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Stochastic Cooperative Games Application to the Analysis of Economic Agent's Interaction

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Abstract The article deals with the development of the theory of stochastic cooperative games and also possible future directions of practical applications of this class of games are considered. The principal feature of the proposed approach to stochastic cooperative games is that it is based on the definition of the imputation as a vector, which provides the conditions of individual and group rationality with a certain (given) α the probability. Unlike previous approaches, that consider imputation in stochastic cooperative game as "fixed" proportions, our view is to consider total utility of the coalition as a random variable, distributed among its participants. This approach introduces the concept of $\alpha-$ core games and consideres a number of problems that can be formulated with respect to the properties of this object.

1. Introduction

Among the most actual and significant problems which the modern economic science faces, the problem of modeling and research of the cooperative effects arising in case of interaction of the economic actors possessing diverse and uncoordinated systems of interests can be called.

It is simple to notice those common features and regularities by which the coalition behavior of economic actors in situations which in the substantial plan are rather far from each other is characterized. In particular, the reasons proving achievement of arrangements between certain people have many common features with arguments owing to which there are agreements between huge corporations, regional associations or parliamentary parties.

These circumstances determine efficiency and fruitfulness of application in modern economic researches of methods of the theory of cooperative games. Actively developing, since the beginning of the 50-th years of the XX century, they found broad application in the most various spheres.

Quite often at the level of daily reasonings it is possible to meet doubts concerning feasibility of use of so difficult mathematical apparatus what the theory of cooperative games, to the analysis of specific economic situations is. However facing processes and the phenomena to which disproportionate and spasmodic changes in the utility of their subjects are objectively peculiar, we with inevitability come to conclusion about lack of alternative of methods of the theory of cooperative games.

As examples potential spheres of application of game-theoretic cooperative models it is possible in addition to such obvious examples as cooperation of large financial players in case of implementation of joint investment projects, to call interaction of subjects of leasing transactions or implementation of socially significant projects in the media sphere.

At the same time it should be noted those basic difficulties which noticeably complicate processes of practical implementation of the mathematical models constructed on the device of cooperative games. Among them the key place is taken by a problem of creation of characteristic function. In practice it is the extremely difficult to characterize effect (consequences) from consolidation of players in the coalition. Especially when it is about hypothetical estimates of the coalitions which are never realized in practice. The assumption about possibility of the description of potential prizes of the potential coalitions from expected project participants by means of the determined sizes looks very disputable. And attractiveness of the hypothesis of their accidental nature is represented much more realistic. However thus we lose possibility of direct use of tools of classical cooperative games. Transition to tools of stochastic cooperative games becomes one of possible methods of overcoming of this difficulty.

2. Literature review

The term stochastic cooperative games has a long history in professional game-theoretic literature. However, it should be emphasized that various authors define it differently.

Two of the first papers devoted to stochastic cooperative games were Charnes and Granot (1973, 1977). They introduced the hypothesis that the values of the characteristic function of a cooperative game are random variables, and proposed a two-step procedure for constructing the imputation of the income distribution of a full coalition. To begin with, the so-called fair payoffs are formed. Typically, these distributions are interpreted as market expectations of players. In the second stage occurs a posteriori adjustment of promised values of imputation in accordance with the factual imputation of the realized value of payoff. The focuses of these papers are devoted to the tasks of correcting the imputations in the second stage.

Another essential step in the development of the theory of stochastic cooperative games is based on a series of publications in the late 1990s, by the author Jeroen Suijs (Suijs and Borm, 1999; Suijs, 1999a; Suijs et al., 1995; Suijs et al., 1999; Suijs et al., 1998). The main difference between Suijs and Charnes and Granot was the introduction of assumptions about how to set preferences in relation to stochastic payoffs for all players; as well as an extended version of the game model which provides a choice of different options for coalitions.

