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Two Modes of Vaccination Program in Controlled SIR Model*

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Abstract The problem of forming herd immunity to an infectious disease, i.e. influenza, which is optimal to the population, is often considered as a modification of the classical Susceptible-Infected-Recovery model. However the annual vaccination of the total population is quite expensive and is not obligatory for every individual. Any agent in population has a choice: whether or not to participate in the vaccination program. So each epidemic season society confronts a dilemma: how to maintain the necessary immunization level which is subject to individual choice. Apparently each available alternative incurs different costs and benefits for an individual agent and the population in total. We compare social and individual benefits and expenses in two cases: optimal vaccination policy is used to preserve the optimal herd immunity; agents participate in the vaccination campaign, considering only individual benefits. It's supposed that agent choices do not depend only on the cost generated by agents' choices during the epidemic period. Agents also take into account all available information, received from neighbors, media and former experience. Every agent compares it's own preferences and the alternatives, chosen by neighbors and can update its choice every season. We study the influence of information about previous epidemics on the decision making process. We investigate an optimal control problem to study the optimal vaccination behavior during an epidemic period based on classical Susceptible-Infected-Recovery model and present a procedure for making vaccination decisions.

Keywords: SIR model, vaccination problem, evolutionary games, optimal control, epidemic process.

1. Introduction

The human population faces an influenza epidemic almost every year hence it is necessary to estimate the social and medical costs which can occur during epidemic period.

Originally Susceptible-Infected-Recovered (SIR) model and its modification describe a fast spreading process, such as flu-like epidemic or other forms of respiratory viral diseases which are circulated in an urban population. The total population is divided into three subgroups: Susceptible, Infected and Recovered. Susceptible is

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a group, where people are not infected, Infected is a group of people having the disease, and Recovered is a group, where all members have immunity to the disease (Mehlhorn, H. et al., 2008, Conn, 2006, Kolesin et al., 2014).

Recent literature has seen a large amount of interest in using optimal control and game-theoretic methods to study disease control of influenza for public health. First, this research problem was referenceed in (Kermack and Mc Kendrick, 1927), where an Susceptible-Infected-Recovered model has been proposed to study the epidemic

In (Behncke, 2000) many variants of optimal control models of SIR-epidemics are investigated for the application of medical vaccination and health promotion campaigns. In (Fu et al., 2010) the vaccination problem is studied from the point of view of individual agents.

Moreover epidemic models can be applied to different fields of human activity, for instance in (Altman et al., 2010, Khouzani et al., 2011), optimal control methods have been used to study the class of epidemic models in mobile wireless networks, and Pontryagin's maximum principle is used to quantify the damage that the malware can inflict on a network by deploying optimum decision rules.

Over the years many medical methods such as preventive measures, intensive treatment, etc. have been developed to protect the entire population during annual epidemics. All these methods share the fundamental strategy to resist the viral propagation. For this reason preventive measures or medical treatment can be considered as an external influence on the epidemic process and can be used as control parameters for the dynamic system.

In a general case an epidemic period continues until there are no more newly infected individuals and vaccination program occurs before the seasonal epidemic begins. It is necessary to take into account the reaction of the agent's immune system to the vaccination, because in the post-vaccination period the human organism becomes weak and it is not able to properly resist against other viruses.

As it has been shown, total vaccination is very expensive and can be applied only in exceptional cases, when i.e. pathogen is extreme virulent. Usually only partial vaccination is used each ordinary season. Since the vaccination campaign is not obligatory, we assume that before epidemic period each agent of population chooses whether use the vaccination or not. Every agent choice incurs appropriate costs and influences the future agent's benefit. If an agent participates in a vaccination program then she pays the vaccination costs and estimates the probable contraindications if vaccine is not effective. Evidently, if an agent prefers to avoid the vaccination then she has to evaluate possible treatment expenses. Besides each agent in a population in each epidemic season is influenced by the information about the previous vaccination campaign, whether or not it was successful.

However, it is very important to maintain the optimal level of herd immunity, which protects the entire population from those who forgo vaccination and at the same time grants the possibility to make an individual choice to every agent. If a sufficient number of agents decide to evade vaccination then such scenario leads to the critical situation as far as numbers of susceptible increase and the virus spreads faster in the total population.

