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Quantum Entanglement in a Zero-Sum Game

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Abstract We consider a class of simple games that emphasizes one important aspect of the game of bridge: what a player consisting of two persons (in terms of (von Neumann and Morgenstern, 1953)) can do when the direct communication is prohibited between them, and how they play against their opponent acting under similar circumstances[1]. We find optimal strategies for this class of games and show how the effect of quantum nonlocality can improve the players' performance.
Quantum nonlocality, or quantum entanglement, is widely known in quantum game theory. In some games, the payoff of players properly equipped with entangled quantum bits can be up to exponentially bigger in comparison with ordinary players. However, all known nonlocal games are quite artificial and, besides, they are fully "cooperative": there are no opponents as such, but all players should strive for the same goal. The introduced game favorably differs from them. Firstly, it has been derived from quite natural problem (the game of bridge); secondly, there is an apparent presence of competition in the game (because it is a zero-sum one!); and, finally, its analysis does not require deep understanding of the heavy mathematical formalism of quantum information theory.

1. Introduction

In modern game theory many useful concepts and tools were developed for the analysis of traditional games. As a rule, such developments began with some simplest possible models which represented, however, very fundamental questions. This article is largely inspired by the famous case, one of the earliest in the study of game theory, namely, the analysis of highly simplified version of poker made by von Neumann and Morgenstern in (von Neumann and Morgenstern, 1953). In the "Poker and Bluffing" section, the authors on the basis of a fairly simple model manage to reveal the mathematical sense of bluffing in some games and show its soundness from the point of view of game theory. This brilliant result was a very great first step in developing the mathematical model of poker, which subsequently gained extensive development.

In this article we shall also consider one special aspect of one popular and well-researched card game. It is the game of bridge, in which one of the main challenges for players is coordinated actions of partners in the best possible way, assuming that no one is able to see the cards of the partner. At the same time, the players must resist the opposing pair of players, who are acting in the similar conditions.

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[1] The present paper was first presented at GTM'2014 under the name "Quantum Entanglement Can Help in the Game of Bridge" in order to attract possibly more card players' attention to quantum information processing, but since then the paper (Muhammad et al., 2014) appeared to do the same job in a more efficient manner.

Bridge today is probably the most intellectual game in which element of gambling is virtually minimized. The very serious attitude to this game can be illustrated by the fact that the game of bridge is the only card game being an Olympic sport (along with such samples of intellectual games like chess and go). So, of course, our intention is not to build anything claiming to help to play bridge, but instead we will perform some basic analysis of only one special aspect of this game.

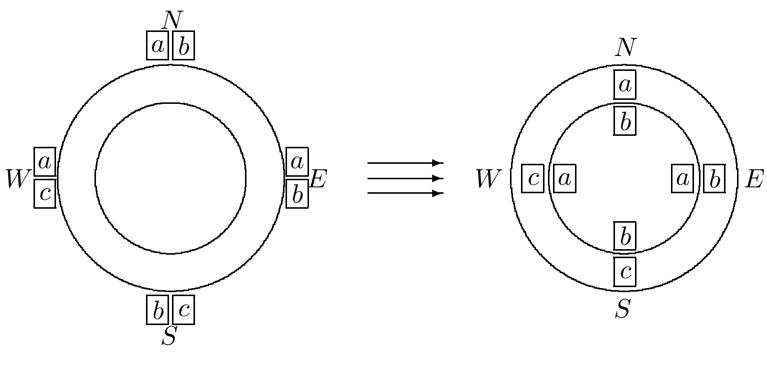
Another source of inspiration for this analysis is the proximity of this problem to the issues that currently, perhaps, are leading in the quantum game theory. We mean the coordination of agents who are prohibited to transfer any signal between them, which is also closely connected with communication complexity of functions and similar topics.

To illustrate the difference in calculating "classical" and "quantum" strategies, we sometimes provide rather bulky formulae that should be taken no more than as illustrations or perhaps, as an auxiliary material for a separate research[2].

2. Formulation of the Rules

Now, consider the following zero-sum card game Γ, which in simplified form illustrates only one, albeit very important, aspect of the game of bridge — namely coordination of a pair of players, assuming that any exchange of information is prohibited between them. Hereafter we call it *Coordination Game*. Just like the game of bridge, this game is played at the same table by two pairs of players (say by pairs N-S and W-E), each pair of players sitting opposite each other. The table is divided into two circles — internal and external. Some set of cards is used for the game, and each player gets two cards from this set at random (of course, each player knows the values of his cards only). Then each player must place one card to the inner circle and the other — to the outer circle. After that, the cards are turned over. If at any circle — internal or external — a card is detected with the value strictly greater than the values of both opponent's cards of the same circle, then that pair which owns this dominant card gets all four cards of the circle; otherwise, no one gets these four cards. If a pair manages to get some cards, then the sum of their values matches their income, which of course is collected from the opposing pair.

Example of play:



[2] Working *Wolfram Mathematica*® script is available upon request by email or at `http://home.lu.lv/~sd80008/bridge/`

Here $a < b < c$ are values of three types of cards used in the game. Hereafter we restrict ourselves to the case of only three types of cards. In addition, for the reason of simplicity we shall assume that neither player receives two cards of the same type. But similar considerations will be correct also for the cases without these restrictions.

In this play there are *W-E*'s cards a, a and *N-S*'s cards b, b at the inner circle; since pair *N-S* has a dominant card b (more precisely, even two such cards), it gets all four cards of the inner circle. At the outer circle there are *W-E*'s cards c, b and *N-S*'s cards a, c; since neither pair has a card which would dominate all the opponents' cards, the outer circle is not captured by anyone. The total result of the play: pair *W-E* pays amount $a + a + b + b$ to pair *N-S*. (Note also that if this amount is not negative, then pair *W-E* ought to regret about their strategy, because if players W and E both had swapped their cards around, they would gain $3b - 3a > 0$.)