A problem arising from both Suijs and Charnes and Granot was the construction of analogues for the superadditivity and convexity concepts for stochastic games. Problems also arose in regards to its core and the definition of the conditions of its existence. The authors introduced the concept of certainty equivalents for stochastic games and formulated statements regarding the relationship of superadditivity, convexity, and non-emptiness in the core of the original game and the game-deterministic analog. The dissemination of the concept of nucleolus to stochastic cooperative games has been developed in the later works of Suijs (1999a, 1999b).

In some papers (Suijs et al., 1998) practical aspects of stochastic cooperative games were considered and stochastic cooperative games were applied to problems of insurance and reinsurance. As an example in the later works of Jeroen Suijs and his coauthors (Gao et al., 2008) is devoted to the spread of the classical concept of the Shapley value to stochastic cooperative games.

Another relatively independent and intensively developing feature of game-theoretic studies is the concept of differential games with stochastic parameters. This aspect of stochastic cooperative games has been explored by Yeung, Petrosyan and Zenkevich (Yeung and Petrosyan, 2006; Yeung and Petrosyan 2004; Zenkevich and

Kolabutin 2007). This theory is best described as the development of the theory of differential games with their integration into their stochastic parameters. Mathematical models of these specific types of practical applications, such as the formalization of the dynamic process management of joint ventures, where the results of this joint venture are subject to further distribution between independent economic entities. The technology change in this joint venture operation is described by the Ito differential equation.

A common feature of the Yeung, Petrosyan, and Zenkevich studies (Yeung and Petrosyan, 2006; Yeung and Petrosyan, 2004; Zenkevich and Kolabutin, 2007) are that they determine the imputation and concepts of stochastic games, and show how they are guided by the values of the expectations of utilities or the values of characteristic function. With this approach, the parameters which were originally declared random at a very early stage of the analysis are replaced by their nonrandom substitutes. This undoubtedly distorts the objective stochastic nature of the simulated situation. In this paper, we will develop an alternative concept, suggesting directly binding definitions and sharing solutions to probabilistic characteristics of the random parameters of the game, not to their expectations.

3. Stochastic cooperative games, imputations in stochastic cooperative games

The baseline definition and parameters of a *stochastic cooperative game* (SCG) is a pair of sets Γ=(I, $\tilde{v}$) where

- I= {$1,\ldots,m$} - set of participants
- $\tilde{v}(S)$ - random variables with known density functions $p_{\tilde{v}(S)}$, which are interpreted as income (utility, payoffs), and are received by the corresponding coalitions $S \subset I$.

Consider in more detail issues relating to the approach to defining of imputation in stochastic cooperative games *(I,v)*. In usual non-stochastic games the imputation refers to vector $x \in R^m$, where $m = |I|$ satisfies the following conditions:

- individual rationality

$$(\forall i \in I)\; x_i \geq v_i \tag{1}$$

- group rationality

$$\sum_{i=1}^{m} x_i = v(I) \tag{2}$$

One possible approach to the definition of the imputation concepts in stochastic cooperative games is built on the principle that fulfillment of the analogues of conditions (1) and (2) with the probability (Zuofeng et al., 2008). Later in the stochastic cooperative game, imputation is a vector $\boldsymbol{x}(\alpha) \in R^n$ satisfying:

$$(\forall i \in I)\; \mathbf{P}\{x_i(\alpha) \geq \tilde{v}(i)\} \geq \alpha \tag{3}$$

- stochastic analog of individual rationality (1),

$$\mathbf{P}\{\sum_{i=1}^{m} x_i(\alpha)\} \geq \alpha \tag{4}$$

– stochastic analog of group rationality (2).

