases and under $\nu \to \infty$ converges to $2ae(1-2a)/A^2$, while knowledge in each separate peripheral node converges to 0. Knowledge in the central node converges to $e(1-2a)/A$.

Remark 9.2. If $\nu = 2$ then the star turns into a chain of three nodes.

4. Cycle of k nodes ($k \geq 3$), to each of which a bundle of ν "hanging" nodes is added. Equilibria in this network will be studied in Section 10 below.

5. Network shown in Fig. 7. The network of this type has order not less than 6 and divisible by 2. The types of nodes are characterized by vectors $\boldsymbol{l}(1) = (2,2)$ $\boldsymbol{l}(2) = (2,0)$. The equations (22) – (23) turn into

$$k_1 = \frac{e(1-2a)(A+2a)}{A^2+8Aa-4a^2}, \tag{29}$$

$$k_2 = \frac{e(1-2a)(2a-A)}{A^2+8Aa-4a^2}. \tag{30}$$

Positivity $k_i > 0$, $i = 1,2$ is equivalent to the absence of productivity ($A < 2a$) and fulfillment of the inequality

$$A^2 + 8Aa - 4a^2 > 0.$$

Conditions $k_i < e$, $i = 1,2$ are then equivalent to

$$A^2 + 8Aa - 4a^2 > (1-2a)(A+2a),$$

$$A^2 + 8Aa - 4a^2 > (1-2a)(2a-A).$$

The system of the latter three inequalities is equivalent to the second of them, which can be written in the form

$$A^2 + 10aA - A - 2a > 0.$$

Ultimately, we obtain necessary and sufficient condition of inner equilibrium:

$$\frac{-10a + 1 + \sqrt{(10a-1)^2 + 8a}}{2} < A < 2a.$$

Let us compare levels of knowledge and utility for the network in Fig. 7 and for the full network.

Proposition 9.2. *If the network of the type depicted in Fig. 7 is completed to become the full network, then, under absence of productivity, knowledge and utility in nodes of type 1 decrease, while knowledge and utility in nodes of types 2 increase.*

Proof. Comparing k_1 and k_2 with knowledge in a node of the full network, $k = e(1-2a)/((n-1)A-2a)$, we see that $k_1 > k$, $k_2 < k$. Comparing environments

$$K_1 = \frac{e(1-2a)(A+10a)}{A^2+8aA-4a^2}, \quad K_2 = \frac{e(1-2a)(A+6a)}{A^2+8aA-4a^2}$$

of the nodes of the initial network with environment of a node of the full network,

$$K = \frac{(n-1)e(1-2a)}{(n-1)A-2a},$$

we see that $K_1 > K$, $K_2 < K$. Theorem 2.2 provides the needed result. □

10. Corner equilibria in networks with two types of nodes

In this section we will study symmetric corner equilibria. The network of type depicted in Fig. 8 can be considered in two ways: as result of addition of ν previously isolated nodes to each node of the cycle of order n, or as result of unification of n stars, each of them with ν peripheral nodes, into one cycle.

Proposition 10.1. *In the network depicted in Fig. 8, inner equilibrium is impossible.*

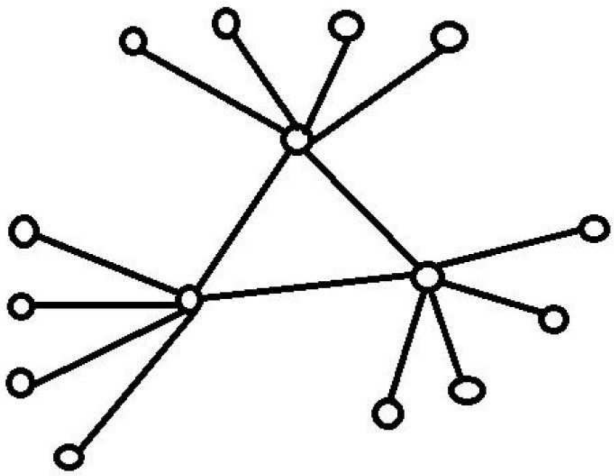

Fig. 8: Cycle of stars.

Under presence of productivity, the corner equilibrium

$$k_1 = \frac{e(1-2a)}{3A-2a}, \quad k_2 = 0,$$

is possible if $A > 1/3$*, and the pure corner equilibrium*

$$k_1 = e, \quad k_2 = 0,$$

is possible if $A \le 1/3$, $2a + \nu A > 1$.

The corner equilibrium

$$k_1 = 0, \quad k_2 = \frac{e(1-2a)}{A-2a},$$

is possible if $A > 1$. *The pure corner equilibrium*

$$k_1 = 0, \quad k_2 = e,$$

is possible if $A \le 1$, $2a + \nu A > 1$.

Besides, under absence of productivity the pure corner equilibrium $k_1 = k_2 = 0$ *is possible.*

In each case, increase in ν *does not influence knowledge and utilities.*

Proof. The types are characterized by vectors $\boldsymbol{l}(1) = (2, \nu)$, $\boldsymbol{l}(2) = (1, 0)$, hence, (22) – (23) turn into

$$k_1^s = \frac{e(1-2a)(A-2a-\nu A)}{(A-2a)(3A-2a)-\nu A^2},$$

$$k_2^s = \frac{2e(1-2a)(A-a)}{(A-2a)(3A-2a)-\nu A^2}.$$

We see that the numerator of the expression for k_1^s is negative, and the numerator of the expression for k_2^s is positive, hence, independently on the sign of the denominator, k_1^s and k_2^s have different signs. Hence, inner equilibrium is impossible.

