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Strategic Stability of Coalitions Technological Alliance Parameters: a Two-Level Cooperation

Nikolay V. Kolabutin

St.Petersburg State University,
Faculty of Applied Mathematics and Control Processes,
Bibliotechnaya pl. 2, St.Petersburg, 198504, Russia
E-mail: kolabutin_nik@mail.ru
URL: ftp://ftp.isdg.ru

Abstract The paper is devoted to two-level cooperation in differential games. Cooperative differential games are one of the fastest growing parts of the game theory. They are widely used for modeling the conflict-controlled processes in various fields, especially in management and economics. The solution of differential game is a cooperative agreement, and the selected principle of optimality, according to which the received payoff is distributed. The main problem of many cooperative solutions is the instability over time. Studies showed that initially selected cooperative solution often loses its optimality over time. Therefore, the question arose about the stability of the co-operative solutions. The stability can be understood as dynamic stability (time consistency), strategic stability or protection from irrational behavior. The concept of dynamic stability was formalized by L.A. Petrosyan. Cooperative solution is dynamically stable, if the principle of optimality, selected early in the game keeps its consistency throughout the gameplay. For dynamic stability is necessary at each moment of time to carry out the regularization of the chosen principle of optimality. For this regularization L.A. Petrosyan proposed to use the redistribution of received payoff in accordance with the "imputation distribution procedure". Petrosjan (1993) and Petrosjan and Zenkevich (1996) presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was introduced to construct time-consistent solutions. Yeung and Petrosjan (2001) designed time-consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosjan (2003) employed the regularization method to construct time-consistent bargaining procedures.

The strategic stability of cooperative solution means, that no individual deviation from the cooperation of each member brings benefits the decline member. This means that the outcome of this cooperative agreement is reached at some Nash equilibrium, which will guarantee the strategic support for such cooperation.

Recently in differential games are studied coalitional solutions in which the coalitions act as individual players. Coalitions can play with each other in a non-cooperative game, then payoff of each coalition is distributed among its members in accordance with some principle of optimality. But coalitions-players can cooperate to increase the joint payoff. In this case the joint payoff is distributed between coalitions according to some principle of optimality then coalition's share of joint payoff is distributed between its participants according to maybe other principle of optimality. This cooperation is called two-level cooperation. Optimality principles of payoff distribution between coalitions and within coalition may be different.

To solve such cooperative models which requires at both levels of the cooperation it is necessary to build the characteristic function and imputation distribution procedure. This paper describes a model of a two-level cooperation in the technological alliance differential game. Participants of the game are the firms with the some technology that brings profit. On the first (lower) level firms form coalitions to increase joint profit. On the second (upper) level coalitions act as individual players and also form the one grand coalition to maximize the joint payoff. The top-level payoff is distributed between coalitions-participants according to some principle of optimality. Thus, each coalition-participants may get more than would receive by playing individually. Then each coalition distributes the its share of joint payoff among its firms-members. This article also presented a stable cooperative solution in this model. For its implementation at every level of cooperation we build the characteristic function and prove its superadditivity. As a principle of optimality the dynamic Shapley value is selected. Proved the dynamic stability and the strategic stability of cooperative soluton. The results are illustrated by a quantitative example.

Keywords: differential game, cooperation, imputation distribution procedure, dynamic stability, strategic stability.

1. Introduction

Consider the cooperative differential game with planned period $[t_0, T]$. Participants are firms producing some technology for which they receive profit. $N = \{1, ..., n\}$ is a set of firms. Parameter of each firm $i \in N$ is the level of its technology, which is denoted by $x_i \in R^+$. This parameter is also called firm's state. The game begins from the state of $x^0 = \left\{x_1^0, ..., x_n^0\right\}$ at the moment t_0 and proceeds the period $T - t_0$, for which firms get some profit from its technology. At the moment T firms liquidate their technology and receive additional profit Kostyunin, 2011.

The firm's profit depends on its technological level, so it seeks to increase its, investing in the development of technology. The firm's i level of investment in its technological development is called firm's strategy in the game and denoted for $u_i \in R^+$. This parameter is also called firm's control.

The dynamics of firm's development proceeds according to the differential equation:

$$\dot{x}_i(s) = \alpha_i \left[u_i(s)x_i(s)\right]^{1/2} - \delta x_i(s) \qquad (1)$$
$$x_i(t_0) = x_i^0, \quad i \in N,$$

where α_i and δ are positive constants. On right side of equation imposed conditions which guarantee the existence, uniqueness and extendibility of solutions for any piecewise continuous controls $u_i(s) \in R^+, \quad s \in [t_0, T]$. Also there is an additional constraint $x_i(s) > 0, \quad s \in [t_0, T]$.

The profit of firm $i \in N$ has the form:

$$H_i\left(x_i^0, T - t_0, u_i\right) = \int\limits_{t_0}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s - t_0)\right] ds + \qquad (2)$$
$$+ \exp\left[-r(T - t_0)\right] q_i \left[x_i(T)\right]^{1/2},$$

where $h_i(s, x_i(s), u_i(s)) = \left[P_i [x_i(s)]^{1/2} - c_i u_i(s)\right]$ is profit of firm i at moment s, state $x_i(s)$ and control $u_i(s)$; P_i, c_i are positive constants; r is discount rate; $q_i [x_i(T)]^{1/2}$ is terminal payoff of firm i at moment T and state $x_i(T)$.

Firms can cooperate to increase joint profit. If firms form a coalition, each firm-participant can gain more abilities from other participants. Therefore the technological dynamics of firms in coalition is changed. Consider coalition K, formed by players from a subset of $K \subseteq N$. The dynamics of the firm in coalition takes the form:

$$\dot{x}_i(s) = \alpha_i [u_i(s)x_i(s)]^{1/2} + \sum_{j\in K,\ j\neq i} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} - \delta x_i(s) \tag{3}$$
$$x_i(t_0) = x_i^0, \quad i \in K,$$

where $b_j^{[j,i]} \geq 0$ is positive constant which mean the technology transfer effect from firm j to firm i. The level of technology of each firm in the coalition K has a positive effect on the rate of technological development, which means that the condition $\partial f_i^K [x_K(s), u_i(s)] / \partial x_j \geq 0,\ j \in K$, where $f_i^K [x_K(s), u_i(s)]$ is the right side of dynamic equation (3). The coalition payoff is the sum of participants' payoffs.

$$\sum_{i\in K} H_i\left(x_i^0, T - t_0, u_i\right) = \tag{4}$$
$$= \sum_{i\in K} \int_{t_0}^{T} h_i(s, x_i(s), u_i(s)) \exp[-r(s-t_0)]\, ds +$$
$$+ \sum_{i\in K} \exp[-r(T-t_0)]\, q_i [x_i(T)]^{1/2}$$

To maximize profit of coalition K we consider the optimal control problem, which maximizes (4) with the boundary conditions (3). This maximization problem is denoted by $\varpi\left[K; t_0; x_K^0\right]$.

A detailed solution of this problem is described in (Yeung and Petrosyan, 2006) with using continuously differentiable function of $W^{(t_0)K}(t, x_K(t))$: $[t_0, T] \times \prod_{j\in K} R^{m_j} \to R$, which determines the max payoff of coalition $K \subseteq N$ on the interval $[t, T]$, where $t \in [t_0, T]$. This function satisfies the Bellman equation:

$$-W_t^{(t_0)K}(t, x_K(t)) = \max_{u_K} \left\{ \sum_{i\in K} h_i(t, x_i(t), u_i(t)) \exp[-r(t-t_0)] + \right.$$
$$\left. \sum_{i\in K} W_{x_i}^{(t_0)K}(t, x_K(t))\, f_i^K [x_K(t), u_K(t)] \right\}$$
$$W^{(t_0)K}(T, x_K(T)) = \sum_{i\in K} \exp[-r(T-t_0)]\, q_i [x_i(T)]^{1/2}$$
$$f^K [x_K(t), u_K(t)] = \left\{ f_i^K [x_K(t), u_i(t)] \right\}_{i\in K} = \{\dot{x}_i\}_{i\in K}, \qquad K \subseteq N$$

The solution of the problem $\varpi\left[K; t_0; x_K^0\right]$ get function $W^{(t_0)K}(t, x_K(t))$ in the following form:

$$W^{(t_0)K}\left(t, x_K(t)\right) = \left[\sum_{i\in K} A_i^K(t)\left[x_i(t)\right]^{1/2} + C^K(t)\right] \exp\left[-r(t-t_0)\right], \tag{5}$$

where values $\{A_i^K(t)\}_{i\in K}$, $C^K(t)$ are solution of the differential equations:

$$\dot{A}_i^K(t) = \left(r + \frac{\delta}{2}\right) A_i^K(t) - \sum_{j\in K, j\neq i} \frac{b_j^{[i,j]}}{2} A_j^K(t) - P_i \tag{6}$$

$$\dot{C}^K(t) = rC^K(t) - \sum_{i\in K} \frac{\alpha_i^2}{16c_i} A_i^K(t)$$

$$A_i^K(T) = q_i, \quad C^K(T) = 0, \quad i \in K$$

In (Yeung and Petrosyan, 2006) has been thoroughly discussed cooperative game where company form a great coalition, maximize joint payoff and share it according to the dynamic Shapley value.