Note that condition (3) essentially means that the imputation $\boldsymbol{x}(\alpha)$ ascribed to the i -player, with a probability not less than α, should be greater than the value of the random variable of the player's individual win. In (3), the i-th value and the imputation vector $\boldsymbol{x}(\alpha)$ is compared with the α-quintile of $F_{\tilde{v}(i)}(x)$ – the distribution function of the random variable $\tilde{v}(i)$. For further compactness, we introduce the following notation

$$v_{(\alpha)}(i) = F^{-1}_{\tilde{v}(i)}(\alpha) \tag{5}$$

– for player i and

$$v_{(\alpha)}(S) = F^{-1}_{\tilde{v}(S)}(\alpha) \tag{6}$$

– for coalition $S \subset I$.

Then condition (3) can be rewritten as

$$(\forall i \in I)\ x_i(\alpha) \geq v_\alpha(i). \tag{7}$$

The transformation of condition (3) to (7) can be justified on the basis of the properties of non-decreasing distribution functions. Indeed, the condition $x_i(\alpha) \geq \tilde{v}(i)$ holds that the probability for level will be carried out for all $\alpha' > \alpha$.

In classical cooperative games, group rationality condition (2) fills the need for full utility distribution for a large or full coalition within the imputation. A modification of the stochastic game (4) means that the large coalition is able to win with a probability of not less than α to ensure the implementation of the imputation. Note that condition (4) is equivalent to

$$\mathbf{P}\{\sum_{i=1}^{m} x_i(\alpha) \geq \tilde{v}(I)\} \geq 1 - \alpha. \tag{8}$$

From (8) we get the result $\sum_{i=1}^{m} x_i(\alpha) \leq \tilde{v}_{1-\alpha}(I)$, if we mark as $v_\alpha(I) = F^{-1}_{\tilde{v}(I)}(\alpha)$ the α -quintile of the $F_{\tilde{v}(I)}(x)$ distribution function.

These alterations can lead to significant differences within stochastic games. If in the conventional cooperative games group rationality condition is defined as strict equality, and thus defines a hyperplane in $m-$dimensional space; the approach proposed here takes the form of inequality and defines a half-space in an $m-$dimensional space. Thus, the nature of $\mathbf{x}$ vectors satisfying the definition of (3) and (4) differs significantly from the nature of imputations in their classical interpretation. Sometimes in order to classify such objects, the term allocation is used.

Separately, we note that the group rationality formulating strict equality provides TU-cooperative games with a set of positive properties that greatly simplifies the process of their analysis as a set in R^n. However, it is impossible not to recognize that the adoption of this assumption significantly changes the properties of the simulated real objects for which the condition less or equal is definitely more appropriate. These theses are a weighty argument in favor of developing our approach to the definition of group rationality.

As a result, the system conditions which determine the imputation in a stochastic game, take the following forms:

$$(\forall i \in I)\ x_i(\alpha) \geq v_\alpha(i), \tag{9}$$

$$\sum_{i=1}^{m} x_i(\alpha) \leq v_{1-\alpha}(I).$$

The naming of variables in modern risk management has steadily entrenched the term value at risk (VaR); best explained in the articles Jorion (2006a, 2006b). Thus, among the advantages of approaches (9) and (10) to the definition of the concepts of stochastic imputations in cooperative games can be attributed the fact that it connects the values of imputation with the values of the VaR in the random parameters of the game. This potentially opens up opportunities for meaningful interpretation of the subsequent results of studies and the properties of this class of games and the concepts which determine their outcomes.

For example, in the studies that develop the theory of cooperative games according to Sujis interpretation originally introduced the concept of allocation.

Under the distribution of payoff $\tilde{v}(S)$ of the random coalition Sproduces the vector, $(\mathbf{d}, \mathbf{r}) \in R^{|S|} \times R^{|S|}$ for example

$$\sum_{i \in S} d_i \leq 0;\ (\forall i \in S) \sum_{i \in S} r_i = 1,\ r_i \geq 0.$$

When the first player wins in accordance with the regulations of the imputation $(\mathbf{d}, \mathbf{r})$ correlates to $d_i + \tilde{v}(S) \cdot r_i$. Accordingly, the terms of individual rationality (for a coalition S) are formulated as

$$(\forall i \in S)\ d_i\ + \tilde{v}(S) \cdot r_i \geq \tilde{v}_i. \tag{10}$$

With this definition, the utility that the imputation promises to $i-$th player is a random variable. It consists of two terms: $d_i \leq 0$ – a priori determined absolute values and $r_i \cdot \tilde{v}(S)$ – regulation component, which is determined as a share of factual coalition utility. The condition provides the distribution of this amount without a rest.