As we mentioned above we present a procedure of making decisions which influence the epidemic process in a urban population.

At the present time each agent has many possibilities to estimate its benefit from the participation in a vaccination program. The agent can evaluate the vaccination cost, feasible complications in post-vaccination period, and she can indirectly estimate the herd immunity of the population. We suppose an agent can not necessarily identify the exact information, but she estimates the average number of its contacts, the current epidemic situation, which is reported in mass media, etc. Meanwhile the collective result of vaccination decisions determines the level of population immunity and the strain of the epidemic in the current period. When the number of vaccinated agents in total population is increased then even agents who are unvaccinated might have less infection risk. Then we assume that every agent having this information, might decide to decline the vaccination this year and thereby reduce her vaccination costs. However, agents can have incomplete information which includes some rumors from neighbors or friends, or they also may estimate the epidemic situation incorrectly. Thus this scenario leads into a problem of increasing the numbers of unvaccinated individuals, inducing the diminution of herd immunity in the future and as a result the frequency of meeting with infected agents is increased. Thereby the aggregated costs will arise during the epidemic period, because each unvaccinated agent may become infected and then she must pay treatment costs, which include healthcare expenses, lost productivity and the possibility of pain. Usually treatment costs exceed the vaccination expenses.

Thus in the current work we suppose that each agent chooses between two possible alternatives: to vaccination or not to vaccinate during an epidemic period. However all these alternatives can lead to deferent negative effects.

In contrast to previous studies, Based on Susceptible-Infected-Recovered (SIR) model we formulate an optimal control problem and receive a structure of optimal control, where the intensity of vaccination is used as a control parameter. In this case we receive the aggregated system costs which are optimal in the sense of the optimal control problem. We also propose a rule which dictates to an agent how she should behave before the epidemic period to achieve maximum protection from the disease. We integrated an uncontrolled Susceptible-Infected-Recovered model into the decision procedure and evaluate the aggregated system costs generated by this complex process. We compare the aggregated costs for the total population for both scenarios of vaccination.

The paper is organized as follows: Section 2. presents the mathematical model of epidemics and formulates the optimal control problem. Section 3. shows the structure of optimal control and main results, using Pontryagin's maximum principle. Section 3.1., introduces the procedure of making decision about vaccination behavior.

2. Controlled Susceptible-Infected-Recovered Model

In this paragraph we formulate the Susceptible-Infected-Recovered (SIR) model to describe epidemiological process in an urban population. The current model shows that vaccination program affects the population and hence we can consider it as control parameter. Then, at time t, n_s, n_I, n_R correspond to fractions of the population who are susceptible, infected and recovered. For all t, condition $N = n_s + n_I + n_R$ is justified. Define

$$S(t) = \frac{n_S}{N},\ I(t) = \frac{n_I}{N},\ R(t) = \frac{n_R}{N},\ (R(t) = 1 - S(t) - I(t))$$

as portions of susceptible, infected and recovered in the population.

Below the system of nonlinear differential equations, describing the epidemic process is presented (Kermack and Mc Kendrick, 1927, Khatri et al., 2003):

$$\begin{aligned} \frac{dS}{dt} &= -\delta S(t)I(t) - \varepsilon u(t)S(t); \\ \frac{dI}{dt} &= \delta S(t)I(t) - \sigma I(t). \end{aligned} \tag{1}$$

Here parameter δ is the transmission rate from state S to I

$$\delta = \delta_0 m(\frac{n_I}{N}) = \delta_0 m I, \tag{2}$$

where δ_0 is a transmissibility of the disease and m is a number of contacts per time unit, the transmission rate from state I to R is defined as $\sigma = \frac{1}{T}$, variable $u(t) \in (0,1)$ is a control parameter which is interpreted as the intensity of vaccination(agents\ per day), $\varepsilon \geq 0$ is a fraction of agents, who were involved in vaccination campaign.

Objective Function: We will minimize aggregated cost in time interval $[0,T]$, hence at any given t following costs exist in the system: $f(I(t))$ these are individual's treatment costs, which are non-decreasing and twice-differentiable, convex function, such as $f(0) = 0,\ f(I(t)) > 0,\ i = \overline{1,N}$ for $I(t) > 0$; function $l(R(t))$ are agent's benefit rate, which arise when an infected agent becomes recovered, $l(R(t))$ is non-decreasing and differentiable function and $l(0) = 0$; function $h(u(t))$ describe vaccination costs, $h(u(t))$ is twice-differentiable and increasing function in $u(t)$ such as $h(0) = 0$, $h(x) > 0,\ i = \overline{1,N}$ when $u(t) > 0$.