In this paper is an advanced model where members of the gameplay are not individual firms $\{i\}_{i\in N}$ but their coalitions which act as individual players (Petrosyan et al., 2006, Petrosyan et al., 2010).

2. The model of Game

Consider the cooperative differential game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$ with duration period $T-t_0$, where $N = \{1, ..., n\}$ is a set of firms. Parameter of each firm $i \in N$ or its state is the level of its technology $x_i \in R^+$. Strategy of each firm $i \in N$ or its control is level of investment in its technological development $u_i \in R^+$. The game begins from the state of $x^0 = \left\{x_1^0, ..., x_n^0\right\}$ at the moment t_0. Dynamic of firm's technological development proceeds according to differential equation (1). The payoff of each firm calculate according to (2). Firms can form coalitions to increase the joint payoff.

Let $\Delta = \{K_1, K_2, ..., K_m\}$ – coalition partition of the game, ie $K_{l1} \bigcap K_{l2} = \Omega$, $l1 \neq l2$, $\bigcup\limits_{l=1}^{m} K_l = N$, $|K_l| = n_l$, $\sum\limits_{l=1}^{m} n_l = n$. Denote by $M = \{1, ..., m\}$ the set of partition indexes.

Introduce some notations:

$x_{K_l}(s) = \{x_i(s)\}_{i\in K_l}$, $l \in M$ is a state of coalition K_l at moment $s \in [t_0, T]$, defined by a set of states of its participants; $x_{K_l}^0 = \{x_i^0\}_{i\in K_l}$ is initial state coalition K_l; $u_{K_l}(s) = \{u_i(s)\}_{i\in K_l}$, $l \in M$ is control of coalition K_l at the moment s, which is a set of controls of its participants; $\breve{N}$ is coalition formed by all elements from partition Δ; $\breve{K} \subseteq \breve{N}$ is any coalition formed by a subset of partition Δ elements; $V^{\Delta(t_0)} = V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ is the characteristic function of the game.

Dynamics of participants of coalition $K_l \subset \Delta$ proceeds according to the system of differential equations:

$$\dot{x}_i(s) = \alpha_i \left[u_i(s)x_i(s)\right]^{1/2} + \sum_{j\in K_l,\ j\neq i} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} - \delta x_i(s) \quad (7)$$

$$x_i(t_0) = x_i^0, \quad i \in K_l$$

Denote the right side of the equation by $f_i^{K_l}\left[x_{K_l}(s), u_{K_l}(s)\right]$

The payoff of coalition K_l is equal to the sum of participants' payoffs:

$$H_{K_l}\left(x_{K_l}^0, T-t_0, u_{K_l}\right) = \sum_{i\in K_l} H_i\left(x_i^0, T-t_0, u_i\right) = \quad (8)$$

$$= \sum_{i\in K_l} \int_{t_0}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s-t_0)\right] ds +$$

$$+ \sum_{i\in K_l} \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2},$$

where $h_i\left(s, x_i(s), u_i(s)\right) = \left[P_i\left[x_i(s)\right]^{1/2} - c_i u_i(s)\right]$ is profit of firm i at moment $s \in [t_0, T]$.

3. Cooperation of coalitions

Coalitions $K_l \subset \Delta$ can cooperate to raise the joint profit. This joint profit coalitions share between themselves in accordance with a some optimality principle. Consider the coalition $\breve{K} = K_{l_1} \bigcup K_{l_2} \bigcup ... \bigcup K_{l_k}$, where $K_{l_1}, K_{l_2}, ..., K_{l_k} \subset \Delta$. The evolution of technological level of firm from $K_l \subset \breve{K}$ satisfy the following system:

$$\dot{x}_i(s) = \alpha_i\left[u_i(s)x_i(s)\right]^{1/2} + \sum_{j\in K_{l_1}} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} + \quad (9)$$

$$+ \sum_{j\in K_{l_2}} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} + ... + \sum_{j\in K_l,\ j\neq i} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} + ... +$$

$$+ \sum_{j\in K_{l_k}} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} - \delta x_i(s)$$

$$x_i(t_0) = x_i^0, \quad i \in K_l \subset \breve{K} \subseteq \breve{N}$$

Sums $\sum_{j\in K_{l_k}} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2}$ represent the total effect of transfer of technology to the firm i from the corresponding coalition. Thus, a synergistic effect of technological development of the firm i is obtained as by members of the coalition, which it originally owned, and by members of other coalitions from coalition $\breve{K}$. Simplify an expression (9):

$$\dot{x}_i(s) = \alpha_i\left[u_i(s)x_i(s)\right]^{1/2} + \sum_{j\in \breve{K},\ j\neq i} b_j^{[j,i]}[x_j(s)x_i(s)]^{1/2} - \delta x_i(s) \quad (10)$$

$$x_i(t_0) = x_i^0, \quad i \in \breve{K} \subseteq \breve{N}$$

The payoff of coalition $\breve{K}$ is its profit, which, as before, is calculated as the sum of the profits of its participants $K_{l_1}, K_{l_2}, ..., K_{l_k}$:

$$H_{\breve{K}}\left(x^0_{\breve{K}}, T-t_0, u_{\breve{K}}\right) = \sum_{\xi=1}^{k} H_{K_{l_\xi}}\left(x^0_{K_{l_\xi}}, T-t_0, u_{K_{l_\xi}}\right) = \tag{11}$$

$$= \sum_{\xi=1}^{k}\left(\sum_{i\in K_{l_\xi}}\int_{t_0}^{T} h_i\left(s, x_i(s), u_i(s)\right)\exp\left[-r(s-t_0)\right]ds + \right.$$

$$\left. + \sum_{i\in K_{l_\xi}} \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}\right)$$

Coalition $\breve{K} = \Delta$, formed by all participants of the game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$, will be called the ***the coalitions technological alliance***.

To find the the solution of the coalition in the game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$ is required to find characteristic function $V^{\Delta(t_0)} = V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ and define the imputation distribution procedure.

4. The calculation of the characteristic function $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ of the game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$

Note that participants of the game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$ are not the individual firm but their coalitions, therefore it is necessary to consider only subsets of $K_l \subset \Delta$ and their associations.

Since in this model the formation of coalitions leads only to a change in the dynamics of the game, and formed coalitions did not interact with each other, therefore any firm j, is not a member of the coalition of K_l, does not affect its development. In this case the characteristic function $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ is equal max payoff of coalition $\breve{K}$.

To calculate the characteristic function $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ the following problem of maximization has to be solved:

$$V^{\Delta(t_0)} = V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = \max_{u_{\breve{K}}}\left(H_{\breve{K}}\left(x_{\breve{K}}(t), T-t, u_{\breve{K}}(t)\right)\right) =$$

$$\max_{u_{\breve{K}}}\left(\sum_{K_l\subset\breve{K}} H_{K_l}\left(x_{K_l}(t), T-t, u_{K_l}(t)\right)\right) =$$

$$\max_{u_{\breve{K}}}\left(\sum_{K_l\subset\breve{K}}\int_{t}^{T}\sum_{i\in K_l} h_i\left(s, x_i(s), u_i(s)\right)\exp\left[-r(s-t_0)\right]ds + \right.$$

$$\left.\sum_{K_l\subset\breve{K}}\sum_{i\in K_l}\exp\left[-r(T-t_0)\right] q_i\left[x_i(T)\right]^{1/2}\right)$$

$W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}(t)\right)$ – continuously differentiable function, which determines the maximum payoff of coalition $\breve{K}$ in the time interval $[t, T]$, $t \in [t_0, T]$.

Dynamics of development of coalition $\breve{K}$ participants proceed in accordance system (10).