Value d_i defines the preliminary rules of distribution of expected utility between the players. The coefficient r_i determines the mechanism of a posteriori redistribution (taking into account actually achieved values). In other words, imputation in this approach promises a player a fixed share from an unknown or random result. Or vice versa, definitions (3) and (4) assume the initial announcement of a nonrandom absolute value of utility or payoff, which is received by the player with the specified level of probability.

Important specific feature of stochastic cooperative games has substantial modification superadditivity concepts in them. For conventional or non-stochastic TU-cooperative games, a situation in which the union of the two coalitions S and T leads only to a simple summation of their utilities.

$$v(S \cup T) = v(S) + v(T).$$

This appears trivial, and the association looks meaningless. At the same time the stochastic game provides a similar amount

$$\tilde{v}^{+}(S \cup T) = \tilde{v}(S) + \tilde{v}(T),$$

which is also a random variable. So, the general equation balances out as

$$\tilde{v}_{\alpha}^{+}(S \cup T) \neq v_{\alpha}(S) + v_{\alpha}(T).$$

Note the studies of the relationship between patterns of a quintile sum of random variables and sums of quintiles diverge in relation to probability theory, not game theory (Watson et al., 1986, Liu et al., 1989).

Thus, in stochastic cooperative games even simple inter-coalitional agreements on the summation of income (utility) can bring additional effects. In connection with the circumstances in this class of games it makes sense to distinguish between two types of utility coalition of associations:

- the usefulness of combining coalitions S and T into the coalition $S \cup T$ as a new random variable $\tilde{v}(S \cup T)$ with the distribution function $F_{\tilde{v}(S \cup T)}(x)$ generates a meaningful specificity of simulated situation. Though, we have a similar situation in the case of classical cooperative games, and when the values $v(S)$ and $v(T)$, on the one hand, and $v(S \cup T)$, on the other hand, are considered exogenous;
- the usefulness of the joint coalition $S \cup T$ in the sum of $\tilde{v}^+(S \cup T) = \tilde{v}(S) + \tilde{v}(T)$. A situation with a content point of view of interest solely in the context of stochastic cooperative games.

These properties can be used in the construction of solution concepts for stochastic cooperative games. In particular the analysis of rationality or acceptance for imputation in some of the coalitions, we may have to consider not only the occasional payoff $\tilde{v}(S)$ received bycoalition as characterized by VaR $v_\alpha(S) = F^{-1}_{\tilde{v}(S)}(\alpha)$, but the amount of random players in $S \subset I$ utilities

$$\tilde{v}^+_\alpha(S) = \sum_{i \in S} \tilde{v}(i)), \tag{11}$$

that are characterized by VaR $v^+_\alpha(S) = F^{-1}_{\tilde{v}^+(S)}(\alpha)$, In this case the conditions of coalition rationality become

$$x(\alpha, S) \geq \tilde{\alpha}(S) \geq v^+_\alpha(S), \tag{12}$$

where $x(\alpha, S) = \sum_{i \in S} x_i(\alpha)$. Note that these issues have been considered in more detail by Konyukhovskiy (2012).

4. Concepts of decisions for stochastic cooperative game

Traditionally, the core in the classical cooperative is defined as the set of non-dominated imputations, which is equivalent to the condition

$$(\forall S \subset I,\ S \neq \not\subset, S \neq I)\ x(S) \geq v(S),\ x(I) = v(I). \tag{13}$$

We note that despite the external similarity designation $x(S)$ should be distinguished from $\mathbf{x}(\alpha)$ by content. In the former $(x(S))$ the amount prescribed by the imputation to the coalition S, i.e. scalar quantity, while in the later $(\mathbf{x}(\alpha))$ the stochastic vector corresponds to a certain level of probability α.

