Contributions to Game Theory and Management, VIII, 99–110

Consistent Conjectural Variations Equilibrium in an Optimal Portfolio Model*

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Abstract In this paper, a general multi-sector, multi-instrument model of financial flows and prices is developed, in which the utility function for each sector is assumed to be quadratic, while the constraints satisfy a certain identity that appears in flow-of-funds accounts. Each sector uses conjectures about its influence upon the prices of the instruments. The equilibrium conditions are first derived, and then the governing variational inequality problems are deduced. Subsequently, a qualitative analysis of the model is conducted, and a concept of consistent conjectures is introduced and examined as well.

Keywords: conjectural variations equilibrium, consistent conjectures, consistent equilibrium, optimal portfolio models

1. Introduction

Consider an economy consisting of m sectors, with a typical sector denoted by i, and with n instruments, with a typical instrument denoted by j. Denote the volume of instrument j held in sector i's portfolio as an asset, by x_{ij}, and the volume of instrument j held in sector's i's portfolio as a liability, by y_{ij}. The assets in sector i's portfolio are grouped into a column vector $x_i \in R^n$, and the liabilities are grouped into the column vector $y_i \in R^n$. Further group the sector asset vectors into the column vector $x \in R^{mn}$, and the sector liability vectors into the column vector $y \in R^{mn}$.

* This work was financially supported by the Research Department of the Tecnológico de Monterrey (ITESM), Campus Monterrey, and by the SEP-CONACYT project CB-2013-01-221676, Mexico. The second author was also supported by the PAICYT project No. CE250-09.

Each sector's utility can be defined as a function of the expected future portfolio value. The expected value of the future portfolio may be described by two characteristics: the expected mean value and the uncertainty surrounding the expected mean. In this model, the expected mean portfolio value of the next period is assumed to be equal to the market value of the current period portfolio. Each sector's uncertainty, or assessment of risk, with respect to the future value of the portfolio is based on a variance-covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument. The $2n \times 2n$ variance-covariance matrix associated with sector i's assets and liabilities is denoted by Q^i.

Since each sector's expectations are formed by reference to current market activity, sector utility maximization can be written in terms of optimizing the current portfolio. Sectors may trade, issue, or liquidate holdings in order to optimize their portfolio compositions.

In this model, it is assumed that the total volume of each balance sheet side is exogenous. Let r_j denote the price of instrument j, and group the prices into the column vector $r \in R^n$. In contrast to the model by (Nagurney, 1999) that makes use of the assumption of perfect competition, i.e., supposes that each sector will behave as if its actions cannot affect the instruments' prices and thus the behavior of the other sectors, we examine a simple oligopoly model. In the latter, we search *conjectural variations equilibrium* (CVE); in other words, we assume that each sector i expects the price of instrument j to grow up together with the (positive) gap $y_{ij} - x_{ij}$ between his liability and asset holdings, and the rate of this grow is $w_{ij} \geq 0$ that will be referred to as sector i's *influence coefficients* upon the price of instrument j. This CVE approach is the main novelty of our model compared to that studied in (Nagurney, 1999).

In this paper, we follow the customary line of dealing with conjectural variations equilibrium exercised in the authors' previous papers (Kalashnikov et al., 2011), (Bulavsky and Kalashnikov, 2012), and (Kalashnikov et al., 2014). Namely, under general enough assumptions, the CVE existence and uniqueness (for each fixed matrix of the influence coefficients) are established. Such CVE are referred to as *exterior* ones. Based upon these results, we introduce the notion of *consistent* conjectures and the related consistent CVE (also called as *interior* equilibrium). The existence theorem about the consistent conjectural variations equilibrium is also established.

The rest of the paper is organized as follows. The quadratic-type portfolio optimization problem is specified and studied in Section 2. Section 3 deals with the exterior conjectural variations equilibrium (CVE) in the financial model with general utility functions of the sector, and the exterior CVE properties are examined in Section 4. Finally, a new concept of consistent (interior) conjectural variations equilibrium is introduced and discussed in Section 5. Concluding remarks, acknowledgments, and a list of references complete the paper.