The problem of maximizing the coalition payoff was described in (Yeung and Petrosyan, 2006). The function $W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}(t)\right)$ satisfies the Bellman equation:

$$-V_t^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = \tag{12}$$

$$= \max_{u_{\breve{K}}}\left\{\sum_{i\in\breve{K}} h_i\left(t, x_i(t), u_i(t)\right)\exp\left[-r(t-t_0)\right] + \right.$$

$$\left. + \sum_{K_l\subset\breve{K}} V_{K_l}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) f_{K_l}^{\breve{K}}\left[x_{\breve{K}}(t), u_{\breve{K}}(t)\right]\right\}$$

$$V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(T), T\right) = \sum_{i\in\breve{K}} \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}$$

Here:

$$V_{K_l}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = gradV_{K_l}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) =$$

$$= \left\{V_{x_i}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)\right\}_{i\in K_l}$$

$$f_{K_l}^{\breve{K}}\left[x_{\breve{K}}(t), u_{K_l}(t)\right] = \left\{f_i^{\breve{K}}\left[x_{\breve{K}}(t), u_i(t)\right]\right\}_{i\in K_l} = \left\{\dot{x}_i(t)\right\}_{i\in K_l} \qquad K_l\subset\breve{K}$$

Expression $V_{K_l}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) f_{K_l}^{\breve{K}}\left[x_{\breve{K}}(t), u_{K_l}(t)\right]$ can be written as amount:

$$V_{K_l}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) f_{K_l}^{\breve{K}}\left[x_{\breve{K}}(t), u_{K_l}(t)\right] = \tag{13}$$

$$= \sum_{i\in K_l} V_{x_i}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) f_i^{\breve{K}}\left[x_{\breve{K}}(t), u_i(t)\right]$$

Substituting (13) in (12), get:

$$-V_t^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = \tag{14}$$

$$= \max_{u_{\breve{K}}}\left\{\sum_{i\in\breve{K}} h_i\left(t, x_i(t), u_i(t)\right)\exp\left[-r(t-t_0)\right] + \right.$$

$$\left. + \sum_{i\in\breve{K}} V_{x_i}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) f_i^{\breve{K}}\left[x_{\breve{K}}(t), u_i(t)\right]\right\}$$

$$V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(T), T\right) = \sum_{i\in\breve{K}} \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2}$$

Taking the partial derivatives of the values $\{u_i\}_{i\in\breve{K}}$ from the expression under the sign of max, and equating them to zero, obtain the formula for the optimal controls:

$$u_i = \frac{\alpha_i^2}{4(c_i)^2}\left[V_{x_i}^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)\exp\left[r(t-t_0)\right]\right]^2 x_i, \quad i\in\breve{K} \tag{15}$$

Substituting (15) in (14) and solving the resulting equation, get:

$$V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = \left[\sum_{i\in\breve{K}} A_i^{\breve{K}}(t)\left[x_i(t)\right]^{1/2} + C^{\breve{K}}(t)\right]\exp\left[-r(t-t_0)\right] \tag{16}$$

Values $\left\{A_i^{\breve{K}}(t)\right\}_{i\in\breve{K}}$ and $C^{\breve{K}}(t)$ are solutions of differential equations:

$$\dot{A}_i^{\breve{K}}(t) = \left(r+\frac{\delta}{2}\right)A_i^{\breve{K}}(t) - \sum_{j\in\breve{K}, j\neq i}\frac{b_j^{[i,j]}}{2}A_j^{\breve{K}}(t) - P_i$$

$$\dot{C}^{\breve{K}}(t) = rC^{\breve{K}}(t) - \sum_{i\in\breve{K}}\frac{\alpha_i^2}{16c_i}A_i^{\breve{K}}(t),$$

$$A_i^{\breve{K}}(T) = q_i, \quad C^{\breve{K}}(T) = 0, \quad i\in\breve{K}$$

Thus, the characteristic function of $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ is equal to the Bellman function $W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}(t)\right)$. Optimal control of the coalition $\breve{K}$ have the form:

$$u_{\breve{K}}^*(t) = \{u_i^*(t)\}_{i\in\breve{K}} = \left\{\frac{\alpha_i^2}{16(c_i)^2}\left[A_i^{\breve{K}}(t)\right]^2\right\}_{i\in\breve{K}} \tag{17}$$

Dynamics of coalition development takes the form:

$$\dot{x}_i^*(s) = \frac{\alpha_i^2}{4c_i}A_i^{\breve{K}}(s)\left[x_i^*(s)\right]^{1/2} + \sum_{j\in\breve{K}, j\neq i} b_j^{[j,i]}\left[x_j^*(s)x_i^*(s)\right]^{1/2} - \delta x_i^*(s) \tag{18}$$
$$x_i^*(t_0) = x_i^0, \quad i\in\breve{K}, \quad s\in[t_0, T]$$

Thus the characteristic function $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right)$ is defined as follows:

$$V^{\Delta(t_0)}\left(K, x_K(t), T-t\right) = \begin{cases} 0 & K=\emptyset \\ W^{(t_0)K_l}\left(t, x_{K_l}(t)\right) & K=K_l\subset\Delta \\ W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}(t)\right) & K=\breve{K}\subseteq\breve{N}, \end{cases} \tag{19}$$

where $W^{(t_0)K}(t, x_K(t))$ is defined by (5)

Establish superadditivity of function $V^{\Delta(t_0)}(K, x_K(t), T-t)$.

Given (19) if the function $W^{(t_0)K}(t, x_K(t))$ is superadditive, then superadditivity of function $V^{\ Delta(t_0)}(K, x_K(t), T-t)$ is obvious.

Firstly give without proof theorem about the comparison of solutions (Chaplygin, 1950)

Theorem 1 (about the comparison of solutions). *Given two Cauchy problem:*

$$\dot{y}_1(t) = f_1(t, y_1(t)), \quad y_1(t_0) = y_1^0$$

$$\dot{y}_2(t) = f_2(t, y_2(t)), \quad y_2(t_0) = y_2^0$$

For each problem the conditions of existence and uniqueness solutions is maintained, and furthermore the condition:

$$f_1(t, u(t)) \geq f_2(t, u(t)), \quad \forall (t, u(t))$$

Assume $y_1^0 \geq y_2^0$. Then for all $t \geq t_0$ the following condition is maintained:

$$y_1(t, t_0, y_1^0) \geq y_2(t, t_0, y_2^0)$$

Proof the superadditivity of function $W^{(t_0)K}(t, x_K(t))$.

Theorem 2 (about superadditivity of function). *Characteristic function $W^{(t_0)K}(t, x_K(t))$ defined by (5) is superadditive.*

Proof. Consider two coalitions $S_1, S_2 \subset N$, $S_1 \cap S_2 = \emptyset$ and their union $S_1 \cup S_2$. For each of these coalitions function $W^{(t_0)K}(t, x_K(t))$ takes the following form::

$$W^{(t_0)S_1\cup S_2}(t, x_{S_1\cup S_2}(t)) =$$
$$\max_{u_{S_1\cup S_2}} \left(\sum_{i\in S_1\cup S_2} \int_t^T \left[P_i [x_i(s)]^{1/2} - c_i u_i(s) \right] \exp[-r(s-t_0)]\, ds + \right.$$
$$\left. \sum_{i\in S_1\cup S_2} \exp[-r(T-t_0)]\, q_i [x_i(T)]^{1/2} \right)$$

$$W^{(t_0)S_1}(t, x_{S_1}(t)) =$$
$$\max_{u_{S_1}} \left(\sum_{i\in S_1} \int_t^T \left[P_i [x_i(s)]^{1/2} - c_i u_i(s) \right] \exp[-r(s-t_0)]\, ds + \right.$$
$$\left. \sum_{i\in S_1} \exp[-r(T-t_0)]\, q_i [x_i(T)]^{1/2} \right)$$

$$W^{(t_0)S_2}\left(t, x_{S_2}(t)\right) = \\ = \max_{u_{S_1}} \left(\sum_{i \in S_1} \int_t^T \left[P_i \left[x_i(s)\right]^{1/2} - c_i u_i(s) \right] \exp\left[-r(s-t_0)\right] ds + \right. \\ \left. + \sum_{i \in S_1} \exp\left[-r(T-t_0)\right] q_i \left[x_i(T)\right]^{1/2} \right).$$

The equations of dynamics for coalitions have the form:

$$\dot{x}_i(s) = f_i^{S_1 \cup S_2}\left[s, x_{S_1 \cup S_2}(s), u_i(s)\right] = \alpha_i \left[u_i(s) x_i(s)\right]^{1/2} + \\ + \sum_{j \in S_1 \cup S_2,\ j \neq i} b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s) \\ x_i(t_0) = x_i^0, \quad i \in S_1 \cup S_2, s \in [t_0, T] \tag{20}$$

$$\dot{x}_i(s) = f_i^{S_1}\left[s, x_{S_1}(s), u_i(s)\right] = \alpha_i \left[u_i(s) x_i(s)\right]^{1/2} + \\ + \sum_{j \in S_1,\ j \neq i} b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s) \\ x_i(t_0) = x_i^0, \quad i \in S_1, s \in [t_0, T] \tag{21}$$

$$\dot{x}_i(s) = f_i^{S_2}\left[s, x_{S_2}(s), u_i(s)\right] = \alpha_i \left[u_i(s) x_i(s)\right]^{1/2} + \\ + \sum_{j \in S_2,\ j \neq i} b_j^{[j,i]} [x_j(s) x_i(s)]^{1/2} - \delta x_i(s) \\ x_i(t_0) = x_i^0, \quad i \in S_2, s \in [t_0, T] \tag{22}$$

From condition $x_i(s) > 0$ for $\forall i \in N$ obtained, that for any u_i:

$$f_i^{S_1 \cup S_2}\left[s, x_{S_1 \cup S_2}(s), u_i(s)\right] \geq f_i^{S_1}\left[s, x_{S_1}(s), u_i(s)\right], \quad i \in S_1 \\ f_i^{S_1 \cup S_2}\left[s, x_{S_1 \cup S_2}(s), u_i(s)\right] \geq f_i^{S_2}\left[s, x_{S_2}(s), u_i(s)\right], \quad i \in S_2$$

Let solutions of equations(20),(21) and (22) respectively $\left\{x_i^{S_1 \cup S_2}(s)\right\}_{i \in S_1 \cup S_2}$, $\left\{x_i^{S_1}(s)\right\}_{i \in S_1}$ and $\left\{x_i^{S_2}(s)\right\}_{i \in S_2}$.