3. Search for Optimal Strategies

A player's strategy in this game is just some set of rules according to which the player having received cards i and j, decides where to put the card i and where to put the card j (for each $i < j : i, j \in \{a, b, c\}$). Let us assume the initial hand to consist of the lower card i lying at the inner circle, and the higher card j lying at the outer circle. Then pure strategies of all the four players

$$\begin{aligned} X_N &= (x_{ba}, x_{ca}, x_{cb})\,, \\ X_S &= (x_{ab}, x_{ac}, x_{bc})\,, \\ Y_W &= (y_{ba}, y_{ca}, y_{cb})\,, \\ Y_E &= (y_{ab}, y_{ac}, y_{bc}) \\ &\quad (\text{where} \quad \forall i, j \in \{a, b, c\}\, (i \neq j) : x_{ij}, y_{ij} \in \{0, 1\}) \end{aligned}$$

must determine for each player whether to swap the cards around or leave them as is. For the pairs of players we shall represent pure strategies respectively as

$$\begin{aligned} X &= (x_{ba}, x_{ca}, x_{cb},\; x_{ab}, x_{ac}, x_{bc})\,, \\ Y &= (y_{ba}, y_{ca}, y_{cb},\; y_{ab}, y_{ac}, y_{bc})\,. \end{aligned}$$

We say that, for example, $x_{ba} = 0$ means that player N, having card b at the outer circle and card a at the inner circle, leaves the cards as is. And we say that $x_{ba} = 1$ means that in this case player N shifts the card a to the outer circle, and the card b — to the inner circle.

As in the game of bridge, we shall assume that a pair of players may communicate before the play is started, so that they can develop a common strategy. In particular, this means that the players may have arbitrary shared randomness. Therefore[3], we must consider their mixed strategies of type

$$\widetilde{X} = \sum_{w \in \{0,1\}^6} \xi_w X_w \text{ and}$$

$$\widetilde{Y} = \sum_{w \in \{0,1\}^6} \eta_w Y_w,$$

[3] If the pairs of players had no shared randomness, then their mixed strategies would look like $\displaystyle\sum_{u \in \{0,1\}^3} \sum_{w \in \{0,1\}^3} \xi_{N_u} \xi_{S_w} X_{(u,w)}$ and $\displaystyle\sum_{u \in \{0,1\}^3} \sum_{w \in \{0,1\}^3} \eta_{W_u} \eta_{E_w} Y_{(u,w)}$.

where $X_{w_1w_2w_3w_4w_5w_6} = Y_{w_1w_2w_3w_4w_5w_6} = (w_1, w_2, w_3, \; w_4, w_5, w_6)$, and $\xi_w, \eta_w \geq 0$ are the probabilities of selecting each of the 64 pure strategies: $\sum \xi_w = \sum \eta_w = 1$ (it is easy to see that for a pair of players there are exactly $2^6 = 64$ different pure strategies). Now let us consider what the players can benefit from the shared randomness.

Lemma 1. *In a Coordination Game there always exists an optimal mixed strategy (in the sense of von Neumann and Morgenstern solution or Nash equilibrium), that ends with mutual swap-around of cards by each of the two players with probability 50%. That is, formally speaking, there always exist optimal strategies of type*

$$\widetilde{X}_+ = \sum_{w \in \{0,1\}^6} \xi_w \left(\frac{1}{2} X_w + \frac{1}{2} X_{\overline{w}} \right) \text{ and}$$

$$\widetilde{Y}_+ = \sum_{w \in \{0,1\}^6} \eta_w \left(\frac{1}{2} Y_w + \frac{1}{2} Y_{\overline{w}} \right),$$

where $\overline{w} = (1, 1, 1, 1, 1, 1) - w$.

Proof. Briefly, in this symmetric game it is always useful (at least not harmful) to confuse the opponent by changing the outer circle cards to the inner circle cards and vice versa — with probability 50%.

Now let us give a formal proof of this fact. Suppose some pair, say *N-S* (but due to the symmetry of the game similar arguments hold also for pair *W-E*), has an optimal strategy

$$\widetilde{X}_+ = \sum_{w \in \{0,1\}^6} \xi_w X_w.$$

Let us consider this strategy together with its dual strategy

$$\widetilde{X}_- = \sum_{w \in \{0,1\}^6} \xi_w X_{\overline{w}} \quad \text{(where } \overline{w} = (1, 1, 1, 1, 1, 1) - w\text{)}$$

— so that if $\widetilde{X}_- = \widetilde{X}_+$ then the proof is complete. Next, we are going to show that all the strategies of the form

$$p\widetilde{X}_+ + (1 - p)\,\widetilde{X}_- \quad \text{(where } 0 \leq p \leq 1\text{)}$$

are also optimal (this fact immediately implies the statement of the Lemma). For this we recall first that the optimality criterion for a mixed strategy $\widetilde{X}_{\text{opt}}$ is equality

$$\min_Y \varGamma \left(\widetilde{X}_{\text{opt}}, Y \right) = \max_{\widetilde{X}} \min_Y \varGamma \left(\widetilde{X}, Y \right),$$

where Y belongs to the set of the opposing pair's pure strategies, and $\varGamma \left(\widetilde{X}, Y \right)$ is the payoff for the first pair applying the mixed strategy $\widetilde{X}$ against the opponents' pure strategy Y in the game $\varGamma$.

Suppose that a strategy $\widetilde{X}_-$ is not optimal:

$$\varGamma \left(\widetilde{X}_-, Y_- \right) < \max_{\widetilde{X}} \min_Y \varGamma \left(\widetilde{X}, Y \right) \quad \text{for some opponents' pure strategy } Y_-.$$

Then strategy $\widetilde{X}_+$ also is non-optimal:

$$\Gamma\left(\widetilde{X}_+, Y_+\right) = \Gamma\left(\widetilde{X}_-, Y_-\right) < \max_{\widetilde{X}} \min_{Y} \Gamma\left(\widetilde{X}, Y\right),$$

where Y_+ is strategy's Y_- dual strategy.