In developing the proposed approach, we can similarly introduce the concept of α-core[1], and define it as the set of imputations. For Example, vectors $(I, \tilde{v})$ satisfy (9) and (10), which supports the following statements

$$(\forall S \subset I,\ S \neq \not\subset, S \neq I)\ x(\alpha, S) \geq v_\alpha(S) \tag{14}$$

[1] the core of the game $(I, \tilde{v})$ for probability level α.

or

$$C_\alpha(\tilde{v}) = \{\mathbf{x} \in R^m | \, \forall S \subset I, S \neq \not\subset, S \neq I : \; x(\alpha, S) \geq v_\alpha(S); \\ x(\alpha, I) \leq v_{1-\alpha}(I)\}. \tag{15}$$

In other words, the division that belongs to αcore prescribes to any coalition a share not less than the VaR utility of this coalition for a given level . In this division, imputation reach is provided by the conditions in accordance with the share prescribed by imputation for a great coalition, as long as it does not exceed VaR and its utility.

Due to the fact that the distribution functions of random variables $F_{\tilde{v}(S)}(x)$ are incremental, an increase in implies an increase in value $v_\alpha(S)$, or a decrease in value $v_{1-\alpha}(I)$, by which we obtain

$$\alpha' < \alpha'' \Rightarrow C_{\alpha'} \subset C_{\alpha''(v)} \tag{16}$$

indicating, payment for implementation, when αcore is more likely to decrease its size.

Moreover, there may be situations in which a sufficiently large αcore is empty. Based on property (16) we can identify the problem of *selecting the highest probability level in which there is a non-empty $\alpha-$core.*

Another natural problem posed by property (16) is determining the dependence of the size αcore based on the choice of α. This in turn gives rise to the problem of choosing a measure of volume. Objective complexity of its solutions are defined in such a way that a change in can cause a change the dimension of the αcore.

More detail is required when analyzing a number of specific aspects of the transition from the classical procedure of cooperative games with transferable utility to their stochastic counterparts. In essence, we must resolve the issue of technology transition from deterministic $v(S)$ to stochastic $\tilde{v}(S)$ utilities. Consider a relatively simple but realistic situation, in which $\tilde{v}(S)$ can be considered as random variables distributed according to the normal law

$$\tilde{v}(S) \in N(\bar{v}(S), \sigma_S^2). \tag{17}$$

Hypothesis (17) is the basic logical development of traditional approaches. As a rule, the process of constructing the characteristic functions for specific applications of cooperative game-theoretic models provides a way to replace indicators which objectively have a random nature to their deterministic counterparts. It's a different method for averaging operations, in which the $v(S)$ latently is identified with $\bar{v}(S)$. In this case, we abandon this simplification and do not lose additional characteristics of random variables with the variation σ_S^2. Also note that the structures of models that are based on hypothesis (17) have serious structural advantages. It allows visual comparison of the results of their analysis with the results obtained in the framework of deterministic model counterparts.

Of course, the hypothesis relating to the distribution of the value $\tilde{v}(S)$ for normal law is inherently arguable. Moreover, its defense is only possible in cases where the provision is a concrete specification of the simulated object. However, at that level of generality, in which we present the problems of stochastic cooperative games, we can resort to general considerations concerning the merits of the normal distribution as a typical representative of a universal and continuous distributions.

By assuming $\tilde{v}(S) \in N(\bar{v}(S),\ \sigma_S^2)$, the following VaR equation is possible

$$v_\alpha(S) = \bar{v}(S) + \sigma_S \cdot \Phi^{-1}(\alpha) \tag{18}$$

where $\Phi(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ also known as the Laplace integral.