2. Model Specification

Define each sector's portfolio optimization problem as follows. Sector i seeks to determine its optimal composition of instruments held as assets and as liabilities, so as to minimize the risk while at the same time maximizing the value of its asset holdings and minimizing the value of its liabilities. The portfolio optimization problem for sector i is, hence, given by:

$$\text{Minimize} \qquad \begin{pmatrix} x_i \\ y_i \end{pmatrix}^T Q^i \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \sum_{j=1}^{n} r_{ij} \left(x_{ij} - y_{ij} \right) \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, \ldots, m; \tag{2}$$

$$\sum_{j=1}^{n} y_{ij} = s_i, \quad i = 1, \ldots, m; \tag{3}$$

and

$$x_{ij} \geq 0, \quad y_{ij} \geq 0, \quad j = 1, \ldots, n, \tag{4}$$

where

$$r_{ij} = r_{ij} \left(x_{ij}, y_{ij} \right) = r_j - w_{ij} \left(x_{ij} - y_{ij} \right), \quad i = 1, \ldots, m; \; j = 1, \ldots, n. \tag{5}$$

Here, the instrument price vector $r \in R^n$ is exogenous to the individual sector optimization problem, whereas the quotient $w_{ij} \geq 0$ reflects the degree of influence of sector i on the price of instrument j conjectured (perceived) by this sector. That is, sector i conjectures the expected price for its liability to equal r_{ij} determined by (5).

Constraints (2) and (3) represent the accounting identity meaning that the accounts for sector i must balance, where s_i is the total financial volume held by sector i. Constraints (4) are the usual nonnegativity restrictions. Let Ω_i denote the closed convex subset of (x_i, y_i) formed by constraints (2) – (4). Since Q^i is a variance-covariance matrix, it will be assumed to be positive definite and, therefore, the objective function for each sector i's portfolio optimization problem is strictly convex.

Necessary and sufficient conditions for a portfolio $(x_i^*, y_i^*) \in \Omega_i$ to be optimal is that it solves the following *linear complementarity problem* (LCP).

For each instrument j ($j = 1, \ldots, n$):

$$\varphi_{ij}^1 \left(x^*, y^* \right) \equiv 2{Q^i_{(11)j}}^T x_i^* + 2{Q^i_{(21)j}}^T y_i^* - r_j^* + 2w_{ij} \left(x_{ij}^* - y_{ij}^* \right) - \mu_i^1 \geq 0, \tag{6}$$

$$\varphi_{ij}^2 \left(x^*, y^* \right) \equiv 2{Q^i_{(22)j}}^T y_i^* + 2{Q^i_{(12)j}}^T x_i^* + r_j^* - 2w_{ij} \left(x_{ij}^* - y_{ij}^* \right) - \mu_i^2 \geq 0, \tag{7}$$

$$\text{and} \quad x_{ij}^* \cdot \varphi_{ij}^1 \left(x^*, y^* \right) = 0, \quad y_{ij}^* \cdot \varphi_{ij}^2 \left(x^*, y^* \right) = 0, \tag{8}$$

where r_j^* denotes the price for instrument j, which is assumed to be fixed from the perspective of all the sectors. Note that each Q^i has been partitioned as

$$Q^i = \begin{pmatrix} Q^i_{(11)} & Q^i_{(12)} \\ Q^i_{(21)} & Q^i_{(22)} \end{pmatrix}, \quad i = 1, 2,$$

and is symmetric. Furthermore, $Q^i_{(\alpha\beta)j}$ denotes the j-th column of $Q^i_{(\alpha\beta)}$, with $\alpha = 1, 2; \beta = 1, 2$. The terms μ_i^1 and μ_i^2, $i = 1, \ldots, m$, are the Lagrange multipliers associated with constraints (2) and (3), respectively.

Each of the m sectors has to solve an appropriate LCP of form (6)–(8).

2.1. Definition of Equilibrium

The inequalities governing the instrument prices in the economy are now described. These prices provide the feedback from the economic system to the sectors regarding the equilibration of the total assets and total liabilities of each instrument. Here, it is assumed that there is free disposal and, hence, the instrument prices will be nonnegative. The economic system conditions ensuring market clearance then take on the following form.