Let $A < 2a$, $A > 1/3$. If $k_2 = 0$ then the nodes of the cycle receive no externalities from the hanging nodes; as in the usual cycle, $k_1 = e(1-2a)/(3A-2)$. But if $k_1 = e(1-2a)/(3A-2a)$ then $k_2^s = e(1-2a)(2A-2a)/(A-2a) < 0$ and, hence, $k_2 = 0$.

Let $A < 2a$, $A \leq 1/3$, $2a + \nu A > 1$. If $k_2 = 0$, then $k_1 = e$. If $k_1 = e$, then $k_2^s = e(1-2a-\nu A)/(3A-2a) < 0$ and, correspondingly, $k_2 = 0$.

Let $A > 2a$, $A > 1$. If $k_1 = 0$, then $k_2 = e(1-2a)/(A-2a)$. If $k_2 = e(1-2a)/(A-2a)$, then $k_1^s = e(1-2a)[(1-\nu)A-2a]/(3A-2a)(A-2a) < 0$ and, hence, $k_1 = 0$.

Let $A > 2a$, $A \leq 1$, $2a + \nu A > 1$. If $k_1 = 0$, then $k_2 = e$. If $k_2 = e$, then $k_1^s = e(1-2a-\nu A)/(3A-2a) < 0$ and, hence, $k_1 = 0$. □

Proposition 10.2. *Let centers of several stars, each with ν peripheral nodes, being initially in inner equilibrium, be unified into one cycle. Under absence of productivity and $A > 1/3$, knowledge in each node in equilibrium declines, and, moreover, each peripheral nodes becomes passive. Utility in each node declines.*

Under $A > 1$ (which implies presence of productivity), each central nodes becomes passive, while knowledge in each periphery node decreases if $\nu = 1$, and increases if $\nu \geq 2$. Utility in each central node decreases and in each periphery node increases.

Proof. According to (26)–(27), before unification, the equilibrium level of knowledge in each central node was

$$k_1^* = \frac{e(1-2a)[(\nu-1)A+2a]}{\nu A^2 - (A-2a)^2},$$

and in each peripheral node,

$$k_2^* = \frac{2ea(1-2a)}{\nu A^2 - (A-2a)^2}.$$

The environments were

$$K_1^* = \frac{e(1-2a)[(\nu-1)A+2a+2\nu a]}{(\nu-1)A^2+4aA-4a^2},$$

$$K_2^* = \frac{e(1-2a)[(\nu-1)A+4a]}{(\nu-1)A^2+4aA-4a^2}.$$

Under absence of productivity and $A > 1/3$, after unification, the level of knowledge in each central node becomes

$$k_1^{**} = \frac{e(1-2a)}{3A-2a},$$

and in each peripheral node

$$k_2^{**} = 0.$$

The environment in each central node becomes

$$K_1^{**} = \frac{3e(1-2a)}{3A-2a}.$$

We see that $k_1^* > k_1^{**}$, $K_1^* > K_1^{**}$. According to Theorem 2.2, utility in each central node decreases. Evidently, utility in each peripheral node also decreases, since the node becomes passive.

If $A > 1$ (which implies presence of productivity), after unification, the level of knowledge in each central node becomes

$$k_1^{**} = 0,$$

by Proposition 10.1; and in each peripheral node:

$$k_2^{**} = \frac{e(1-2a)}{A-2a}.$$

The environment in each peripheral node becomes

$$K_2^{**} = \frac{e(1-2a)}{A-2a}.$$

We see that $k_2^* > k_2^{**}$ if $\nu = 1$, $k_2^* < k_2^{**}$ if $\nu \geq 2$, $K_2^* < K_2^{**}$. By Theorem 2.2, utility in each peripheral node increases. In each central node, utility decreases. □

Remark 10.1. Strategic complementarity is observed when the stars are unified under absence of productivity. After unification, investments in central nodes decrease, and in peripheral nodes it is not profitable to make investments in equilibrium. As result, the knowledge in the central nodes is the same as if there are no peripheral nodes at all. Strategic substitutability takes place under presence of productivity. After unification, investments of peripheral nodes increase if $\nu \geq 2$ and it is not profitable for central nodes to make investment. As result, in periphery nodes the knowledge is the same as if there are no central nodes.

11. Conclusions

Our model describes situations in which agents in a network make investments of some resource (such as money or time) on the first stage (period 1 in the model), and obtain a gain on the second stage (period 2). Such situations are typical in life of families, communities, firms, countries, international organizations, etc. Thus, the model can have numerous applications in analysis of equilibria in various economic, social and political systems.

In framework of the model, we consider questions which concern relations between network structure, incentives, behavior of the agents, and the equilibrium state of economic or social system in terms of welfare of the agents.

We introduce new concepts and develop techniques which can be used in such kind of analysis. In particular, we provide some results of studying the model, among them results describing consequences of appearance of new links in networks and of adjunctions of components. We introduce the concept of types of nodes, propose classification of networks based on this concept, describe an algorithm of subdivision of networks into types, and demonstrate the role of types in characterizing inner equilibria.

Interesting questions for further research could be relations between different possible concepts of equilibrium, and dynamics of formation of new equilibrium after adjunction of components or after rise of new links.

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