Given theorem 1, for any valid $u_i(s)$ the level of firm's technology $x_i^{S_1 \cup S_2}(s) \geq x_i^{S_1}(s), \quad i \in S_1$ and $x_i^{S_1 \cup S_2}(s) \geq x_i^{S_2}(s), \quad i \in S_2$.

Given the firm's payoff formula (2) obtain that for any valid $u_i(s)$ payoff of firm $H_i\left(x_i(t), T-t, u_i\right)$ in coalition $S_1 \cup S_2$ is greater than payoff of the same firm in the coalition S_1 or S_2.

$$H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i(t)\right)\geq H_i\left(x_i^{S_1}(t),T-t,u_i(t)\right)\quad i\in S_1$$
$$H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i(t)\right)\geq H_i\left(x_i^{S_2}(t),T-t,u_i(t)\right)\quad i\in S_2$$

Summing firms' payoffs for coalitions, get

$$\sum_{i\in S_1}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i(t)\right)\geq\sum_{i\in S_1}H_i\left(x_i^{S_1}(t),T-t,u_i(t)\right)\tag{23}$$
$$\sum_{i\in S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i(t)\right)\geq\sum_{i\in S_2}H_i\left(x_i^{S_2}(t),T-t,u_i(t)\right)$$

Denote by $\left\{u_i^{S_1}(s)\right\}_{i\in S_1}$ and $\left\{u_i^{S_2}(s)\right\}_{i\in S_2}$ controls, maximizing payoffs' sum in coalitions S_1 and S_2. Substituting them in (23), get:

$$\sum_{i\in S_1}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_1}(t)\right)\geq\sum_{i\in S_1}H_i\left(x_i^{S_1}(t),T-t,u_i^{S_1}(t)\right)\tag{24}$$
$$\sum_{i\in S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_2}(t)\right)\geq\sum_{i\in S_2}H_i\left(x_i^{S_2}(t),T-t,u_i^{S_2}(t)\right)$$

Adding the inequality (24), get:

$$\sum_{i\in S_1}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_1}(t)\right)+\sum_{i\in S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_2}(t)\right)\geq$$
$$\geq\sum_{i\in S_1}H_i\left(x_i^{S_1}(t),T-t,u_i^{S_1}(t)\right)+\sum_{i\in S_2}H_i\left(x_i^{S_2}(t),T-t,u_i^{S_2}(t)\right)$$

Denote by $\left\{u_i^{S_1\cup S_2}\right\}_{i\in S_1\cup S_2(s)}$ controls, maximizing summs of payoffs in coalition $S_1\cup S_2$. Then:

$$\sum_{i\in S_1\cup S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_1\cup S_2}(t)\right)=\tag{25}$$
$$=\max_{u_i,i\in S_1\cup S_2}\left\{\sum_{i\in S_1\cup S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i(t)\right)\right\}\geq$$
$$\geq\sum_{i\in S_1}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_1}(t)\right)+\sum_{i\in S_2}H_i\left(x_i^{S_1\cup S_2}(t),T-t,u_i^{S_2}(t)\right)\geq$$
$$\geq\sum_{i\in S_1}H_i\left(x_i^{S_1}(t),T-t,u_i^{S_1}(t)\right)+\sum_{i\in S_2}H_i\left(x_i^{S_2}(t),T-t,u_i^{S_2}(t)\right)=$$
$$=\max_{u_i,i\in S_1}\left\{\sum_{i\in S_1}H_i\left(x_i^{S_1}(t),T-t,u_i(t)\right)\right\}+$$
$$+\max_{u_i,i\in S_2}\left\{\sum_{i\in S_2}H_i\left(x_i^{S_2}(t),T-t,u_i(t)\right)\right\}$$

By definition:

$$\max_{u_i, i\in S_1\cup S_2}\left\{\sum_{i\in S_1\cup S_2} H_i\left(x_i^{S_1\cup S_2}(t), T-t, u_i(t)\right)\right\} = W^{(t_0)S_1\cup S_2}\left(t, x_{S_1\cup S_2}(t)\right) \quad (26)$$

$$\max_{u_i, i\in S_1}\left\{\sum_{i\in S_1} H_i\left(x_i^{S_1}(t), T-t, u_i(t)\right)\right\} = W^{(t_0)S_1}\left(t, x_{S_1}(t)\right)$$

$$\max_{u_i, i\in S_2}\left\{\sum_{i\in S_2} H_i\left(x_i^{S_2}(t), T-t, u_i(t)\right)\right\} = W^{(t_0)S_2}\left(t, x_{S_2}(t)\right)$$

Substituting (26) in (25) obtain:

$$W^{(t_0)S_1\cup S_2}\left(t, x_{S_1\cup S_2}(t)\right) \geq W^{(t_0)S_1}\left(t, x_{S_1}(t)\right) + W^{(t_0)S_2}\left(t, x_{S_2}(t)\right),$$

as required.

5. The imputation distribution procedure in the coalitions technological alliance

Suppose that the participants of coalitions technological alliance share joint payoff in accordance with the dynamic Shapley value (Petrosyan and Zaccour, 2003). Note calculated the share of coalition $K_l \subset \Delta$. The formula for the components of the Shapley value takes the following form:

$$\nu_{K_l}(V) = \sum_{\breve{K}\subseteq\breve{N}} \frac{(k-1)!(m-k)!}{m!}\left[V(\breve{K}) - V(\breve{K}\setminus K_l)\right] \quad (27)$$

where $\breve{K} = K_{l_1}\bigcup K_{l_2}\bigcup ...\bigcup K_{l_k}$ is subset of coalitions from partition Δ, K_{l_ξ}, $\xi = \overline{1,...,k}$, k is number of coalitions included in coalition $\breve{K}$.

To maximize revenue of technological alliance, the players on the interval $[t_0, T]$ will use a set of controls defined by the formula (17), and implement the optimal cooperative trajectory (18) in the case of $\breve{K} = \breve{N}$. It is anticipated that the division of the joint income of the players will use the Shapley value, the components of which are calculated by the formula (27). At the initial time t_0, the coalition's K_l share of joint payoff is equal to:

$$\nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) = \sum_{\breve{K}\subseteq\breve{N}} \frac{(k-1)!(m-k)!}{m!}\left[V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}^0, T-t_0\right) - \right. \quad (28)$$

$$\left. - V^{\Delta(t_0)}\left(\breve{K}\setminus K_l, x_{\breve{K}\setminus K_l}^0, T-t_0\right)\right]$$

Given that $V^{\Delta(t_0)}\left(\breve{K}, x_{\breve{K}}(t), T-t\right) = W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}(t)\right)$, expression (28) can be rewritten:

$$\nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) = \sum_{\breve{K} \subseteq \Delta} \frac{(k-1)!(m-k)!}{m!} \Big[W^{(t_0)\breve{K}}\left(t_0, x_{\breve{K}}^0\right) - \\ - W^{(t_0)\breve{K} \setminus K_l}\left(t_0, x_{\breve{K} \setminus K_l}^0\right)\Big] \tag{29}$$

The Shapley value has to be maintained throughout the game. Therefore, in every moment the next condition has to be maintained:

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) = \sum_{\breve{K} \subseteq \Delta} \frac{(k-1)!(m-k)!}{m!} \Big[W^{(t_0)\breve{K}}\left(t, x_{\breve{K}}^*(t)\right) - \\ - W^{(t_0)\breve{K} \setminus K_l}\left(t, x_{\breve{K} \setminus K_l}^*(t)\right)\Big]$$

For realization the dynamic Shapley is necessary to determine the imputation distribution procedure (Petrosyan, 1977), to compensate transient changes. Define the imputation distribution procedure as a function of $B_\Delta(t) = \{B_{K_l}(t)\}_{t=t_0}^T$, such that:

$$\nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) = \int_{t_0}^{T} B_{K_l}(s) \exp\left[-r(s-t_0)\right] ds + \\ + \exp\left[-r(T-t_0)\right] \sum_{i \in K_l} q_i \left[x_i^*(T)\right]^{1/2} \tag{30}$$

The function $B_{K_l}(t)$ is a payment received by a coalition K_l at the moment t after redistribution of joint payoff. For dynamic stability the next equality has to be maintained at each moment:

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) = \int_{t}^{T} B_{K_l}(s) \exp\left[-r(s-t_0)\right] ds + \\ + \exp\left[-r(T-t_0)\right] \sum_{i \in K_l} q_i \left[x_i^*(T)\right]^{1/2} \tag{31}$$

From (30) and (31):

$$\nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) = \int_{t_0}^{t} B_{K_l}(s) \exp\left[-r(s-t_0)\right] ds + \nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) \tag{32}$$

This condition means *temporary solvency* or a *dynamic stability* (Petrosyan, 1977, Zenkevich and Petrosyan, 2007) decision on coalition parties $\{K_l\}$. But it is also necessary to show the dynamic stability of the solution with respect to each individual firm. This will be done below.