Therefore aggregated system costs is:

$$J = \int_0^T (f(I(t)) - l(R(t)) + h(u(t)))dt. \tag{3}$$

3. Structure of optimal control

We use Pontryagin's maximum principle (Pontryagin et al., 1962), to find the optimal control $u(t)$ to the problem described above in Section 2. Define the associated Hamiltonian H and adjoint functions λ_i as follows:

$$\begin{aligned} H = -\lambda_0(f(I(t)) - l(R(t)) + h(u(t)))+ \\ \lambda_S(-\delta S(t)I(t) - \varepsilon u(t)S(t)) + \lambda_I(\delta S(t)I(t) - \sigma I(t)) = \\ -\lambda_0(f(I(t)) - l(R(t)) + h(u(t)) - \delta S(t)I(t)(\lambda_S(t) - \\ \lambda_I(t)) - \lambda_S(t)\varepsilon u(t)S - \lambda_I(t)I(t)\sigma, \end{aligned} \tag{4}$$

where $\lambda_0 = 1$.

Adjoint system is:

$$\begin{aligned} \dot{\lambda}_I(t) &= f'(I(t)) + \delta S(\lambda_S(t) - \lambda_I(t)) + \lambda_I(t)\sigma; \\ \dot{\lambda}_S(t) &= \delta I(t)(\lambda_S(t) - \lambda_I(t)) - \lambda_S(t)u(t)\varepsilon; \end{aligned} \tag{5}$$

and transversality conditions are

$$\lambda_I(T) = 0,\ \lambda_S(T) = 0,\ \lambda_R(T) = 0. \tag{6}$$

According to Pontryagin's maximum principle, there exist continuous and piecewise continuously differentiable co-state functions λ_i that at every point $t \in [0, T]$ where u is continuous, satisfy (5) and (6). In addition, we have $\lambda(t) = (\lambda_0(t), \lambda_S(t), \lambda_I(t), \lambda_R(t))$

$$u \in \arg\max_{\underline{u} \in [0,1]} H(\overline{\lambda}, (S, I, R), \underline{u}), \tag{7}$$

here $\underline{u} \in [0, 1]$ is admissible control.

Rewrite a Hamiltonian

$$\begin{aligned} H = -(f(I(t)) - l(R(t)) - \delta S(t)I(t)(\lambda_S(t) - \lambda_I(t)) - \\ (\lambda_S(t)\varepsilon u(t)S + h(u(t))) - \lambda_I(t)I(t)\sigma. \end{aligned} \tag{8}$$

According to the general method of principle maximum we consider a derivative $\frac{\partial H}{\partial u}$:

$$\frac{\partial H}{\partial u} = -h'(u) - \lambda_S = -(h'(u) + \varepsilon \lambda_S S) \tag{9}$$

and Hamiltonian reaches maximum if and only if the next condition is satisfied:

$$(h'(u) + \varepsilon \lambda_S S) < 0. \tag{10}$$

Since $h(u)$ is non-increasing function, then $h'(u) \geq 0$, $S \geq 0$ as a fraction of susceptible agents and $\varepsilon \geq 0$ as a fraction of agents, who were involved in vaccination campaign, then condition (10) is satisfied only if $\psi < 0$, where

$$\psi = \lambda_S S \tag{11}$$

is defined as switching function.

Lemma 1. *Function ψ is increasing over the time interval $[0, T)$.*

Proof. Functions λ_S and state function S are continuous and differentiable at each $t \in [0, T)$ then we can consider a time derivative of ψ:

$$\begin{aligned} \dot{\psi} = \dot{\lambda}_S(t)S + \dot{S}(t)\lambda_S(t) = \\ (\delta I(\lambda_S - \lambda_I) - \lambda_S u\varepsilon)S + \lambda_S(-\delta SI - \varepsilon u(t)S(t)) = \\ \delta I(\lambda_S - \lambda_I) - \lambda_S S(2u\varepsilon + \delta I). \end{aligned}$$

Since $u(t)$ is piecewise continuous and all variables S, I, u, ε, δ are nonnegative then $\dot{\psi} \geq 0$ and the statement of lemma is satisfied if $\lambda_I - \lambda_S < 0$ and $\lambda_S < 0$.