If we take into account $\sigma_S > 0$ and $\Phi^{-1}(\alpha) > 0$ with, we calculate

$$v_\alpha(S) > \bar{v}(S)$$

for all probability levels where $\alpha > 0.5$. Further, comparing condition (13), which must be completed by imputations belonging to core in non-stochastic games, with condition (14), we determine that they belong to $\alpha-$core in stochastic game. Therefore, we conclude that

$$x(S, \alpha) > x(S),$$

where $x(S, \alpha) = \sum_{i \in S} x_i(\alpha)$ is the amount, distributed between members of coalition S by the imputation $\mathbf{x}(\alpha), x(S) = \sum_{i \in S} \bar{v}_i$.

The problem of determining the maximum level of probability , on which there exists a non-empty $\alpha-$core for stochastic games based on the premise that (17) takes the form of

$$\Phi^{-1}(\alpha) \to max, \tag{19}$$

where

$$C_\alpha(\tilde{v}) = \{\mathbf{x} \in R^m | \ \forall S \subset I, S \neq \not\subset, S \neq I : \sum_{i \in S} x_i \geq \bar{v}(S) + \sigma_S \cdot \Phi^{-1}(\alpha); \\ \sum_{i \in I} \leq \bar{v}(I) - \sigma_I \cdot \Phi^{-1}(\alpha)\} \neq \emptyset. \tag{20}$$

As possible constructive versions of the solution of this problem we will note two following approaches.

First, a certain interest is represented by the estimates which are based on differences between maximum and minimum values of the shares provided by the imputations belonging $\alpha-$core to each of players.

In other words, for each of players $i \in I$ values can be determined

$$x_i^- = \min_{x \in C_\alpha} \{x_i\}, x_i^+ = \max_{x \in C_\alpha} \{x_i\}, \Delta x_i = x_i^+ - x_i^-$$

Δx_i in essence characterize the range of opportunities (hopes) of each of participants of game within the kernel which is implemented at the level of probability α. Points in dimension R^n in which the values $x_i^-, x_i^+ (i \in I)$, are occur, define so called parallelepiped of hopes. Of course this parallelepiped is greater than $\alpha-$core (it is its subset). However its amount which is calculated rather simply, can be considered as the approximate (hardened) characteristic of a kernel in case of the set level α.

Secondly, for games with rather small number of participants as the characteristic $\alpha-$core the quantity, the points which got to it lying on some discrete grid can be used.

5. Extension of the excess concept on coalition stochastic cooperative games

Briefly discuss the issues associated with a possible extension to stochastic cooperative game solution concepts that are based on the notion of excess coalition.

Consider a distribution vector ($\mathbf{x} \in R^m$) of utilities among the participants of the game. One of its characteristics is the figure.

$$\alpha(\mathbf{x}, (I)) = Pr(\tilde{v}(\{I\}) \geq x(I)) = 1 - F_{\tilde{v}(\{I\})}(x(I)), \tag{21}$$

where $x(I) = \sum_{i \in I} x_i$, i.e. the probability that a random realization of the payoff of full (large) coalition $\tilde{v}(\{I\})$ will be able to provide the distribution that vector $\mathbf{x}$ promises to game participants. Substantially $\alpha(x(I))$ − it is the probability, with which $\mathbf{x}$ is stochastic pre-imputation (satisfies the condition of group rationality with α probability).