This contradicts our assumption about the optimality of $\widetilde{X}_+$, so we conclude that the strategy $\widetilde{X}_-$ necessarily is optimal. And since both

$$\Gamma\left(\widetilde{X}_+, Y'\right) \geq \max_{\widetilde{X}} \min_{Y} \Gamma\left(\widetilde{X}, Y\right) \quad \text{and}$$
$$\Gamma\left(\widetilde{X}_-, Y'\right) \geq \max_{\widetilde{X}} \min_{Y} \Gamma\left(\widetilde{X}, Y\right)$$

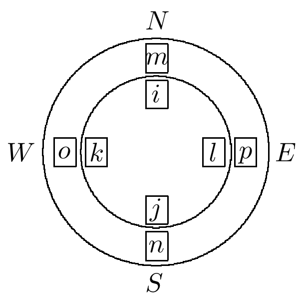
for each opponents' pure strategy Y', it follows that also

$$\begin{aligned} &\Gamma\left(p\widetilde{X}_+ + (1-p)\,\widetilde{X}_-, Y'\right) = \\ &\quad = p\Gamma\left(\widetilde{X}_+, Y'\right) + (1-p)\,\Gamma\left(\widetilde{X}_-, Y'\right) \geq \\ &\qquad \geq \max_{\widetilde{X}} \min_{Y} \Gamma\left(\widetilde{X}, Y\right) \end{aligned}$$

for each opponents' pure strategy Y'. □

We have just shown that for each pair it is worthwhile to swap the cards around mutually — with probability 50%. In order to simplify our further arguments, let us slightly amend the rules of the game. Namely, suppose that the card dealing system itself performs the above-mentioned task instead of the players: sometimes (with probability 50%) it swaps the cards around for a pair of players (i.e. for N-S or W-E).

Now let us temporarily forget about mixed strategies and consider only pure strategies and the results that can be achieved by using them in this new amended game. In order to calculate the payoff of the first pair $\Gamma(X, Y)$ (for some pure strategies X and Y), one should consider all $3^4 = 81$ different cases[4] of the following type:



where
$i, j, k, l, m, n, o, p \in \{a, b, c\}$,
$m > i, \quad j < n,$
$o > k, \quad l < p.$

[4] Since each of the 4 players may receive one of 3 different hands: (a, b), (a, c) or (b, c).

Then the payoff of the first player can be calculated by the formula

$$\Gamma(X,Y) =$$

$$= \frac{1}{81} \sum_{\substack{m>i,\\ j<n,\\ o>k,\\ l<p}} \begin{cases} \frac{Q(i,j,k,l)+Q(m,n,o,p)}{2} + \frac{Q(i,j,o,p)+Q(m,n,k,l)}{2}, & \text{if } x_{mi}=x_{jn}, y_{ok}=y_{lp}; \\ \frac{Q(i,j,k,p)+Q(m,n,l,o)}{2} + \frac{Q(i,j,l,o)+Q(m,n,k,p)}{2}, & \text{if } x_{mi}=x_{jn}, y_{ok}\neq y_{lp}; \\ \frac{Q(i,n,k,l)+Q(j,m,o,p)}{2} + \frac{Q(i,n,o,p)+Q(j,m,k,l)}{2}, & \text{if } x_{mi}\neq x_{jn}, y_{ok}=y_{lp}; \\ \frac{Q(i,n,k,p)+Q(j,m,l,o)}{2} + \frac{Q(i,n,l,o)+Q(j,m,k,p)}{2}, & \text{if } x_{mi}\neq x_{jn}, y_{ok}\neq y_{lp}, \end{cases}$$

where $Q(x_1, x_2, y_1, y_2) = (x_1 + x_2 + y_1 + y_2) \cdot \operatorname{sgn}(\max(x_1, x_2) - \max(y_1, y_2))$. This expression can be rewritten in an unconditional form:

$$\Gamma(X,Y) = \frac{1}{162} \sum_{\substack{m>i,\\ j<n,\\ o>k,\\ l<p}} \begin{aligned} &(Q(i,j,k,l)+Q(m,n,o,p)+Q(i,j,o,p)+Q(m,n,k,l))(1-x_{mi}\oplus x_{jn})(1-y_{ok}\oplus y_{lp}) \\ &+(Q(i,j,k,p)+Q(m,n,l,o)+Q(i,j,l,o)+Q(m,n,k,p))(1-x_{mi}\oplus x_{jn})(y_{ok}\oplus y_{lp}) \\ &+(Q(i,n,k,l)+Q(j,m,o,p)+Q(i,n,o,p)+Q(j,m,k,l))(x_{mi}\oplus x_{jn})(1-y_{ok}\oplus y_{lp}) \\ &+(Q(i,n,k,p)+Q(j,m,l,o)+Q(i,n,l,o)+Q(j,m,k,p))(x_{mi}\oplus x_{jn})(y_{ok}\oplus y_{lp}). \end{aligned} \tag{1}$$

This sum is a fourth degree polynomial of twelve 0-1-valued arguments (x_{ba},x_{ca},x_{cb}, x_{ab},x_{ac},x_{bc}, y_{ba},y_{ca},y_{cb}, y_{ab},y_{ac},y_{bc}), which consists of 180 terms[5].

If we fix two values $x_{ba} = y_{ba} = 0$, this expression can be simplified to a polynomial consisting of only 94 terms, which can be represented as follows:

$$\begin{aligned}
&81\ \Gamma(X,Y) = \\
&(64a+20b+12c)\ (x_{ac}+x_{ca}+y_{ac}y_{ca}-y_{ac}-y_{ca}-x_{ac}x_{ca}) \\
&+(46a-6b)\ (x_{ab}+y_{ab}y_{ca}-y_{ab}-x_{ab}x_{ca}) \\
&+(18a+26b+12c)\ (x_{bc}+x_{cb}+y_{ac}y_{cb}+y_{bc}y_{ca}-y_{bc}-y_{cb}-x_{ac}x_{cb}-x_{bc}x_{ca}) \\
&+(14a-14b)\ (y_{ab}y_{cb}-x_{ab}x_{cb}) \\
&+(4a+40b+12c)\ (y_{bc}y_{cb}-x_{bc}x_{cb}) \\
&+(6a+2b)\ (x_{ac}y_{ab}+x_{bc}y_{ab}+x_{bc}y_{ac}+x_{bc}y_{ca}+x_{ca}y_{ab}+x_{cb}y_{ab}+x_{cb}y_{ac}+x_{cb}y_{ca}+x_{ab}x_{ca}y_{ac} \\
&\quad +x_{ab}x_{ca}y_{bc}+x_{ab}x_{ca}y_{ca}+x_{ab}x_{ca}y_{cb}+x_{ac}x_{ca}y_{bc}+x_{ac}x_{ca}y_{cb}+x_{ab}y_{ac}y_{ca} \\
&\quad +x_{ab}y_{ac}y_{cb}+x_{ab}y_{bc}y_{ca}+x_{ac}y_{ac}y_{cb}+x_{ac}y_{bc}y_{ca}+x_{ca}y_{ac}y_{cb}+x_{ca}y_{bc}y_{ca} \\
&\quad +x_{ac}x_{ca}y_{ab}y_{ca}+x_{ac}x_{cb}y_{ab}y_{ca}+x_{ac}x_{cb}y_{ac}y_{ca}+x_{bc}x_{ca}y_{ab}y_{ca}+x_{bc}x_{ca}y_{ac}y_{ca} \\
&\quad -y_{ac}x_{ab}-y_{bc}x_{ab}-y_{bc}x_{ac}-y_{bc}x_{ca}-y_{ca}x_{ab}-y_{cb}x_{ab}-y_{cb}x_{ac}-y_{cb}x_{ca}-y_{ab}y_{ca}x_{ac} \\
&\quad -y_{ab}y_{ca}x_{bc}-y_{ab}y_{ca}x_{ca}-y_{ab}y_{ca}x_{cb}-y_{ac}y_{ca}x_{bc}-y_{ac}y_{ca}x_{cb}-y_{ab}x_{ac}x_{ca} \\
&\quad -y_{ab}x_{ac}x_{cb}-y_{ab}x_{bc}x_{ca}-y_{ac}x_{ac}x_{cb}-y_{ac}x_{bc}x_{ca}-y_{ca}x_{ac}x_{cb}-y_{ca}x_{bc}x_{ca} \\
&\quad -y_{ac}y_{ca}x_{ab}x_{ca}-y_{ac}y_{cb}x_{ab}x_{ca}-y_{ac}y_{cb}x_{ac}x_{ca}-y_{bc}y_{ca}x_{ab}x_{ca}-y_{bc}y_{ca}x_{ac}x_{ca}) \\
&+(2a-2b)\ (x_{ab}y_{ab}y_{cb}+x_{ac}y_{ab}y_{cb}+x_{ca}y_{ab}y_{cb}+x_{ab}x_{cb}y_{ab}y_{ca}+x_{ab}x_{cb}y_{ac}y_{ca} \\
&\quad -y_{ab}x_{ab}x_{cb}-y_{ac}x_{ab}x_{cb}-y_{ca}x_{ab}x_{cb}-y_{ab}y_{cb}x_{ab}x_{ca}-y_{ab}y_{cb}x_{ac}x_{ca}) \\
&+(4a+4b)\ (x_{ab}y_{bc}y_{cb}+x_{ac}y_{bc}y_{cb}+x_{ca}y_{bc}y_{cb}+x_{bc}x_{cb}y_{ab}y_{ca}+x_{bc}x_{cb}y_{ac}y_{ca} \\
&\quad -y_{ab}x_{bc}x_{cb}-y_{ac}x_{bc}x_{cb}-y_{ca}x_{bc}x_{cb}-y_{bc}y_{cb}x_{ab}x_{ca}-y_{bc}y_{cb}x_{ac}x_{ca}).
\end{aligned} \tag{2}$$

Simplification by fixing $x_{ba} = y_{ba} = 0$ is valid, as in the amended version of the game any dual strategies Z_+ and Z_- are equivalent, and therefore for each strategy with $x_{ba} = 1$ ($y_{ba} = 1$) there is an equivalent dual strategy with $x_{ba} = 0$ ($y_{ba} = 0$).

Thus, it remains to consider only $2^5 = 32$ pure strategies. Their values can be written in the form of a round-robin tournament (by substituting all x_{ij} and y_{ij} of polynomial (2) with appropriate values), i.e. in the form of 32×32 skew-symmetric matrix with elements of type $\alpha a + \beta b + \gamma c$.

It is interesting to note that only five such strategies may be optimal for some values $a < b < c$:

[5] In order to build this polynomial one should substitute all non-linear Boolean expressions of type $a \oplus b$ by appropriate quadratic trinomials: $a \oplus b = |a - b| = (a - b)^2 = a^2 + b^2 - 2ab = a + b - 2ab$.

x_{ba}	x_{ca}	x_{cb}	x_{ab}	x_{ac}	x_{bc}
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	1	1	1
0	0	1	0	0	0
0	1	1	1	0	0

Moreover, we shall not consider strategy X_{001000}, since it is equivalent to strategy X_{000001} up to the permutation of players in the pair. For remaining four potentially optimal strategies X_{000000}, X_{000001}, X_{000111} and X_{011100}, let us write down their results in their plays against all 32 pure strategies $Y_{000000}, \ldots, Y_{011111}$. In Table 1 we present values of $81\,\Gamma(X, Y)$ for corresponding strategies X and Y.

For each of the four columns of Table 1 the following holds: *if one requires each of* 32 *expressions of the column to be non-negative, then the system of inequalities describes some nondegenerate unbounded convex polytope.* And for all other strategies which are not equivalent to the potentially optimal ones, if one wrote down their results in the similar way, then corresponding systems of inequalities would be contradictory (that is, they would have no solution).

System of inequalities for potentially optimal strategies can be further simplified by getting rid of redundant inequalities. In fact, the only meaningful inequalities correspond to the six expressions shown in bold in Table 1. Indeed, if bold-shown expressions in a column are non-negative, then all other expressions also are non-negative (given that $a < b < c$, of course).