Below we formulate an auxiliary lemma which shows the signs of expressions $(\lambda_I - \lambda_S)$ and λ_S.

Lemma 2. *For all t, $0 < t < T$ following conditions hold $(\lambda_I(t) - \lambda_S(t)) < 0$ and $\lambda_S(t) < 0$.*

The proof of lemma is presented in section 6.

Then to establish the optimal vaccination policy we formulate the next proposition.

Proposition 1. *The optimal vaccination policy has following structure:*
If $h(\cdot)$ *are concave then exists time moment* $t \in [0, T]$ *such as:*

$$u(t) = \begin{cases} 0, & \textit{if } -\psi < h(u_{max}); \\ 1, & \textit{if } -\psi > h(u_{max}). \end{cases} \tag{12}$$

If $h(\cdot)$ *is strictly convex, then exists* t_0, t_1, $0 < t_0 < t_1 < T$*:*

$$u(t) = \begin{cases} 0, & -\psi \le h'(0); \\ h'^{-1}(-\psi), & h'(0) < -\psi \le h'(u_{max}); \\ u_{max}, & h'(u_{max}) < -\psi. \end{cases} \tag{13}$$

Proof. Proof of proposition 1 is divided into two parts and we consider two cases, which depend on the properties of function h:

1) h is concave.

Since function h is concave ($h'' < 0$), then ($\psi u - h(u)$), are convex functions of u. Hamiltonian H is a strictly convex function according to (8) and for any $t \in [0, T]$ and it reaches its maximum either at $u = 1$ or $u = 0$.

$$u(t) = \begin{cases} 0, & \text{if } -\psi < h(u_{max}); \\ 1, & \text{if } -\psi > h(u_{max}). \end{cases} \tag{14}$$

2) h is strictly convex.

If functions h are strictly convex ($h'' > 0$) then ($u\psi - h(u)$) and Hamiltonian is concave function, then ($\frac{dH}{du} = \psi - \dot{h}u = 0$, $u \in [0, 1]$). Then

$$u(t) = \begin{cases} 0\ , & \text{if } -\psi \le \frac{dh(0)}{du}; \\ \frac{dh^{-1}(-\psi)}{du}\ , & \text{if } \frac{dh(0)}{du} < -\psi \le \frac{dh(u_{max})}{du}; \\ 1\ , & \text{if } -\psi > \frac{dh(u_{max})}{du}, \end{cases} \tag{15}$$

functions ψ, h', u are continuous at all $t \in [0, T]$. In this case h is strictly convex and h' is strictly increasing function then $h'(0) < h'(u_{max})$. Thus there exists such moments t_0, t_1 ($0 < t_0 < t_1 < T$) such as conditions (13) are satisfied.

3.1. Decision making process

If an agent participates in vaccination program then she pays vaccination costs, which contain the immediate monetary cost, indirect cost of time spent in medical institution during a vaccination program and any health effects. We also suppose that a vaccination is not absolutely effective and a vaccination program should be finished before the epidemic starts. We will call as preepidemic period the time interval before the epidemic starts. It is supposed that agents can only switch from one action to another during this period.

In the current section we describe the decision procedure which depicts the behavior of agents of population at each epidemic season. Every season an agent adopts one of alternatives, which determines whether or not he is vaccinated. Chosen decision incurs appropriate costs and benefits and at the end of the season each agent can reevaluate its choice and switch on from one behavior to another.

Consider a human population of size N, where each member receives information from H information sources such as: public opinion, advertisement, social information, own experience.

Define as $M_i,\ i \in N$ an information set of agent i, $M_i = \{m_1, \ldots, m_h\},\ h \in H, i \in N$, where variable m_i is binary. This variable shows whether or not individual agent accepts information.

$$m_i = \begin{cases} 1, \text{if agent receives corresponding message;} \\ 0, \text{if agent does not receive corresponding message;} \end{cases}$$

If an agent takes into account "own experience" then

$$m_i = \begin{cases} 1, \text{if agent receives positive experience;} \\ 0, \text{if agent receives negative experience;} \end{cases}$$

Then an agent can calculate a number of positive elements in his\her information set, which is denoted as M_i^+. Based on this information he\ she uses a rule of making decision about vaccination.