For each instrument j; $j = 1, \ldots, n$:

$$\sum_{i=1}^{m}(x_{ij}^* - y_{ij}^*) \begin{cases} = 0, & \text{if } r_j^* > 0; \\ \geq 0, & \text{if } r_j^* = 0. \end{cases} \tag{9}$$

In other words, if the price is positive, then the market must clear for the instrument; if there is an excess supply of an instrument in the economy, then its price must be zero. Combining the above sector and market inequalities and equalities yields the following definition; here, the term *exterior* is used in the same sense that in the authors' previous paper (Kalashnikov et al., 2011).

Definition 1. (*Exterior Equilibrium in the Financial Model.*)

For a fixed set of conjectures described by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m\ \ n}$, a vector $(x^*, y^*, r^*) \in \prod_{i=1}^m \Omega_i \times R_+^n$ is called the *exterior equilibrium* of the financial model if and only if it satisfies the system of equalities and inequalities (6) − (9), for all sectors $1 \leq i \leq m$, and for all instruments $1 \leq j \leq n$, simultaneously.

Now we are ready to deduce the variational inequality problem equivalent to the exterior equilibrium conditions of our financial model.

Theorem 1. *(Variational Inequality Formulation of Exterior Financial Equilibrium.)*

For a fixed set of conjectures described by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m\ \ n}$, a vector $(x^*, y^*, r^*) \in \prod_{i=1}^m \Omega_i \times R_+^n$ of sector assets, liabilities, and instrumental prices is the exterior financial equilibrium if and only if it solves the following variational inequality problem: Determine a vector $(x^*, y^*, r^*) \in \prod_{i=1}^m \Omega_i \times R_+^n$, satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2{Q^i_{(11)j}}^T x_i^* + 2{Q^i_{(21)j}}^T y_i^* - r_j^* + 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(x_{ij} - x_{ij}^*\right) +$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2{Q^i_{(22)j}}^T y_i^* + 2{Q^i_{(12)j}}^T x_i^* + r_j^* - 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(y_{ij} - y_{ij}^*\right) +$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(r_j - r_j^*\right) \geq 0, \quad \forall (x, y, r) \in \prod_{i=1}^{m}\Omega_i \times R_+^n. \tag{10}$$

Proof. A. Necessity.

Assume that $(x^*, y^*, r^*) \in \prod_{i=1}^{m} \Omega_i \times R_+^n$ is an exterior equilibrium state. Then inequalities and equalities (6) – (9) hold for all i and j. Hence, from (6) one has that

$$\left[2{Q^i_{(11)j}}^T x_i^* + 2{Q^i_{(21)j}}^T y_i^* - r_j^* + 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right) - \mu_i^1\right]\left(x_{ij} - x_{ij}^*\right) \geq 0,$$

from which it follows, after summing up those inequalities with respect to $j = 1, \ldots, n$, and applying constraints (2), that

$$\sum_{j=1}^{n}\left[2{Q^i_{(11)j}}^T x_i^* + 2{Q^i_{(21)j}}^T y_i^* - r_j^* + 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(x_{ij} - x_{ij}^*\right) \geq 0. \tag{11}$$

Similarly, summing up by j the inequalities deduced from (7)

$$\left[2{Q^i_{(12)j}}^T x_i^* + 2{Q^i_{(22)j}}^T y_i^* + r_j^* - 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right) - \mu_i^2\right]\left(y_{ij} - y_{ij}^*\right) \geq 0,$$

and now making use of (3) one yields

$$\sum_{j=1}^{n}\left[2{Q^i_{(22)j}}^T y_i^* + 2{Q^i_{(12)j}}^T x_i^* + r_j^* - 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(y_{ij} - y_{ij}^*\right) \geq 0. \tag{12}$$

Now summing up inequalities (11) and (12) through all i, one concludes that for $(x^*, y^*) \in \prod_{i=1}^{m} \Omega_i$,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2{Q^i_{(11)j}}^T x_i^* + 2{Q^i_{(21)j}}^T y_i^* - r_j^* + 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(x_{ij} - x_{ij}^*\right) +$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2{Q^i_{(22)j}}^T y_i^* + 2{Q^i_{(12)j}}^T x_i^* + r_j^* - 2w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\right]\left(y_{ij} - y_{ij}^*\right) \geq 0,$$
$$\forall (x, y) \in \prod_{i=1}^{m} \Omega_i. \tag{13}$$