Note that in every moment of $s \in [t_0, T]$ occurs only redistribution of the joint profits, but the sum of players payoffs does not changes:

$$\sum_{K_l \subset \Delta} B_{K_l}(s) = \sum_{K_l \subset \Delta} \sum_{i \in K_l} h_i \left(s, x_i^*(s), u_i^*(s)\right) =$$
$$= \sum_{K_l \subset \Delta} \sum_{i \in K_l} \left[P_i \left[x_i^*(s)\right]^{1/2} - c_i u_i^*(s) \right]$$

In this case the function $B_{K_l}(s)$ has the form:

$$B_{K_l}(s) = \sum_{\breve{K} \subseteq \Delta} \frac{(k-1)!(m-k)!}{m!} \left\{ \left[W_t^{(s)\breve{K}} \left(s, x_{\breve{K}}^*(s)\right) - W_t^{(s)\breve{K} \setminus K_l} \left(s, x_{\breve{K} \setminus K_l}^*(s)\right) \right] + \right.$$
$$+ \sum_{j \in \breve{K}} \left[W_{x_j}^{(s)\breve{K}} \left(s, x_{\breve{K}}^*(s)\right) \right] f_j^N \left[x_N^*(s), u_j^*(s)\right] -$$
$$\left. - \sum_{h \in \breve{K} \setminus K_l} \left[W_{x_h}^{(s)\breve{K} \setminus K_l} \left(s, x_{\breve{K} \setminus K_l}^*(s)\right) \right] f_h^N \left[x_N^*(s), u_h^*(s)\right] \right\}$$

6. The payoff distribution within the coalition K_l

Coalition's K_l share of joint payoff has to be distributed between firms-participants. Suppose that firms act cooperatively within the coalition. Define the cooperative game $\Gamma^{K_l}\left(x^0, T - t_0\right)$, where K_l is the set of [layers, and $V^{(t_0)K_l}\left(L, x_L(t), T - t\right)$, $L \subseteq K_l$ is characteristic function, calculated on the assumption, that firms, not included in K_l, use their optimal coalition strategies, and firms, not included in L use their optimal individual strategies in game $\Gamma^{K_l}\left(x^0, T - t_0\right)$.

To calculate the payoff share of firm $i \in K_l$ is necessary to calculate the value of the characteristic function in the game $\Gamma^{K_l}\left(x^0, T - t_0\right)$ and determine the imputation distribution procedure.

The characteristic function will be calculated as follows. First, we calculate the value of the characteristic function for the coalition K_l, then for any coalition $L \subset K_l$.

In calculating the characteristic function for the coalition K_l necessary to consider that it participates in the game of coalitions $\Gamma^{\Delta}\left(x^0, T - t_0\right)$. Therefore it gets more payoff than could be obtained by playing individually. Since any subcoalition $K \subset K_l$ is not included in the partition Δ, assume that it does not have those bonuses, which are available to the coalition K_l, and its characteristic function will be calculated without considering of coalitions game.

The value of the characteristic function $V^{K_l(t_0)}\left(K_l, x_{K_l}(t), T - t\right)$ should be equal to the maximum payoff that coalition K_l can receive. If coalition played alone, its maximum payoff would be equal to the function $W^{(t_0)K_l}\left(t, x_{K_l}^*(t)\right)$, defined formula (19). This function determines the maximum winning coalition K_l in the case of individual development of the game of coalitions. But since the coalition combined into a technological alliance, the result of profit distribution in the upper level, each coalition K_l gets a share of revenue equal to component of the Shapley value $\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right)$, which are calculated according to the formula (29).

Due to individual rationality $\nu_{K_l}^{(t_0)}(t, x_N^*(t)) \geq W^{(t_0)K_l}\left(t, x_{K_l}^*(t)\right)$, payoff of coalition is higher than in individual development. Therefore, the characteristic function $V^{K_l(t_0)}\left(K_l, x_{K_l}^*(t), T-t\right)$ is equal to the payoff of coalition K_l in the game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$, ie component of the Shapley value:

$$V^{K_l(t_0)}\left(K_l, x_{K_l}^*(t), T-t\right) = \nu_{K_l}^{(t_0)}(t, x_N^*(t))$$

Calculate the characteristic function of any coalition $L \subset K_l$. The function $V^{K_l(t_0)}(L, x_L(t), T-t)$ is found by solving the following optimization problem:

$$\begin{aligned}
V^{K_l(t_0)}(L, x_L(t), T-t) &= \max_{u_L}\left(H_L\left(x_L(t), T-t, u_L(t)\right)\right) = \\
&= \max_{u_L}\left(\sum_{i \in L} H_i\left(x_i(t), T-t, u_i(t)\right)\right) = \\
= \max_{u_L}\Bigg(\sum_{i \in L} \int_t^T & h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s-t_0)\right] + \\
&+ \sum_{i \in L} \exp\left[-r(T-t_0)\right] q_i\left[x_i(T)\right]^{1/2}\Bigg), L \subset K_l
\end{aligned}$$

Function $V^{K_l(t_0)}(L, x_L(t), T-t)$ satisfies the Bellman equation:

$$\begin{aligned}
&-V_t^{K_l(t_0)}(L, x_L(t), T-t) = \\
&= \max_{u_L}\Bigg\{\sum_{i \in L} h_i\left(t, x_i(t), u_i(t)\right) \exp\left[-r(t-t_0)\right] + \\
&+ \sum_{i \in L} V_{x_i}^{K_l(t_0)}(L, x_L(t), T-t)\, f_i^L\left[x_L(t), u_i(t)\right]\Bigg\}
\end{aligned}$$

$$V^{K_l(t_0)}\left(L, x_N^0, T\right) = \sum_{i \in L} \exp\left[-r(T-t_0)\right] q_i\left[x_i(T)\right]^{1/2}$$

$$h_i\left(t, x_i(t), u_i(t)\right) = \left[P_i[x_i(t)]^{1/2} - c_i u_i(t)\right]$$

$$f_i^L\left[x_L(t), u_i(t)\right] = \alpha_i\left[u_i(s) x_i(s)\right]^{1/2} + \sum_{j \in L,\ j \neq i} b_j^{[j,i]}[x_j(s) x_i(s)]^{1/2} - \delta x_i(s)$$

Using a similar technique to that described above, receive:

$$V^{K_l(t_0)}(L, x_L(t), T-t) = \left[\sum_{i \in L} A_i^L(t)\left[x_i(t)\right]^{1/2} + C^L(t)\right] \exp\left[-r(t-t_0)\right]$$

Values $A_i^L(t)$ and $C^L(t)$ are solutions of differential equations:

$$\dot{A}_i^L(t) = \left(r + \frac{\delta}{2}\right) A_i^L(t) - \sum_{j \in L, j \neq i} \frac{b_j^{[i,j]}}{2} A_j^L(t) - P_i$$

$$\dot{C}^L(t) = rC^l(t) - \sum_{i \in L} \frac{\alpha_i^2}{16c_i} A_i^L(t)$$

$$A_i^L(T) = q_i, \quad C^L(T) = 0, \quad i \in L \subset K_l$$

From the equations above is formula of partial derivatives:

$$V_t^{K_l(t_0)}(L, x_L(t), T-t) = \left(\left[\sum_{i \in L} \dot{A}_i^L(t)\left[x_i(t)\right]^{1/2} + \dot{C}^L(t)\right] - \right.$$
$$\left. - r\left[\sum_{i \in L} A_i^L(t)\left[x_i(t)\right]^{1/2} + C^L(t)\right]\right) \exp\left[-r(t-t_0)\right]$$

$$V_{x_i}^{K_l(t_0)}(L, x_L(t), T-t) = \frac{1}{2} A_i^L(t)\left[x_i(t)\right]^{-1/2} \exp\left[-r(t-t_0)\right]$$

It is easy to verify that the characteristic function $V^{K_l(t_0)}(L, x_L(t), T-t)$ coincides in this case with the function $W^{(t_0)L}(t, x_L(t))$, which determines the maximum payoff of coalition L in game of firms (Yeung and Petrosyan, 2006). Thus, the characteristic function of the game $\Gamma^{K_l}\left(x^0, T-t_0\right)$ has the following form:

$$V^{K_l(t_0)}(K, x_K(t), T-t) = \begin{cases} 0 & K = \emptyset \\ \nu_{K_l}^{(t_0)}(t, x_N(t)) & K = K_l \\ W^{(t_0)i}(t, x_i(t)) & K = \{i\} \in K_l \\ W^{(t_0)L}(t, x_L(t)) & K = L \subset K_l \end{cases} \tag{33}$$

The superadditivity of function $\Gamma^{K_l}\left(x^0, T-t_0\right)$ obviously follows from superadditivity of function $W^{(t_0)L}(t, x_L(t))$ (Theorem 2) and the condition $\nu_{K_l}^{(t_0)}(t, x_N^*(t)) \geq W^{(t_0)K_l}\left(t, x_{K_l}^*(t)\right)$.