Each of coalitions $S \subset I (S \neq \oslash, I)$ following its understanding of rationality, will compare $\alpha(x(I))$ probability of receiving $x(S) = \sum_{i \in S} x_i$, that is promised by $\mathbf{x}$ if this small coalition enter the full (large) coalition, with probability of receiving equal utility (without association with other players)

$$\alpha((x(S))) = Pr(\tilde{v}(\{S\}) \geq x(S)) = 1 - F_{\tilde{v}(\{S\})}(x(S)). \tag{22}$$

In this case, the characteristics of the pre-imputation $\mathbf{x}$ following figures can be used

$$e_\alpha(S, \mathbf{x}) = \alpha(x(S)) - \alpha(x(I)), (\forall S \neq \oslash, I). \tag{23}$$

We will denote $e(S, \mathbf{x})$ − probabilistic excesses of coalitions. From content point of view $e_\alpha(S, \mathbf{x})$ reflects satisfaction level ($e_\alpha(S, \mathbf{x}) < 0$) or, vice versa, dissatisfaction ($e_\alpha(S, \mathbf{x}) > 0$) with probabilistic characteristics of the distribution $\mathbf{x}$. Here there is a direct parallel with the concept of excess of coalition for deterministic cooperative games, in which the excess is defined as a measure of satisfaction (dissatisfaction) the volume of the utility, which receives coalition

$$e(S, \mathbf{x}) = v(S) - x(S). \tag{24}$$

Based on the concept of probabilistic excess, such concepts as nucleolus and kernel can be extended for stochastic cooperative games.

As is known the concept of nucleolus for classical cooperative games with transferable utility based on the idea of finding the imputation, which is achieved by the lexicographic minimum excesses coalitions. It can naturally be extended to stochastic games.

Let us a little more detail on the issues of matching two imputations. From the properties of the distribution functions of random variables, it follows that if we compare some vectors $\mathbf{x}$ and $\mathbf{y}$, about which it is known that

$$x(I) > y(I)(\sum_{i \in I} x_i = \sum_{i \in I} y_i), \tag{25}$$

then, by virtue of the fact that for continuous random variables

$$\alpha' < \alpha" \to v_{\alpha'}(I) < v_{\alpha"}(I), \tag{26}$$

receive

$$Pr(x(I) \geq \tilde{v}(\{I\})) < Pr(y(I) \geq \tilde{v}(\{I\})). \qquad (27)$$

From the content point of view (27) reflects obvious fact: pre-imputation $\mathbf{y}$ provides a full coalition with less utility than $\mathbf{x}$ but more likely.

In case of comparison of two imputations $\mathbf{x}$ and $\mathbf{y}$ so that $x(I) = y(I)$, as a criterion can be used values of probability of excesses (23).

In particular, it seems reasonable that the task of finding the pre-imputation (or, in general, a plurality of pre-imputations) for which exists the minimum for most likely excess of coalition. It is easy to see that this problem is a problem of finding the smallest analogue of core (least core) in classical deterministic games.

From (23) taking in account (21) we get

$$e_\alpha(S, \mathbf{x}) = Pr(\tilde{v}(S) \geq x(S)) - Pr(\tilde{v}(I) \geq x(I)) =$$

$$= 1 - F_{\tilde{v}(S)}(x(S)) - (1 - F_{\tilde{v}(I)}(x(I))) = F_{\tilde{v}(I)}(x(I)) - F_{\tilde{v}(S)}(x(S)) \qquad (28)$$

for $S \neq \oslash, I$. Under the assumption that utility players and coalitions in the game $(I, \tilde{v})$ are random variables, that are normally distributed with parameters $\tilde{v}(S)$ and σ_S $(v(S) \in N(\tilde{v}(S), \sigma_S)))$ equation (28) will be following

$$e_\alpha(S, \mathbf{x}) = \varPhi\left(\frac{x(I) - \tilde{v}(I)}{\sigma_S}\right) - \varPhi\left(\frac{x(S) - \tilde{v}(S)}{\sigma_S}\right), S \neq \oslash, I, \qquad (29)$$

where $\varPhi(x)$ − Laplace function.

Denote that $X(\alpha)$ set of pre-imputation, realized with probability α, i.e.

$$X(\alpha) = \{\mathbf{x} \in R^n | Pr(\tilde{v}(I) \geq x(I)) = \alpha\}. \qquad (30)$$

We emphasize that on the one hand for a particular vector pre-imputation $\mathbf{x}$ we can determine the probability of its implementation $\alpha(x(I))$ and, vice versa, having probability level α and knowing the distribution function for utility of a full coalition $F_{\tilde{v}(I)}(x)$, we can find $X(\alpha)$ see (30).