It is also easy to notice that all the bold-shown expressions form three symmetric pairs that actually correspond to the three planes in three-dimensional space with coordinates a, b, c. These planes split the unbounded wedge specified by the two inequalities $a < b < c$ into four parts, within each of which one of the four above-mentioned strategies is optimal (and within each of these four planes two strategies are optimal).

All these facts imply the following theorem.

Theorem 1. *In any amended Coordination Game (with cards' values $a < b < c$) there exists an optimal pure strategy.*

We leave this theorem without a formal proof, which would be rather bulky. However, we have already managed to describe essentially all the basic steps of the proof.

In this section we have shown that for any cards' types $a < b < c$ in the amended game there is an optimal pure strategy, so that mixed strategies, about which we temporarily forgot in the middle of the chapter, can be now forgotten forever (as a mixed strategy is optimal only if all its nonzero components are optimal pure strategies). These potentially optimal pure strategies belong to the four disjoint families of mutually equivalent strategies, and in order to determine a family of optimal strategies for given values a, b, c, one only needs to check a few linear inequations (i.e. to find which of Table 1 columns contain(s) non-negative bold-shown expressions).

Table 1:
Results of potentially optimal strategies in a round-robin tournament

	X_{000000}	X_{000001}	X_{000111}	X_{011100}
Y_{000000}	0	$\boldsymbol{18a+26b+12c}$	$128a+40b+24c$	$68a+60b+24c$
Y_{000001}	$\boldsymbol{-18a-26b-12c}$	0	$98a+10b+12c$	$\boldsymbol{44a+32b+12c}$
Y_{000010}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{000011}	$-82a-46b-24c$	$-58a-18b-12c$	$34a-10b$	$-16a+16b$
Y_{000100}	$-46a+6b$	$-22a+34b+12c$	$94a+50b+24c$	$32a+72b+24c$
Y_{000101}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{000110}	$-110a-14b-12c$	$-80a+16b$	$30a+30b+12c$	$-28a+56b+12c$
Y_{000111}	$-128a-40b-24c$	$-98a-10b-12c$	0	$\boldsymbol{-52a+28b}$
Y_{001000}	$-18a-26b-12c$	0	$98a+10b+12c$	$44a+32b+12c$
Y_{001001}	$-32a-12b-12c$	$-14a+14b$	$80a+28b+12c$	$28a+48b+12c$
Y_{001010}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{001011}	$-78a-6b-12c$	$-54a+22b$	$46a+38b+12c$	$-8a+60b+12c$
Y_{001100}	$-50a-34b-12c$	$-26a-6b$	$82a+2b+12c$	$24a+28b+12c$
Y_{001101}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{001110}	$-96a-28b-12c$	$-66a+2b$	$48a+12b+12c$	$-12a+40b+12c$
Y_{001111}	$-110a-14b-12c$	$-80a+16b$	$30a+30b+12c$	$-28a+56b+12c$
Y_{010000}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010001}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010010}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010011}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010100}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010101}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010110}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{010111}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{011000}	$-82a-46b-24c$	$-58a-18b-12c$	$34a-10b$	$-16a+16b$
Y_{011001}	$-78a-6b-12c$	$-54a+22b$	$46a+38b+12c$	$-8a+60b+12c$
Y_{011010}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{011011}	$-60a+20b$	$-36a+48b+12c$	$76a+68b+24c$	$16a+88b+24c$
Y_{011100}	$-68a-60b-24c$	$\boldsymbol{-44a-32b-12c}$	$\boldsymbol{52a-28b}$	0
Y_{011101}	$-64a-20b-12c$	$-40a+8b$	$64a+20b+12c$	$8a+44b+12c$
Y_{011110}	$-50a-34b-12c$	$-26a-6b$	$82a+2b+12c$	$24a+28b+12c$
Y_{011111}	$-46a+6b$	$-22a+34b+12c$	$94a+50b+24c$	$32a+72b+24c$

4. Quantum Strategies

Now let us turn to the notion of quantum strategy. In the theory of nonlocal quantum games one traditionally considers the case when the players before the start of the play can share not only arbitrary classical information (thus defining their strategies and creating some shared randomness), but they are allowed also to share some amount of quantum information.

The easiest comprehensible case of quantum nonlocality is traditionally described by the famous CHSH game (Clauser et al., 1969), where the players (say N and S) before the start of the play create an EPR pair — a pair of entangled qubits in the Bell state[6]

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle_N \otimes |0\rangle_S + \frac{1}{\sqrt{2}}|1\rangle_N \otimes |1\rangle_S .$$

Then, during the play, not being able to communicate with each other, they perform measurements — each player measures his qubit in some special way, according to the current position in the game. As a result of the measurement each player receives one bit of information (equiprobably 0 or 1). This information can be called *quantum shared randomness.*

If prior to the measurement one of the players turns his qubit by some angle α and the other — by some angle β, then with probability $\cos^2 \frac{\alpha-\beta}{2}$ the results of their measurements will coincide, and with probability $\sin^2 \frac{\alpha-\beta}{2}$ the results will be different. More precisely, these probabilities can be represented in the following table:

Table 2: Results of measuring an EPR pair

N's result	S's result	Probability
0	0	$\frac{1}{2}\cos^2 \dfrac{\alpha-\beta}{2}$
0	1	$\frac{1}{2}\sin^2 \dfrac{\alpha-\beta}{2}$
1	0	$\frac{1}{2}\sin^2 \dfrac{\alpha-\beta}{2}$
1	1	$\frac{1}{2}\cos^2 \dfrac{\alpha-\beta}{2}$

By manipulating with angles' values α and β in CHSH game, players can increase the probability of winning from 75% to as much as $\dfrac{\cos^2 \frac{\pi}{4} + 1}{2} = 85.3553\ldots\%$. Quantum players can have up to exponentially bigger advantage comparing to classical players in some games of this kind (Mermin, 1990, Ardehali, 1992, Ambainis et al., 2012a, Briët and Vidick, 2012, Ambainis et al., 2012b). Similar technique can be applied also in the amended *Coordination Game*. Since this game, exactly as CHSH game, requires each player to make only a one-bit move (to swap one's cards around or not to swap), it is sufficient to have only one qubit per player (i.e. one EPR pair per pair of players).