From relationships (9), one can further conclude that $r_j^* \geq 0$ must satisfy

$$\sum_{i=1}^{m}\left(x_{ij}^* - y_{ij}^*\right)\left(r_j - r_j^*\right) \geq 0, \quad \forall r_j \geq 0, \tag{14}$$

and, therefore, $r^* \in R_+^n$ is such that

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\left(x_{ij}^* - y_{ij}^*\right)\left(r_j - r_j^*\right) \geq 0, \quad \forall r \in R_+^n. \tag{15}$$

Again summing up inequalities (13) and (15), one obtains the variational inequality (10).

B. Sufficiency.

Now we establish that any solution to variational inequality (10) will also satisfy the equilibrium conditions (6) – (9).

If $(x^*, y^*, r^*) \in \prod_{i=1}^m \Omega_i \times R_+^n$ is a solution of variational inequality (10), let $x_i = x_i^*$; $y_i = y_i^*$, for all i. Then one has that

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{m} \left(x_{ij}^* - y_{ij}^* \right) \right] \left(r_j - r_j^* \right) \geq 0, \quad \forall r \in R_+^n,$$

which implies condition (9).

Next, select $r_j = r_j^*$, for all j, and substitute it into (10) thus yielding

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[2{Q_{(11)j}^i}^T x_i^* + 2{Q_{(21)j}^i}^T y_i^* - r_j^* + 2w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \right] \left(x_{ij} - x_{ij}^* \right) +$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[2{Q_{(22)j}^i}^T y_i^* + 2{Q_{(12)j}^i}^T x_i^* + r_j^* - 2w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \right] \left(y_{ij} - y_{ij}^* \right) \geq 0,$$

$$\forall (x, y) \in \prod_{i=1}^{m} \Omega_i, \tag{16}$$

which clearly implies (6) – (8). The proof is complete. □

A qualitative analysis of the variational inequality (10) governing the financial equilibrium model introduced in this section is presented in the next section within the framework of a more general model, of which the quadratic model examined here is a special case.

3. General Utility Functions

In this section, the above quadratic financial model is extended, and a variational inequality formulation of the (exterior) equilibrium conditions presented.

Assume that each sector seeks to maximize its utility, where the utility function, $U_i(x_i, y_i, r_i)$, is given by:

$$U_i(x_i, y_i, r_i) = u_i(x_i, y_i) + \langle r_i, x_i - y_i \rangle, \tag{17}$$

where $u_i : R^n \times R^n \to R$ is a differentiable function, and $r_i = r_i(x_i, y_i) \in R^n$ is the conjectured price vector with its components r_{ij}, $j = 1, \ldots, n$, defined in (5). Taking that into account, we can rewrite (17) as follows:

$$U_i(x_i, y_i, r_i) = u_i(x_i, y_i) + \langle r_i, x_i - y_i \rangle - \sum_{j=1}^{n} w_{ij} \left(x_{ij} - y_{ij} \right)^2. \tag{18}$$

Then the optimization problem for sector i can be specified as:

$$\text{Maximize}_{(x_i, y_i) \in \Omega_i} \quad U_i(x_i, y_i, r_i), \tag{19}$$

where Ω_i is a closed, convex, nonempty, and bounded subset of R^{2n}, denoting the feasible set of asset and liability choices. Note that in this model, we no longer require the constraint set Ω_i to be of the form defined by equalities (2) – (3) and inequalities (4). Nevertheless, the model introduced in this section captures the general financial equilibrium model of Section 2. as a special case, with $u_i(x_i, y_i) = -\begin{pmatrix} x_i \\ y_i \end{pmatrix}^T Q^i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