Introduce the imputation distribution procedure in the game. Assume that firms share the payoff of coalition K_l according to the dynamic Shapley value $\nu^{(t_0)K_l}\left(t_0, x_{K_l}^0\right) = \left\{\nu_i^{(t_0)K_l}\left(t_0, x_{K_l}^0\right)\right\}_{i \in K_l}$ again. Since coalition K_l involved in the game of coalitions $\Gamma^{\Delta}\left(x^0, T-t_0\right)$, then its participants maximize joint payoff $W^{(t_0)N}\left(t_0, x_N^0\right)$, using a set of optimal controls $\{u_i^*(t)\}_{i \in N}$, obtained by the formula (17) on the interval $[t_0, T]$ and implement corresponding optimal trajectory for the case $\breve{K} = \breve{N}$.

At the initial moment t_0, the share of firm's $i \in K_l$ cooperative profit is equal to:

$$\nu_i^{(t_0)K_l}\left(t_0, x^0_{K_l}\right) = \sum_{K\subseteq K_l} \frac{(k-1)!(k_l-k)!}{k_l!}\Big[V^{K_l(t_0)}\left(K, x^0_K, T-t_0\right) - \\ - V^{K_l(t_0)}\Big(K\setminus i, x^0_{K\setminus i}, T-t_0\Big)\Big],$$

where $k_l = |K_l|$ is number of coalition members K_l.

The Shapley value has to be maintained throughout the game At moment $t \in [t_0, T]$ and state $x^*_{K_l}(t) \in x^*_N(t)$ for firm $i \in K_l$ has to be maintaned the following principle:

$$\nu_i^{(t_0)K_l}\left(t, x^*_{K_l}(t)\right) = \sum_{K\subseteq K_l} \frac{(k-1)!(k_l-k)!}{k_l!}\Big[V^{K_l(t_0)}\left(K, x^*_K(t), T-t\right) - \\ - V^{K_l(t_0)}\Big(K\setminus i, x^*_{K\setminus i}(t), T-t\Big)\Big]$$

Given (33), the formula for the components of the Shapley value can be rewritten as follows:

$$\nu_i^{(t_0)K_l}\left(t, x^*_{K_l}(t)\right) = \sum_{K\subset K_l} \frac{(k-1)!(k_l-k)!}{k_l!}\Big[W^{(t_0)K}\left(t, x^*_K(t)\right) - \\ - W^{(t_0)K}\Big(t, x^*_{K\setminus i}(t)\Big)\Big] + \\ + \frac{1}{k_l}\Big[\nu^{(t_0)}_{K_l}\left(t, x^*_{K_l}(t)\right) - W^{(t_0)K_l\setminus i}\Big(t, x^*_{K_l\setminus i}(t)\Big)\Big]$$

For realization the dynamic Shapley is necessary at any moment to make a redistribution of the joint payoff. Define the imputation distribution procedure for allocating sharing (Petrosyan, 1977), as a function of $B^{K_l}(t) = \Big\{B_i^{K_l}(t)\Big\}_{t=t_0}^{T}$, such that:

$$\nu_i^{(t_0)K_l}\left(t_0, x^0_{K_l}\right) = \int\limits_{t_0}^{T} B_i^{K_l}(s)\exp\left[-r(s-t_0)\right]ds + \\ + \exp\left[-r(T-t_0)\right] q_i \left[x_i^*(T)\right]^{1/2} \tag{34}$$

Function $B_i^{K_l}(s)$ is an payment received by firm $i \in K_l$ at the moment $s \in [t_0, T]$. For dynamic stability of solution the next equality has to be maintained at any moment $t \in [t_0, T]$:

$$\nu_i^{(t_0)K_l}\left(t, x^*_{K_l}(t)\right) = \int\limits_{t}^{T} B_i^{K_l}(s)\exp\left[-r(s-t_0)\right]ds + \\ + \exp\left[-r(T-t_0)\right] q_i \left[x_i^*(T)\right]^{1/2} \tag{35}$$

From (34) and (35) receive:

$$\nu_i^{(t_0)K_l}\left(t_0, x^0_{K_l}\right) = \int\limits_{t_0}^{t} B_i^{K_l}(s)\exp\left[-r(s-t_0)\right]ds + \nu_i^{(t_0)K_l}\left(t, x^*_{K_l}(t)\right)$$

Function $B_i^{K_l}(t)$ is determined from the derivative component of the Shapley value $\nu_i^{(t_0)K_l}\left(t_0 x^0_{K_l}\right)$. Note, that characteristic function $V^{K_l(t_0)}\left(K_l, x_{K_l}(t), T-t\right)$ is equal component of Shapley value $\nu^{(t_0)}_{K_l}\left(t, x^*_N(t)\right)$ in the game of coalitions $\Gamma^{\Delta}\left(x^0, T-t_0\right)$, which depends from the states of all participants of the partition Δ. Consequently, partial derivatives of the components $\nu_i^{(t_0)K_l}\left(t, x^*_{K_l}(t)\right)$ for firms' $j \notin K_l$ states are not zero.

In the general case $B_i^{K_l}(s)$ has the form:

$$\begin{aligned}
B_i^{K_l}(s) = \sum_{K\subset K_l} \frac{(k-1)!(k_l-k)!}{k_l!}\Bigg\{\Big[W_t^{(s)K}\left(s, x^*_K(s)\right) - \\
W_t^{(s)K\setminus i}\left(s, x^*_{K\setminus i}(s)\right)\Big] + \sum_{j\in K}\left[W_{x_j}^{(s)K}\left(s, x^*_K(s)\right)\right] f_j^N\left[x^*_N(s), u^*_j(s)\right] - \\
- \sum_{h\in K\setminus i}\left[W_{x_h}^{(s)K\setminus i}\left(s, x^*_{K\setminus i}(s)\right)\right] f_h^N\left[x^*_N(s), u^*_h(s)\right]\Bigg\} + \\
+\frac{1}{k_l}\Bigg\{\left[\nu^{(t_0)}_{K_l}\right]_t\left(s, x^*_N(s)\right) - W_t^{(s)K_l\setminus i}\left(s, x^*_{K_l\setminus i}(s)\right) + \\
+\sum_{j\in N}\left[\nu^{(t_0)}_{K_l}\right]_{x_j}\left(s, x^*_N(s)\right) f_j^N\left[x^*_N(s), u^*_j(s)\right] - \\
- \sum_{h\in N\setminus i}\left[W_{x_h}^{(s)K_l\setminus i}\left(s, x^*_{K_l\setminus i}(s)\right)\right] f_h^N\left[x^*_N(s), u^*_h(s)\right]\Bigg\}
\end{aligned} \quad (36)$$

7. Dynamic stability of coalitions technological alliance

To prove the dynamic stability of the coalition built solution necessary to show that at each moment there is only a redistribution of payoff between all firms $i \in N$, and the total sum of profit remains unchanged, ie it is necessary to prove:

$$\begin{aligned}
\sum_{K_l\subset\Delta}\sum_{i\in K_l} B_i^{K_l}(s) = \sum_{K_l\subset\Delta}\sum_{i\in K_l} h_i\left(s, x^*_i(s), u^*_i(s)\right) = \\
= \sum_{K_l\subset\Delta}\sum_{i\in K_l}\left[P_i\left[x^*_i(s)\right]^{1/2} - c_i u^*_i(s)\right]
\end{aligned} \quad (37)$$

It has already been established that at any time there is a redistribution of profits between coalitions participating game $\Gamma^{\Delta}\left(x^0, T-t_0\right)$. Consequently, equality (37) can be rewritten as:

$$\sum_{K_l \subset \Delta} \sum_{i \in K_l} B_i^{K_l}(s) = \sum_{K_l \subset \Delta} B_{K_l}(s)$$

To prove this equality it's enough to show that $\sum\limits_{i \in K_l} B_i^{K_l}(s) = B_{K_l}(s)$ for any coalition-participant $K_l \subset \Delta$.

Summing component $B_i^{K_l}(s)$, is easy to check, that:

$$\sum_{i \in K_l} B_i^{K_l}(s) = -\left(\left[\nu_{K_l}^{(t_0)}\right]_t (s, x_N^*(s)) + \right.$$

$$\left. + \sum_{j \in N} \left[\nu_{K_l}^{(t_0)}\right]_{x_j} (s, x_N^*(s)) \, f_j^N \left[x_N^*(s), u_j^*(s)\right]\right)$$

The right side of this equation is the total derivative of $\nu_{K_l}^{(t_0)}(t, x_N^*(t))$ for t with the opposite sign, which by definition equal $B_{K_l}(s)$. Thus, the resulting coalition solution is dynamically stable.

8. Strategic stability of coalitions technological alliance

Show the strategic stability of cooperative solution. Recall that the cooperative solution is strategically stable if no individual deviation from the cooperation of each member brings benefits the decline member.