[6] For example, it may be a pair of photons in the zero-spin state (so called *singlet*), as it was performed in a number of physical experiments.

We assume that the result of the measurement fully determines the further actions of a player: a player swaps his cards around when measured 1, and leaves them as is when measured 0.[7] Thus, in the amended game quantum strategy won't be from the binary vector space $\{0,1\}^6$ but from $(-\pi;\pi]^6$.

The formula for calculating the expected value in a play between quantum strategies $\widehat{X} = (X_{ba}, X_{ca}, X_{cb}, X_{ab}, X_{ac}, X_{bc})$ and $\widehat{Y} = (Y_{ba}, Y_{ca}, Y_{cb}, Y_{ab}, Y_{ac}, Y_{bc})$ looks, obviously, as follows:

$$\Gamma\left(\widehat{X},\widehat{Y}\right) = \frac{1}{162} \sum_{\substack{m>i,\\ j<n,\\ o>k,\\ l<p}} \begin{matrix} (Q(i,j,k,l)+Q(m,n,o,p)+Q(i,j,o,p)+Q(m,n,k,l))\cos^2\frac{x_{mi}-x_{jn}}{2}\cos^2\frac{y_{ok}-y_{lp}}{2} \\ +(Q(i,j,k,p)+Q(m,n,l,o)+Q(i,j,l,o)+Q(m,n,k,p))\cos^2\frac{x_{mi}-x_{jn}}{2}\sin^2\frac{y_{ok}-y_{lp}}{2} \\ +(Q(i,n,k,l)+Q(j,m,o,p)+Q(i,n,o,p)+Q(j,m,k,l))\sin^2\frac{x_{mi}-x_{jn}}{2}\cos^2\frac{y_{ok}-y_{lp}}{2} \\ +(Q(i,n,k,p)+Q(j,m,l,o)+Q(i,n,l,o)+Q(j,m,k,p))\sin^2\frac{x_{mi}-x_{jn}}{2}\sin^2\frac{y_{ok}-y_{lp}}{2}, \end{matrix} \tag{3}$$

where $Q(x_1, x_2, y_1, y_2) = (x_1 + x_2 + y_1 + y_2) \cdot \operatorname{sgn}(\max(x_1, x_2) - \max(y_1, y_2))$.

A classical strategy $Z = (z_{ba}, z_{ca}, z_{cb}, z_{ab}, z_{ac}, z_{bc})$ in the amended version of the game will be equivalent to quantum strategy $\widehat{Z} = \pi Z$, so the set of quantum strategies can be seen as superset of the set of classical strategies. Indeed, for $\alpha, \beta \in \{0, \pi\}$ the following equalities hold:

$$\sin^2\frac{\alpha-\beta}{2} = \left(\frac{\alpha}{\pi}\right) \oplus \left(\frac{\beta}{\pi}\right) \qquad \text{and}$$
$$\cos^2\frac{\alpha-\beta}{2} = 1 - \left(\frac{\alpha}{\pi}\right) \oplus \left(\frac{\beta}{\pi}\right),$$

so that formula (3) is equivalent to the formula (1).

Just like we did it for a classical strategy, we can fix $x_{ba} = y_{ba} = 0$ also for a quantum one. This simplification is valid, because adding some value (say, $-x_{ba}$ or $-y_{ba}$) to all angles of the strategy doesn't change the difference of any pair of the angles. Formally speaking, a quantum strategy

$$\widehat{Z} = (z_{ba}, z_{ca}, z_{cb}, z_{ab}, z_{ac}, z_{bc})$$

is equivalent to strategy

$$\widehat{Z}_0 = (0, z_{ca} - z_{ba}, z_{cb} - z_{ba}, z_{ab} - z_{ba}, z_{ac} - z_{ba}, z_{bc} - z_{ba}). \tag{4}$$

Taking this fact into account, one can regroup and simplifty the sum (3) to the following function:

[7] In fact, this kind of behavior is the only one that exploits the quantum nonlocality effect in the most efficient way; any probabilistic deviation from this scheme drives the efficiency of a quantum strategy back to the efficiency of ordinary classical mixed strategies.