Assuming that each sector's utility function is concave, necessary and sufficient conditions for an optimal portfolio (x_i^*, y_i^*), given a fixed vector of instrument prices r^*, are that $(x_i^*, y_i^*) \in \Omega_i$, and satisfy the inequality:

$$-\langle \nabla_{x_i} U_i\left(x_i^*, y_i^*, r_i^*\right), x_i - x_i^* \rangle - \langle \nabla_{y_i} U_i\left(x_i^*, y_i^*, r_i^*\right), y_i - y_i^* \rangle \geq 0, \forall (x_i, y_i) \in \Omega_i, \quad (20)$$

where $\nabla_{x_i} U_i(\cdot)$ denotes the gradient of $U_i(\cdot)$ with respect to x_i, or, equivalently, in view of (17) – (18),

$$-\langle \nabla_{x_i} u_i\left(x_i^*, y_i^*\right) + r^*, x_i - x_i^* \rangle + 2\sum_{j=1}^{n} w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\left(x_{ij} - x_{ij}^*\right) -$$

$$-\langle \nabla_{y_i} u_i\left(x_i^*, y_i^*\right) - r^*, x_i - x_i^* \rangle -$$

$$-2\sum_{j=1}^{n} w_{ij}\left(x_{ij}^* - y_{ij}^*\right)\left(y_{ij} - y_{ij}^*\right) \geq 0, \ \forall (x_i, y_i) \in \Omega_i. \quad (21)$$

A respective variational inequality must hold for each of the m sectors.

The system of equalities and inequalities governing the instrument prices in the economy as in (9) is still valid. Hence, one can immediately write down the following economic system conditions.

For each instrument j; $j = 1, \ldots, n$:

$$\sum_{i=1}^{m} x_{ij}^* - \sum_{i=1}^{m} y_{ij}^* \begin{cases} = 0, & \text{if } r_j^* > 0, \\ \geq 0, & \text{if } r_j^* = 0. \end{cases} \quad (22)$$

In other words, as before, if an instrument in the economy is in excess supply, then its price must be zero; otherwise, if the price of the instrument is positive, then the market for this instrument must clear.

Combining the above sector and market inequalities, as well as the equalities yields the following.

Definition 2. (*Exterior Financial Equilibrium with General Utility Functions.*)

For a fixed set of conjectures described by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m \quad n}$, a vector $(x^*, y^*, r^*) \in \prod_{i=1}^{m} \Omega_i \times R_+^n$ is the *exterior conjectural variations equilibrium* (CVE) in the financial model developed above if, and only if it satisfies inequalities (21) and (22), for all sectors $1 \leq i \leq m$, and for all instruments $1 \leq j \leq n$, simultaneously.

The variational inequality formulation of the equilibrium conditions of the model is now presented. The proof of this theorem is very similar to that of Theorem 1, hence it is omitted.

Theorem 2. *(Variational Inequality Formulation of Exterior Financial Conjectural Variations Equilibrium with General Utility Functions)*

For a fixed set of conjectures described by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m\ \ n}$, a vector of assets and liabilities of the sectors, and instrument prices $(x^*, y^*, r^*) \in \prod_{i=1}^{m} \Omega_i \times R_+^n$ is the exterior financial conjectural variations equilibrium if and only if it solves the variational inequality problem:

$$
\begin{aligned}
&-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i (x_i^*, y_i^*) + r^*, x_i - x_i^* \rangle + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \left(x_{ij} - x_{ij}^* \right) - \\
&- \sum_{i=1}^{m} \langle \nabla_{y_i} u_i (x_i^*, y_i^*) - r^*, y_i - y_i^* \rangle - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \left(y_{ij} - y_{ij}^* \right) + \\
&+ \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij}^* - \sum_{i=1}^{m} y_{ij}^* \right) \left(r_j - r_j^* \right) \geq 0, \quad \forall (x, y, r) \in \prod_{i=1}^{m} \Omega_i \times R_+^n. \qquad (23)
\end{aligned}
$$

4. Qualitative Properties

In this section, certain qualitative properties of the exterior CVE in the model outlined in Section 3 are examined.

Theorem 3. *(Existence Theorem)*

For any fixed set of conjectures described by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m\ \ n}$, if $(x^*, y^*, r^*) \in \prod_{i=1}^{m} \Omega_i \times R_+^n$ is the exterior CVE in the model, that is, it solves the variational inequality (23), then the equilibrium asset and liability vector (x^*, y^*) is a solution to the variational inequality:

$$
\begin{aligned}
&-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i (x_i^*, y_i^*), x_i - x_i^* \rangle + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \left(x_{ij} - x_{ij}^* \right) - \\
&- \sum_{i=1}^{m} \langle \nabla_{y_i} u_i (x_i^* y_i^*), y_i - y_i^* \rangle - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^* \right) \left(y_{ij} - y_{ij}^* \right) \geq 0, \forall (x, y) \in S, \qquad (24)
\end{aligned}
$$

where the subset

$$
S \equiv \left\{ (x, y) \in \prod_{i=1}^{m} \Omega_i : \ \sum_{i=1}^{m} (x_{ij} - y_{ij}) \geq 0; j = 1, \ldots, n \right\}
$$

is nonempty.