Agents decisions. Let be $K_i = \{k_1,\ k_2,\ k_3\}$ the set of possible actions of each agent i see (Weibull 1995). We consider the next possible actions of agent j:

- k_1 – agent uses preventive measures (vitamins, pharmacological products, ets);
- k_2 – agent participates in vaccination program;
- k_3 – agent does not use any preventive measures, including vaccination.

We define a reaction function ϕ as a rule which dictates to every agent his reaction to information signals.

$$k_j^i = \phi(M_i^+), i \in N, j \in K_i. \tag{16}$$

The rule describes a choice of the available action to each agent, depending on received information and may be described as:

$$\phi(M_i^+) = \begin{cases} k_2,\ M_i^+ \geq 2; \\ k_1 \text{ or } k_3,\ M_i^+ < 2; \end{cases} \tag{17}$$

E Every season an agent chooses an action, which determines whether or not she vaccinates and at the end of the season. As we mentioned above, each decision incurs its costs and benefits. In compliance with its costs, each individual may change its behaviour, if he is not satisfied by his benefit. Thereby if agents decides to vaccinate or to use any other preventive measures then she pays for a vaccination or for preventive measures. In both cases we can interpret agent's benefit as the enhancement of the immune system. If agent does not use any preventive measures then she does not have any costs as well as benefits. We define as $c_p(k_l)$ any costs of agent, $l = 1, 2, 3$. Agent's benefits contain salary per full time period without medical certificate induced by the illness and a reduction of costs for preventive measures. Here we define the agent's benefits as $b(k_l)$. Thus we can rewrite the expression (17), supplementing more details about costs and benefits:

$$\phi(M_i^+) = \begin{cases} k_2, \text{ if } \ M_i^+ \geq 2; \\ k_1, \text{ if } \ M_i^+ < 2 \text{ and } c_p(k_1) < c_p(k_3); \\ k_3, \text{ if } \ M_i^+ < 2 \text{ and } c_p(k_3) < c_p(k_1); \end{cases} \tag{18}$$

We may define as $p(k_l) = b(k_l) - c(k_l)$ the difference between the agents benefits and costs, than the greater is the difference the better is the selected action.

An epidemics of influenza occurs almost every year, hence agents of population should make a decision regularly. Each epidemic season, an agent adopts an action, which determines whether or not she vaccinates. Agent using action l imitates behavior j according to the imitation rule $\rho(k_l \to k_j)$:

$$\rho_{lj}(p,x) = x_j(p(k_j) - p(k_l)), \tag{19}$$

here $\rho_{lj}(p,x)$ is the probability of switching from action k_l to action k_j, p is the cost function corresponding to chosen action. This revision probability generates the dynamics of imitation (see (Sandholm 2011, Weibull 1995)) that describes the changes of number of agents which choose one or another action:

$$\dot{x}_{k_l} = x_{k_l} \sum_{k_j \in K} x_{k_j} \left[(p(k_l) - p(k_j)) - (p(k_j) - p(k_l)) \right], \tag{20}$$

here x_{k_l} is a fraction of agents that use action k_l.

Agent are able to use this scheme of making decision only before the epidemic starts. The condition which define the beginning epidemic have introduced in (Kermack and Mc Kendrick, 1927):

$$\delta S - \sigma I \geq 0, S(0) \geq \frac{\sigma}{\delta}. \tag{21}$$

At the end of preepidemic period, for example at time moment $\bar{t}$, according to the rule of imitation dynamics, several groups of agents are formed $x_{k_l}(\bar{t})$, $i = 1,2,3$. Hence, we will recalculate the subgroups of susceptible, infected and recovered agents, using the final information about agents choices: $S_v^0 = S - x_{k_l}$, $R_v^0(u_v) + x_{k_l}$, $I_v^0 = I^0$.