$$
\begin{aligned}
&162\ \Gamma\left(\widehat{X},\widehat{Y}\right)= \\
&(64a+20b+12c)\ (\cos^2\tfrac{y_{ca}-y_{ac}}{2}-\cos^2\tfrac{x_{ca}-x_{ac}}{2}) \\
&+(12a+48b+12c)\ (\cos^2\tfrac{y_{cb}-y_{bc}}{2}-\cos^2\tfrac{x_{cb}-x_{bc}}{2}) \\
&+(16a+8b)\ (\cos^2\tfrac{y_{ab}}{2}-\cos^2\tfrac{x_{ab}}{2}) \\
&+(34a-10b)\ (\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2}-\cos^2\tfrac{x_{ac}}{2}-\cos^2\tfrac{x_{ca}-x_{ab}}{2}) \\
&+(30a+30b+12c)\ (\cos^2\tfrac{y_{ca}-y_{bc}}{2}+\cos^2\tfrac{y_{cb}-y_{ac}}{2}-\cos^2\tfrac{x_{ca}-x_{bc}}{2}-\cos^2\tfrac{x_{cb}-x_{ac}}{2}) \\
&+(18a-18b)\ (\cos^2\tfrac{y_{bc}}{2}+\cos^2\tfrac{y_{cb}-y_{ab}}{2}-\cos^2\tfrac{x_{bc}}{2}-\cos^2\tfrac{x_{cb}-x_{ab}}{2}) \\
&+4a\ (\cos^2\tfrac{x_{ca}-x_{ac}}{2}\cos^2\tfrac{y_{ab}}{2}-\cos^2\tfrac{y_{ca}-y_{ac}}{2}\cos^2\tfrac{x_{ab}}{2}) \\
&+(2a+2b)\ (\ \cos^2\tfrac{x_{cb}-x_{bc}}{2}(\cos^2\tfrac{y_{ab}}{2}+\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2}+\cos^2\tfrac{y_{ca}-y_{ac}}{2}) \\
&\qquad -\cos^2\tfrac{y_{cb}-y_{bc}}{2}(\cos^2\tfrac{x_{ab}}{2}+\cos^2\tfrac{x_{ac}}{2}+\cos^2\tfrac{x_{ca}-x_{ab}}{2}+\cos^2\tfrac{x_{ca}-x_{ac}}{2})) \\
&+(3a+b)\ (\ (\cos^2\tfrac{x_{ca}-x_{bc}}{2}+\cos^2\tfrac{x_{cb}-x_{ac}}{2})(\cos^2\tfrac{y_{ab}}{2}+\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2}+\cos^2\tfrac{y_{ca}-y_{ac}}{2}) \\
&\qquad -(\cos^2\tfrac{y_{ca}-y_{bc}}{2}+\cos^2\tfrac{y_{cb}-y_{ac}}{2})(\cos^2\tfrac{x_{ab}}{2}+\cos^2\tfrac{x_{ac}}{2}+\cos^2\tfrac{x_{ca}-x_{ab}}{2}+\cos^2\tfrac{x_{ca}-x_{ac}}{2}) \\
&\qquad +\cos^2\tfrac{x_{ca}-x_{ac}}{2}(\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2})-\cos^2\tfrac{y_{ca}-y_{ac}}{2}(\cos^2\tfrac{x_{ac}}{2}+\cos^2\tfrac{x_{ca}-x_{ab}}{2})) \\
&+(a-b)\ (\ (\cos^2\tfrac{x_{bc}}{2}+\cos^2\tfrac{x_{cb}-x_{ab}}{2})(\cos^2\tfrac{y_{ab}}{2}+\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2}+\cos^2\tfrac{y_{ca}-y_{ac}}{2}) \\
&\qquad -(\cos^2\tfrac{y_{bc}}{2}+\cos^2\tfrac{y_{cb}-y_{ab}}{2})(\cos^2\tfrac{x_{ab}}{2}+\cos^2\tfrac{x_{ac}}{2}+\cos^2\tfrac{x_{ca}-x_{ab}}{2}+\cos^2\tfrac{x_{ca}-x_{ac}}{2}) \\
&\qquad +\cos^2\tfrac{y_{ab}}{2}(\cos^2\tfrac{x_{ac}}{2}+\cos^2\tfrac{x_{ca}-x_{ab}}{2})-\cos^2\tfrac{x_{ab}}{2}(\cos^2\tfrac{y_{ac}}{2}+\cos^2\tfrac{y_{ca}-y_{ab}}{2}))
\end{aligned}
\tag{5}
$$

Unfortunately, it is hardly possible to do anything with such expression without numerical methods.

5. Quantum Players Outperform Classical Players

In Section 3. we described the method of finding an optimal classical strategy for the entire three-dimensional space of values a, b, c. In this section we shall only be interested in its small subspace, namely in games with parameters $(a, b, c) = (-n, n, n+1)$.

We can now apply our newly acquired method and easily conclude that for all these games exactly two dual pure strategies are optimal:

$$X_+ = (0,1,1,1,0,0) \qquad \text{and} \qquad X_- = (1,0,0,0,1,1)\,,$$

because both bold expressions in the last column of the Table 1 is strictly greater than zero, so that strategies X_+ and X_- strictly dominate all other 62 strategies[8]. Of course, they also strictly dominate any mixed strategy with at least one positive non-optimal component.

The exact results of this strategy against all other strategies can also be derived from the expressions of the last column of Table1.

In Section 5. we justified the matching of any classical strategy of some quantum strategy (where quantum entanglement is used as a source of ordinary classical shared randomness). As we noted at the end of that section, one should use numerical methods in order to find an optimal quantum strategy. Of course, an optimal quantum strategy cannot be worse than the optimal classical strategy (as classical strategies are only a finite subset of the continuous six-dimensional space of quantum strategies). Could it be better? Is it pure or mixed? And if it is mixed, then how many nonzero components does it have?

Numerical optimization answers that for $(a, b, c) = (-n, n, n+1)$ there is essentially one pure optimal strategy.

[8] We also note that the line $(a, b, c) = (-n, n, n+1)$ passes fairly close to the border with that part of the entire space of games, in which the optimal strategies are $X = (0,0,0,0,0,1)$ and the three equivalent ones.

But let us first consider a nearly optimal strategy

$$\widehat{X} = \left(0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, 0, \frac{-\pi}{3}\right).$$

By means of the formula (2) we can find its results against any classical strategy. As we have already deduced, it is sufficient to consider only 32 out of 64 pure classical strategies. We represent these results in Table 3.

Table 3:
Results of strategy $\widehat{X}=\left(0,\frac{\pi}{3},\frac{2\pi}{3},\frac{\pi}{3},0,\frac{-\pi}{3}\right)$ against pure classical strategies

Y	$162\ \Gamma\left(\widehat{X},Y\right)$	Y	$162\ \Gamma\left(\widehat{X},Y\right)$	Y	$162\ \Gamma\left(\widehat{X},Y\right)$	Y	$162\ \Gamma\left(\widehat{X},Y\right)$
Y_{000000}	$50n{+}33$	Y_{001000}	$12n{+}9$	Y_{010000}	$108n{+}9$	Y_{011000}	$70n{-}15$
Y_{000001}	$12n{+}9$	Y_{001001}	$70n{+}9$	Y_{010001}	$108n{+}9$	Y_{011001}	$166n{+}9$
Y_{000010}	$108n{+}9$	Y_{001010}	$108n{+}9$	Y_{010010}	$108n{+}9$	Y_{011010}	$108n{+}9$
Y_{000011}	$70n{-}15$	Y_{001011}	$166n{+}9$	Y_{010011}	$108n{+}9$	Y_{011011}	$204n{+}33$
Y_{000100}	$146n{+}33$	Y_{001100}	$50n{+}9$	Y_{010100}	$108n{+}9$	Y_{011100}	$12n{-}15$
Y_{000101}	$108n{+}9$	Y_{001101}	$108n{+}9$	Y_{010101}	$108n{+}9$	Y_{011101}	$108n{+}9$
Y_{000110}	$204n{+}9$	Y_{001110}	$146n{+}9$	Y_{010110}	$108n{+}9$	Y_{011110}	$50n{+}9$
Y_{000111}	$166n{-}15$	Y_{001111}	$204n{+}9$	Y_{010111}	$108n{+}9$	Y_{011111}	$146n{+}33$

As one can see, the best classical strategy Y_{011100} loses only $12n-15$ to quantum strategy $\widehat{X}$. Additionally, there are asymptotically best classical strategy Y_{000001} and equivalent ones[9], which lose $12n+9$. So that we can conclude that good strategies work better than others, but still not as good as quantum strategies do.