For this set the task of finding the pre-imputation that minimizes the maximum excess coalition is formulated as

$$\min_{x \in X(\alpha)} \{ \max_{S \neq \oslash, I} \{e_\alpha(S, \mathbf{x})\}\}. \qquad (31)$$

In the case of normally distributed utilities (31) looks like following

$$\min_{x \in X(\alpha)} \left\{ \max_{S \neq \oslash, I} \left\{ \varPhi\left(\frac{x(I) - \tilde{v}(I)}{\sigma_I}\right) - \varPhi\left(\frac{x(S) - \tilde{v}(S)}{\sigma_S}\right) \right\} \right\}. \qquad (32)$$

Taking into account, that terms $\varPhi((x(I) - \tilde{v}(I))/\sigma_I)$ for all coalitions $S \neq \oslash, I$ are constant, we receive that problem (32) can be reduced to the problem

$$\max_{x \in X(\alpha)} \left\{ \min_{S \neq \oslash, I} \left\{ \varPhi\left(\frac{x(S) - \tilde{v}(S)}{\sigma_S}\right) \right\} \right\}. \qquad (33)$$

Given the monotonicity $\Phi(x)$ we see that the task (33) is equivalent to a simple optimization problem

$$\max_{x\in X(\alpha)}\left\{\min_{S\neq\oslash,I}\left\{\frac{x(S)-\tilde{v}(S)}{\sigma_S}\right\}\right\}. \tag{34}$$

If we introduce the auxiliary variable x_{m+1}, is an exact lower bound of the set of values $(x(S)-\tilde{v}(S))/\sigma_S$, i.e.

$$x_{m+1}\leq\frac{x(S)-\tilde{v}(S)}{\sigma_S}\ or\ x(S)-\sigma_S\cdot x_{m+1}\geq\tilde{v}(S)\ (\forall S\neq\oslash,I),$$

than the problem (34) will be reduced to a linear programming problem

$$x_{m+1}\to max \tag{35}$$

under the constraints

$$x(S)-\sigma_S\cdot x_{m+1}\geq\tilde{v}(S)\ (\forall S\neq\oslash,I), \tag{36}$$

$$x(I)=\tilde{v}(I)-\sigma_I\cdot\Phi^{-1}(\alpha). \tag{37}$$

Note that the condition (37) reflects the requirements of the desired accessory to the set of vectors $X(\alpha)$ − a plurality of pre-imputation, realized with probability α − on the assumption that the total utility of (large) coalition is normally distributed with parameters $\tilde{v}(I)$ and σ_I.

6. Conclusion

Stochastic cooperative games and the concepts of their solution on the basis of the stochastic $\alpha-$core allow to consider more flexibly risk factors and uncertainty in processes of modeling and research of cooperative interaction of economic actors. It will quite be approved with tendencies to increase of requirements to adequacy of the tools reflecting accidental impacts of environment.

In particular, relying on the concept $\alpha-$core, we have opportunities for an objective exception of consideration for reasons of a "stochastic" inefficiency of a number of options of income distributions between participants of coalition associations of economic actors.

The given examples rather visually reflect properties which are possessed by stochastic cooperative games. Accounting of these properties can significantly influence adjustment of "traditional" mechanisms of decision making as on strategic management, and the organization of the current activities of the coalitions, the unions and partner associations of economic diverse actors.

In particular, stochastic cooperative models allow to take into account asymmetry of a provision of potential participants of partnership relatively risk factors and uncertainty. It promotes further increase of level of accuracy and adequacy of the created stimulating impacts. First of all, the measures aimed at consolidation of common efforts of both the state, and non-state companies, entities and corporations within the large-scale investment projects requiring accumulation of serious material resources.

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