Conversely, if (x^*, y^*) is a solution of (24), there exists an $r^* \in R_+^n$, such that (x^*, y^*, r^*) is a solution of (23), and, thus, the exterior CVE in the financial model.

Proof. A. Necessity.

Having fixed any set of conjectures defined by the $m \times n$ matrix $W = (w_{ij})_{i=1,j=1}^{m\ \ n}$, assume that (x^*, y^*, r^*) is an exterior CVE state. Then (x^*, y^*, r^*) satisfies (23). Select $x_i = x_i^*$; $y_i = y_i^*$, for all i, and $r = 0$; substituting these vectors into (23) one yields:

$$
-\sum_{j=1}^{n} \left[\sum_{i=1}^{m} \left(x_{ij}^* - y_{ij}^* \right) \right] r_j^* \geq 0. \qquad (25)
$$

Now choose $r = r^*$ and insert that into (23) thus obtaining:

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i \left(x_i^*, y_i^*\right) + r^*, x_i - x_i^* \rangle + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(x_{ij} - x_{ij}^*\right) -$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} u_i \left(x_i^*, y_i^*\right) - r^*, y_i - y_i^* \rangle - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(y_{ij} - y_{ij}^*\right) \geq 0,$$

or, after some evident transformations,

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i \left(x_i^*, y_i^*\right), x_i - x_i^* \rangle + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(x_{ij} - x_{ij}^*\right) -$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} u_i \left(x_i^*, y_i^*\right), y_i - y_i^* \rangle - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(y_{ij} - y_{ij}^*\right) \geq$$

$$\geq \sum_{j=1}^{n} r_j^* \left[\sum_{i=1}^{m} (x_{ij} - y_{ij}) - \sum_{i=1}^{m} \left(x_{ij}^* - y_{ij}^*\right) \right]. \tag{26}$$

The right-hand-side term of inequality (26) is nonnegative, because of (25) and the definition of the feasible set S. Thus, we have established that (x^*, y^*) satisfies (23), hence also solves (24).

B. Sufficiency.

Observe that for any matrix of conjectures W there always exists an asset and liability pattern (x^*, y^*) solving (24), since the feasible set S is compact (*cf.*, Isac at al., 2002). Further, by the Lagrange Multiplier Theorem (Ferris and Pang, 1997), one is guaranteed the existence of multipliers $r^* \in R_+^n$, corresponding to the constraints defining S. For this triple (x^*, y^*, r^*), one evidently has

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i \left(x_i^*, y_i^*\right) + r^*, x_i - x_i^* \rangle + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(x_{ij} - x_{ij}^*\right) -$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} u_i \left(x_i^*, y_i^*\right) - r^*, y_i - y_i^* \rangle - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \left(x_{ij}^* - y_{ij}^*\right) \left(y_{ij} - y_{ij}^*\right) +$$

$$+\sum_{j=1}^{n} \left[\sum_{i=1}^{m} x_{ij}^* - \sum_{i=1}^{m} y_{ij}^* \right] \left(r_j - r_j^*\right) \geq 0, \forall (x, y, r) \in \prod_{i=1}^{m} \Omega_i \times R_+^n.$$

The proof is complete. □

At last, we show that if the utility functions u_i are strictly concave for all i, then the exterior equilibrium asset and liability pattern (x^*, y^*) is also unique for any fixed conjectures matrix W.