The precize results of numerical optimization are as follows. For $n \to \infty$ there is exactly one optimal pure quantum strategy of type (4):

$$\widehat{X}_{\text{opt}} = (\begin{array}{r} 0, \\ 0.324347\ldots\pi, \\ 0.661239\ldots\pi, \\ 0.361300\ldots\pi, \\ 0.036953\ldots\pi, \\ -0.299938\ldots\pi \end{array})$$

Its results against the two best classical strategies are as follows:

$$\Gamma\left(\widehat{X}_{\text{opt}}, Y_{000001}\right) = 0.076\ldots n + 0.042\ldots$$
$$\Gamma\left(\widehat{X}_{\text{opt}}, Y_{011100}\right) = 0.084\ldots n - 0.106\ldots$$

To conclude, we note a few facts concerning the considered class of games:

- optimal quantum strategy dominates all the classical strategies (for $n \geq 2$);
- although classical strategy Y_{011100} dominates Y_{000001}, the latter looses less when played against $\widehat{X}_{\text{opt}}$;

[9] Recall that line $(a, b, c) = (-n, n, n+1)$ passes near that region where they are optimal.

– quantum-over-classical advantage may reach at least about 7.5% of the average absolute value of a card;
– that is, quantum entanglement in *Coordination Game* in some cases helps better than any classical shared randomness.

6. Some Open Questions

In the quantum games theory, there are several main directions. Let us shortly describe some most important of them.

— 1. The coin-flipping games were considered beginning with the works (Meyer, 1999, Ambainis, 2002, Kitaev, 2002). In these games players are able to send quantum information to each other. Or, as it is considered in (Jain and Watrous, 2009), players deal with quantum information by sending qubits to a referee. Such games are usually criticized for the fact that they generally aren't comparable to any classical game.

— 2. In a number of works based on the initial idea of (Clauser et al., 1969) (among which we must highlight (Cleve et al., 2004)), nonlocal games are considered. This is probably the most famous type of quantum games, because they force to reconsider some of concepts of cryptography, communication complexity, etc. However, from the game-theoretical point of view, they seem to be not that interesting. Firstly, because their rules are very artificial. And secondly, they are in some sense strategically degenerate games: although they have variations for arbitrary number of players, the players must all have the same goal. They usually are called cooperative, but in terms of traditional game theory they should be viewed as only one-player games with incomplete information.

— 3. Another direction of quantum games originates from (Eisert et al., 1999). Among quite few results in this area let us mention some applied aspects of quantum game theory are discussed in (Dahl and Landsburg, 2011), where the authors consider an example of economic game with quantum strategies. Work (Zhang, 2011) provides with the systematic study of some important classes of quantum games. These articles, among other things, discuss the role of quantum entanglement in correlated equilibria of some bimatrix games. Unfortunately, the initial idea described in (Eisert et al., 1999) was immediately criticized in (Benjamin and Hayden, 2001a, Benjamin and Hayden, 2001b, van Enk and Pike, 2002) and remained without proper attention for a long time. Only in recent years, this model has been to some extent rehabilitated in the works (Zhang, 2011) and (Kravchenko, 2013), where it was slightly modified (in two different ways) so as to satisfy the basic arguments of the criticism.

In this article we were one of the first to consider a two-player zero-sum game, which has a straightforward classical analogue. Oddly enough, the basic case of game theory — a two-player zero-sum game — has not been quantized for so long time.

This *Coordination Game* clearly doesn't belong to the first class of quantum games ("penny-flipping"), as it doesn't assume any transfer of quantum information during the play. Instead it can be properly attributed to the second class of quantum games ("nonlocal"), since it presents exploiting quantum nonlocality effect. But actually it combines the advantages of both the second and the third classes of quantum games. On the one hand, the correspondence between classical and quantum versions of the game is undisputable, and on the other hand it models a real conflict situation, which essentially involves two players with opposing interests.

In this regard, the following general questions arise.

What are games, for which it is possible to find an interesting quantum analogue? Obviously, the *Coordination Game* can be somehow generalized and then successfully quantized as well — so as to demonstrate a significant difference between the classical and the quantum version of the game. But for now this game models only one specific aspect of the game of bridge. **Are there some other aspects of games for which exploiting of quantum effects can lead to different results?**

There is yet another very important aspect in the game of bridge — the so-called *signaling* within a pair of players. It is considered both impossible and unwise to transfer all the information about one's cards to a partner (unwise, because the opposing pair will be able to use this information in their favor). **What could be optimal classical strategies given that some exchange of information is allowed among partners? What could be the effect of exploiting quantum correlation for transferring incomplete and possibly not perfectly accurate classical information?**

Some questions arise also regarding the considered *Coordination Game*. In this game the set of all classical strategies are totally ordered under dominance (i.e. weak dominance relations are antisymmetric, transitive and total). One implication of this fact is the existence of optimal pure strategies, which appeared in this paper as Theorem 1. **Does similar statement hold also for the quantum version of the game? Are there quantum games, where all optimal quantum strategies are mixed or even totally mixed?** In nonlocal games quantum entanglement seems to be sufficient for all purposes of classical shared randomness, so that for this class of games the answer likely is negative. But does the same hold also for other classes of games? Note that this issue has already arisen in connection with criticism of EWL scheme in (Benjamin and Hayden, 2001a), and for that specific case it was proved that no optimal pure quantum strategy exists. It would be interesting to understand whether that case is exceptional or a typical one.

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