Since members of a cooperative game $\Gamma_V^{\Delta}\left(x_N^0, T - t_0\right)$ is not individual firms but coalitions $\{K_l\}_{l \in M}$, consider that a individual firm $i \in K_l$ can not itself get out of the cooperation $\breve{N}$. Only the whole coalition K_l could.

Formulate a theorem on the existence of ε-Nash-equilibrium in the game $\Gamma_V^{\Delta}\left(x_N^0, T - t_0\right)$.

Theorem 3. *In the game $\Gamma_V^{\Delta}\left(x_N^0, T - t_0\right)$ for any $\varepsilon > 0$, there is ε-Nash-equilibrium with payoffs equal components of share $\left\{\nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right)\right\}_{K_l \subset \Delta}$.*

Proof. The proof is done by constructing an optimal control at the class of piecewise program strategies. Consider a family associated with the game $\Gamma_V^{\Delta}(x, T - t)$ games $\Gamma_{K_l, \breve{N} \backslash K_l}(x, T - t)$ from the initial state $x = \{x_i\}_{i \in N}$ duration $T - t$, where coalition K_l is played individually, and the other coalitions-participants united in a coalition of $\breve{N} \backslash K_l$. Since $\Gamma_{K_l, \breve{N} \backslash K_l}(x, T - t)$ is not a zero-sum game between the K_l and $N \backslash K_l$, the maximum payoff of coalition K_l in the game $\Gamma_{K_l, N \backslash K_l}(x, T - t)$ is determined by the function $W^{(t_0) K_l}\left(t, x_{K_l}^*(t)\right)$.

Let $\hat{u}^{\Delta \backslash K_l}(x, t; \cdot)$ - this is $\breve{N} \backslash K_l$ is optimal piecewise program strategy of the coalition $\breve{N} \backslash K_l$ in the game $\Gamma_{K_l, N \backslash K_l}(x, Tt)$. For ease of description denote it N_{-K_l} by optimal strategy. This strategy leads to the maximization of the coalition win $\breve{N} \backslash K_l$.

It is logical to assume that in case of deviation the coalition K_l from the optimal control of the cooperative $u_{K_l}^*(s) \in u_N^*(s)$ and cooperative trajectory

$x^*_{K_l}(s) \in x^*_N(s)$ it stops receiving the bonuses of which gave the alliance and continues to evolve independently. But since piecewise program strategies used, it is not happening at once but when switching $\{t_\xi\}, \quad \xi = 0, 1, \ldots, \theta - 1$.

Let $\hat{x}(\tau) = \{\hat{x}_{K_1}(\tau), \ldots, \hat{x}_{K_m}(\tau)\}$ is a segment of admissible trajectory of players, defined on the interval $[t_0,\ t]$, $t \in [t_0,\ T]$. For each $K_l \in \Delta = \{K_1, \ldots, K_m\}$ define the values: $\bar{t}(l) = \sup\left\{t_l :\ \hat{x}_{K_l}(t_l) = x^*_{K_l}(t_l)\right\}$ and $\bar{\bar{t}}(h) = \min_h \{\bar{t}(l) = \bar{t}(h)\}$. Here $\bar{\bar{t}}(h)$ belongs to one of the intervals $[t_\xi,\ t_{\xi+1}], \quad \xi = 0, 1, \ldots, \theta - 1$. Thus, the interval $[t_0,\ \bar{t}(l)]$ is the period where $\hat{x}_{K_l}(\tau) = x^*_{K_l}(\tau)$, and $\left[t_0,\ \bar{\bar{t}}(h)\right]$ is period, in which $\hat{x}(\tau) = x^*_N(\tau)$. Note that in this case the value of $\hat{x}_{K_l}(\tau)$ is a set of trajectories of firms from coalition K_l, ie $\hat{x}_{K_l}(\tau) = \{\hat{x}_i(\tau)\}_{i \in K_l}$. Therefore, the expression $\hat{x}_{K_l}(\tau) = x^*_{K_l}(\tau)$ means that $\hat{x}_i(\tau) = x^*_i(\tau)$ for all $i \in K_l$.

Define the following strategies for the coalition K_l, in condition $t_\xi \le \bar{\bar{t}}(h) \le t_{\xi+1}$:

$$u^*_{K_l}(\tau) = \begin{cases} u^*_{K_l}(\tau) \quad \hat{x}(\tau) = x^*_N(\tau), \quad \tau \in [t_0, t_\xi] \\ \hat{u}^{\Delta \setminus K_h}_{K_l}(\hat{x}(t_{\xi+1}), t_{\xi+1}), \quad \tau \in (t_\xi, t_{\xi+1}] \\ \forall u_{K_l}(\tau), \quad \tau \in (t_{\xi+1}, T] \end{cases} \tag{38}$$

Prove that situation $u^*(\cdot) = \left\{u^*_{K_l}(\cdot)\right\}_{K_l \subset \Delta}$, where the strategy defined by the formula (38) will be ε-Nash-equilibrium in the game $\Gamma^{\Delta}_V(x^0_N, T - t_0)$.

If all the coalition-participants do not deviate from the conditionally optimal cooperative trajectory throughout the game, the payoff of coalition K_l is equal to the corresponding component of the Shapley value:

$$\begin{aligned} H_{K_l}\left(x^0_{K_l}, T - t_0, u^*_{K_l}\right) = \int_{t_0}^{T} B_{K_l}(s) \exp\left[-r(s - t_0)\right] ds + \\ + \exp\left[-r(T - t_0)\right] \sum_{i \in K_l} q_i \left(x_i(T)\right)^{1/2} = \nu^{(t_0)}_{K_l}(t_0, x^0_N) \end{aligned} \tag{39}$$

Let coalition K_l deviated from the cooperative trajectory at the moment $t \in [t_{\xi-1}, t_\xi]$, changing its strategy on u_{K_l}. Show that:

$$H_{K_l}\left(x^0_{K_l}, T - t_0, u^*_{K_l}\right) \ge H_{K_l}(x^0_{K_l}, T - t_0, u_{K_l}) - \varepsilon, \tag{40}$$

for all $K_l \subset \Delta$ and any strategy u_{K_l}. Assume that the resulting trajectory of $x(\tau)$ is different from the path of $x^*_N(\tau)$. If from the moment t_ξ, coalition $\{\Delta \setminus K_l\}$ will use its N_{-K_l}-optimal strategy for $\hat{u}^{\Delta \setminus K_l}(x, t; \cdot)$, then, from this moment, max payoff of coalition K_l, with discount will be:

$$H_{K_l}(x_{K_l}(t_\xi), T - t_\xi, u_{K_l}) = \exp\left[-r(t_\xi - t_0)\right] *$$
$$* \left(\sum_{i \in K_l} \int_{t_\xi}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s - t_\xi)\right] ds \cdot + \right.$$
$$\left. + \exp\left[-r(T - t_\xi)\right] \sum_{i \in K_l} q_i \left(x_i(T)\right)^{1/2} \right) =$$
$$= \sum_{i \in K_l} \int_{t_\xi}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s - t_0)\right] ds +$$
$$+ \exp\left[-r(T - t_0)\right] \sum_{i \in K_l} q_i \left(x_i(T)\right)^{1/2}$$

Then its total payoff will be:

$$H_{K_l}\left(x^0_{K_l}, T - t_0, u_{K_l}\right) = \int_{t_0}^{t} B_{K_l}(s) \exp\left[-r(s - t_0)\right] ds + \tag{41}$$
$$\int_{t}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s - t_0)\right] ds +$$
$$\sum_{i \in K_l} \int_{t_\xi}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s - t_0)\right] ds +$$
$$\exp\left[-r(T - t_0)\right] \sum_{i \in K_l} q_i \left(x_i(T)\right)^{1/2},$$

where the function $B'_{K_l}(s, x(s))$ is the payment received by a coalition K_l at moment $s \in [t, t_\xi)$ and calculated as result of the imputation distribution procedure where the trajectory $x^*_N(s)$ is replaced by the path $x(s)$. Given that

$$\nu^{(t_0)}_{K_l}\left(t_0, x^0_N\right) = \int_{t_0}^{t} B_{K_l}(s) \exp\left[-r(s - t_0)\right] ds + \nu^{(t_0)}_{K_l}\left(t, x^*_N(t)\right),$$

and besides

$$\sum_{i \in K_l} \int_{t_\xi}^{T} h_i\left(s, x_i(s), u_i(s)\right) \exp\left[-r(s - t_0)\right] ds +$$
$$+ \exp\left[-r(T - t_0)\right] \sum_{i \in K_l} q_i \left(x_i(T)\right)^{1/2} = W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right),$$

the payoff of coalition K_l is equal:

$$H_{K_l}\left(x^0_{K_l}, T - t_0, u_{K_l}\right) = \nu^{(t_0)}_{K_l}(t_0, x^0_N) - \nu^{(t_0)}_{K_l}(t, x^*_N(t)) + \tag{42}$$
$$+ \int_{t}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s - t_0)\right] ds + W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right)$$

Since $t \in [t_{\xi-1}, t_\xi]$, then consequently

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) \geq \nu_{K_l}^{(t_0)}\left(t_\xi, x_N(t_\xi)\right)$$

and

$$\int_t^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s-t_0)\right] ds \leq \int_{t_{\xi-1}}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s-t_0)\right] ds$$

Therefore, the coalition's K_l payoff are not greater than

$$\begin{aligned} H_{K_l}\left(x_{K_l}^0, T-t_0, u_{K_l}\right) \leq \nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) - \nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) + \\ + \int_{t_{\xi-1}}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s-t_0)\right] ds + W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right) \end{aligned} \tag{43}$$

Choosing a value $\delta > 0$ $(t_\xi - t_{\xi-1} = \delta)$ small enough can be ensure that the integral $\int_{t_{\xi-1}}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r(s-t_0)\right] ds$ was less than $\varepsilon/2$.