It is clear that if the functions u_i are strictly concave then the functions U_i defined by (17) or (18) are also strictly concave with respect to the variables (x_i, y_i). Indeed,

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} U_i \left(x_i^1, y_i^1, r_i^1\right) - \nabla_{x_i} U_i \left(x_i^2, y_i^2, r_i^2\right), x_i^1 - x_i^2 \rangle -$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}U_i\left(x_i^1,y_i^1,r_i^1\right)-\nabla_{y_i}U_i\left(x_i^2,y_i^2,r_i^2\right),y_i^1-y_i^2\rangle=$$

$$-\sum_{i=1}^{m}\langle\nabla_{x_i}u_i\left(x_i^1,y_i^1\right)-\nabla_{x_i}u_i\left(x_i^2,y_i^2\right),x_i^1-x_i^2\rangle-$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}u_i\left(x_i^1,y_i^1\right)-\nabla_{y_i}u_i\left(x_i^2,y_i^2\right),y_i^1-y_i^2\rangle+$$

$$+2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left[\left(x_{ij}^1-x_{ij}^2\right)-\left(y_{ij}^1-y_{ij}^2\right)\right]^2>0,\forall\left(x^1,y^1\right)\neq\left(x^2,y^2\right)\in\prod_{i=1}^{m}\Omega_i. \quad (27)$$

Now assume that for some fixed matrix of conjectures W, there are two distinct exterior equilibrium solutions $\left(x^1,y^1,r^1\right)$ and $\left(x^2,y^2,r^2\right)$. Then

$$-\sum_{i=1}^{m}\langle\nabla_{x_i}u_i(x_i^1,y_i^1)+r^1,x_i'-x_i^1\rangle+2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left(x_{ij}^1-y_{ij}^1\right)\left(x_{ij}'-x_{ij}^1\right)-$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}u_i\left(x_i^1,y_i^1\right)-r^1,y_i'-y_i^1\rangle-2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left(x_{ij}^1-y_{ij}^1\right)\left(y_{ij}'-y_{ij}^1\right)+$$

$$+\sum_{j=1}^{n}\left(\sum_{i=1}^{m}x_{ij}^1-\sum_{i=1}^{m}y_{ij}^1\right)\left(r_j'-r_j^1\right)\geq 0,\quad\forall\left(x',y',r'\right)\in\prod_{i=1}^{m}\Omega_i\times R_+^n, \quad (28)$$

and

$$-\sum_{i=1}^{m}\langle\nabla_{x_i}u_i\left(x_i^2,y_i^2\right)+r^2,x_i-x_i^2\rangle+2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left(x_{ij}^2-y_{ij}^2\right)\left(x_{ij}-x_{ij}^2\right)-$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}u_i\left(x_i^2,y_i^2\right)-r^2,y_i-y_i^2\rangle-2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left(x_{ij}^2-y_{ij}^2\right)\left(y_{ij}-y_{ij}^2\right)+$$

$$+\sum_{j=1}^{n}\left(\sum_{i=1}^{m}x_{ij}^2-\sum_{i=1}^{m}y_{ij}^2\right)\left(r_j-r_j^2\right)\geq 0,\quad\forall(x,y,r)\in\prod_{i=1}^{m}\Omega_i\times R_+^n. \quad (29)$$

Now select $(x',y',r')=(x^2,y^2,r^2)$ and substitute it to (28); symmetrically, set $(x,y,r)=(x^1,y^1,r^1)$ and put it into inequality (29). Summing up the resulting inequalities, we come to

$$-\sum_{i=1}^{m}\langle\nabla_{x_i}u_i\left(x_i^1,y_i^1\right)-\nabla_{x_i}u_i\left(x_i^2,y_i^2\right),x_i^1-x_i^2\rangle-$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}u_i\left(x_i^1,y_i^1\right)-\nabla_{y_i}u_i\left(x_i^2,y_i^2\right),y_i^1-y_i^2\rangle+$$

$$+2\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ij}\left[\left(x_{ij}^1-x_{ij}^2\right)-\left(y_{ij}^1-y_{ij}^2\right)\right]^2\leq 0, \quad (30)$$

which contradicts (27). Hence, we have just established the following result.

Theorem 4. *(Uniqueness of Exterior Equilibrium Asset and Liability Pattern.)*

If the utility functions u_i are strictly concave for all sectors i, then for any fixed conjectures matrix W, the exterior conjectural variations equilibrium (CVE) pattern (x^*, y^*, r^*) exists uniquely.