Then the following inequality is maintained:

$$\begin{aligned} H_{K_l}\left(x_{K_l}^0, T-t_0, u_{K_l}\right) \leq \nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) - \nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) + \\ + \int_{t_{\xi-1}}^{t_\xi} B'_{K_l}(s, x(s)) \exp\left[-r\left(s-t_0\right)\right] ds + W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right) \leq \\ \leq \nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) - \nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) + W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right) + \varepsilon/2 \end{aligned} \tag{44}$$

It is incorrect to argue that

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) \geq W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right),$$

because the state $x_{K_l}(t_\xi)$ is not included in cooperative trajectory x_N^*. But because the dynamic Shapley value has the property of individual rationality, ie

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) \geq W^{(t_0)K_l}\left(t_\xi, x_{K_l}^*(t_\xi)\right),$$

then choosing the value $\delta > 0$ $(t_\xi - t_{\xi-1} = \delta)$ small enough can ensure that

$$\nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) \geq W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right) - \varepsilon/2,$$

Therefore

$$\begin{aligned} \nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) - \nu_{K_l}^{(t_0)}\left(t, x_N^*(t)\right) + \\ + W^{(t_0)K_l}\left(t_\xi, x_{K_l}(t_\xi)\right) + \varepsilon/2 \leq \nu_{K_l}^{(t_0)}\left(t_0, x_N^0\right) + \varepsilon \end{aligned} \tag{45}$$

As a result obtain the inequality:

$$H_{K_l}\left(x^0_{K_l}, T-t_0, u_{K_l}\right) \leq \nu^{(t_0)}_{K_l}\left(t_0, x^0_N\right) + \varepsilon = \\ = H_{K_l}\left(x^0_{K_l}, T-t_0, u^*_{K_l}\right) + \varepsilon,$$

as required.

The strategic stability within the coalition proved similarly. Thus the strategic stability of the coalition solution is proved.

9. Quantitative example

Illustrate numerical results on the example of the three firms. On the set of firms $N = \{1, 2, 3\}$ defined partition $\Delta = \{\{1, 2\}, \{3\}\}$ consisting of two coalitions. Given initial parameters: $t_0 = 0$; $T = 20$; $r = 0,1$; $\delta = 0,2$; $P_1 = 0,6; P_2 = 0,3; P_3 = 0,15$; $c_i = 0,5$; $\alpha_i = 0,3$; $b^{[j,i]}_j = 0,05$; $q_i = 0,1$ $i, j =\in N$

Figure 1 shows graphics of coalitions states for a game duration in the case of individual developı

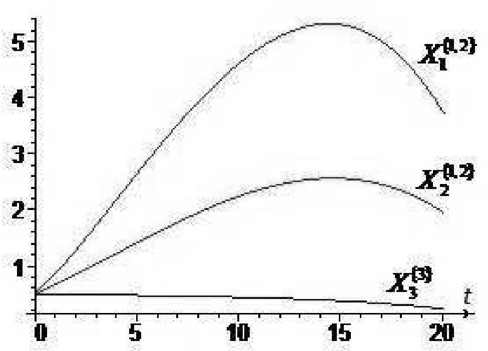

Fig. 1: Graphics of coalitions states in the case of individual development

On the figure 2 here are graphics of coalitions states for a game duration in the case of cooperation:

On the figures 3 and 28 here are graphics of coalitions profit before redistribution ($h_{1,2}(t) = h_1(t) + h_2(t)$ and $h_3(t)$) and after redistribution ($B_{1,2}(t)$ and $B_{\{3\}}(t)$).

Figure 5 illustrate profit's shares of firms 1 and 2 inside coalition $\{1, 2\}$ ($B^{\{1,2\}}_1(t)$ and $B^{\{1,2\}}_2(t)$)

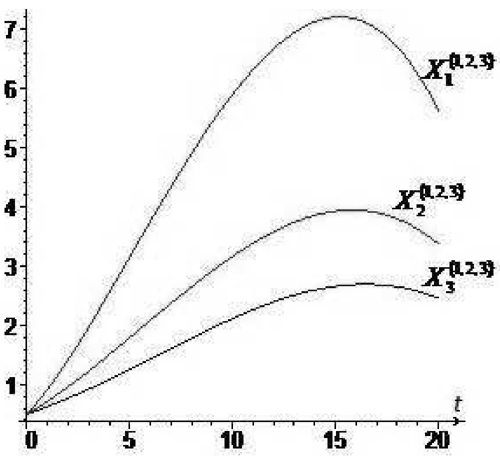

Fig. 2: graphics of coalitions states in the case of technological alliance

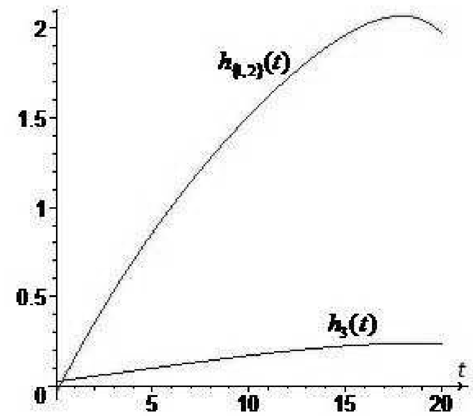

Fig. 3: Graphics of coalitions profit before redistribution

Table 5.3. shows the values of coalitions payoffs in randomly selected time points before and after the payoff redistribution. The values of profits before and after redistribution are different, but the sum does not change.

Table 1: Values of coalitions payoffs before and after redistribution

t	$h_{\{1,2\}}(t)$	$h_{\{3\}}(t)$	$h_{\{1,2\}}(t) + h_{\{3\}}(t)$	$B_{\{1,2\}}(t)$	$B_{\{3\}}(t)$	$B_{\{1,2\}}(t) + B_{\{3\}}(t)$
0	-0,04031	0,02488	**-0,01543**	-0,07842	0,06299	**-0,01543**
5	0,85579	0,09914	**0,95493**	0,81135	0,14358	**0,95493**
10	1,51064	0,16949	**1,68013**	1,46475	0,21538	**1,68013**

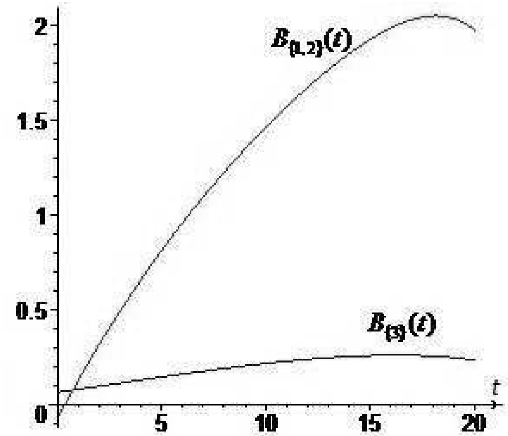

Fig. 4: Graphics of coalitions profit after redistribution

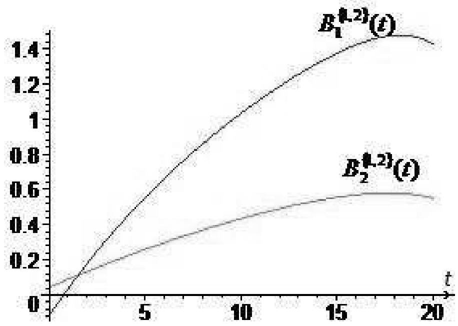

Fig. 5: Graphics of firms profit

Table 2 shows the value of profits of firms 1 and 2 in the coalition $\{1, 2\}$ in a randomly selected time points. At each point equal the sum of the profits of firms profit coalition.

Table 2: Values of firms payoffs after payoff redistribution

t	$B_1^{\{1,2\}}(t)$	$B_2^{\{1,2\}}$	$B_1^{\{1,2\}}(t) + B_2^{\{1,2\}}(t)$	$B_{\{1,2\}}(t)$
0	-0,12022	0,04180	**-0,07842**	**-0,07842**
5	0,55274	0,25861	**0,81135**	**0,81135**
10	1,03134	0,43341	**1,46475**	**1,46475**

Thus the constructed cooperative solution is dynamically stable.

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