Proof. The proof is given above. □

5. Consistent Conjectures

In the previous sections, we implicitly assumed that the conjectures W are given exogenously for the model. However, in a series of recent publications by the authors (Kalashnikov et al., 2011)–(Kalashnikov et al., to appear), a concept of *consistent* conjectures has been proposed and justified. Namely, one of the clue results established in (Kalashnikov et al., to appear) was as follows. Any consistent (in certain sense) conjectural variations equilibrium (CVE) in a classical oligopoly with quadratic cost functions leads to a Cournot-Nash equilibrium in an upper level many person game, which is defined in a way described below.

Because of that, even though it is impossible to apply the concept of consistent conjectural variations equilibrium to our financial model directly (since it differs from the classical oligopoly model), Theorem 4 allows one to generate the *upper level game* and define consistent conjectures W implicitly. The procedure works as follows.

Under assumptions of Theorem 4, define a many person game $\Gamma = (N, W, V)$ by the following rules:

(i) $N = \{1, \ldots, m\}$ is the set of the same sectors as in our financial model;

(ii) the set of feasible conjectures $W = (w_{ij})_{i=1,j=1}^{m \quad n} \in R_+^{m\times n}$ is the set of possible strategies in the upper level game;

(iii) $V = V(W) = (V_1, \ldots, V_m)$ are payoff functions used by the sectors $i = 1, \ldots, m$, in order to estimate their payoffs as the result of being stuck to their strategies $w_i = (w_{i1}, \ldots, w_{in})$. These functions are well-defined via Theorems 3 and 4 as follows: For each matrix W, according to those theorems, there exists uniquely the exterior CVE (x^*, y^*, r^*). Now the payoff function $V_i = V_i(W)$ is well-defined as the optimal value of the utility function $U_i(x^*, y^*, r^*)$, introduced in (18), i.e.,

$$V_i(W) = U_i(x^*, y^*, r^*), \quad i = 1, \ldots, m. \tag{31}$$

Indeed, the payoff values (31) are defined by formula (18), in which the (equilibrium) assets and liabilities holdings (x^*, y^*), as well as the equilibrium price r^*, are the elements of the exterior CVE whose existence and uniqueness is guaranteed by Theorem 4 of Section 4..

Now if the strategies set W in the upper level game is a compact (i.e., closed and bounded) subset of $R_+^{m\times n}$, one can guarantee the existence of the classical (Cournot-Nash) equilibrium in this game (Ferris and Pang, 1997). Then we name this Cournot-Nash equilibrium in the upper level game as the *consistent*, or *interior* CVE in the original financial model.

Definition 3. (*Consistent, or Interior Financial CVE with General Utility Functions*).

The asset, liabilities, and price vector (x^*, y^*, r^*) generated in the Cournot-Nash equilibrium of the upper level game is called the *interior equilibrium* in the financial model, and the corresponding conjectures W^* involved in the upper level game Cournot-Nash equilibrium, are named *consistent*.

The section is finished with the following existence result.

Theorem 5. *(Existence of the Interior CVE in the Financial Model)*
Under assumptions of Theorem 4 and for a compact feasible set of conjectures W, there exists the consistent (interior) CVE in the financial model (19).

Proof. It is straightforward (*cf.,* Ferris and Pang, 1997). □

6. Conclusion

In the present paper, a general multi-sector, multi-instrument model of financial flows and prices is developed, in which the utility function for each sector is assumed to be quadratic and constraints satisfy a certain accounting identity that appears in flow-of-funds accounts. Each sector uses conjectures of its influence upon the prices of instruments. The equilibrium conditions are first derived, and then the governing variational inequality is presented. Subsequently, a qualitative analysis of the model is conducted.

Finally, a criterion of consistency of the influence coefficients w_{ij} is introduced, and the existence of at least one interior (consistent) CVE in the financial model for a compact feasible conjectures set W, is established.

In our forthcoming papers, we are going to study properties of consistent conjectural variations equilibrium (CVE) and conduct comparative static research by comparing it with the classical Nash equilibrium (Nagurney, 1999) related to the examined financial model.

Acknowledgments. The research activity of the authors was financially supported by the Research Department of the Tecnológico de Monterrey (ITESM), Campus Monterrey, and by the SEP-CONACYT project CB-2013-01-221676, Mexico. The second author was also supported by the PAICYT project No. CE250-09.

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