Recalculated values S_v^0, R_v^0, I_v^0 correspond to initial states for uncontrolled SIR model (1)

$$\begin{aligned} \frac{dS}{dt} &= -\delta S(t)I(t); \\ \frac{dI}{dt} &= \delta S(t)I(t) - \sigma I(t); \end{aligned} \tag{22}$$

parameters δ and σ are the same as in section 2. As initial states we use $R(\bar{t}) = R_v^0$, $I(\bar{t}) = I_v^0$, $S(\bar{t}) = 1 - I_v(\bar{t}) - R_v(\bar{t})$, where $\bar{t} \in (0,T)$ it a time moment at which the preepidemic period has finished.

In the current section we define the aggregated system costs which include two parts, vaccination costs and potential treatment costs:

$$\overline{J} = x_{k_2}(\bar{t})c_p(k_2) + \int_{\bar{t}}^{T} f(I(t)) - l(R(t))dt, \tag{23}$$

here $\bar{t}$ is also an instant time moment, when vaccination program should be stopped to avoid complications of vaccination, $x_{k_2}(\bar{t})$ is number of agents who adopt vaccination by the time moment $\bar{t}$. In other words at time moment $\bar{t}$ when condition (21) is satisfied. The first part of (23) corresponds to aggregated vaccination costs and the second part is equal to aggregated treatment costs. Now we consider two functionals (23) and (3) and compare social or personal utility from the vaccination decision. In addition we compare sickness rate $L = \delta SI$ for both cases together with cost functionals. Here we also define as L^* the threshold level of sickness rate and as L_u the sickness rate in controlled case.

Thus we introduce following algorithm, which describes the behavior of agents in human population during an epidemic period:

- an agent, based on the information set, chooses one of possible decisions;
- agents' choices form an initial states for Susceptible-Infected-Recovered model at the current epidemic period;
- Susceptible-Infected-Recovered model defines the epidemic dynamics for current epidemic period;
- the society estimates the difference between the threshold level of the sickness rate and it's current value for each social group;
- the society corrects the information which are available for all agents, for example the advertisement of vaccination programm can be increased (decreased), then the information set of each agent can be changed. Then the process continues at the next period.

4. Numerical simulations

In this section we present numerical simulations to confirm our theoretical results. We depict optimal control for different cases which correspond to different values of parameters. The common parameters of SIR model are: $\delta = 0.7$, $\sigma = 0.06667$, $\varepsilon = 0.1$. Initial states are: $S(0) = 0.9$, $I(0) = 0.1$, $R(0) = 0$. Maximum of intensity rate of vaccination is $u_{\max} = 0.5$, function $h(u)$ is convex.

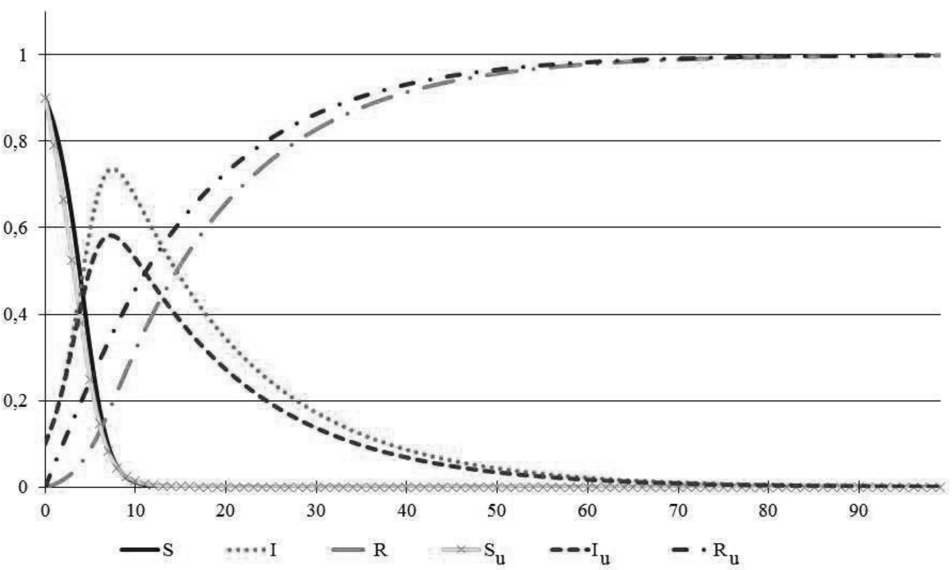

Fig. 1: Controlled and uncontrolled case of epidemic process in urban population. Costs functions are: $f(I) = 10I$, $l(R) = 3R(t)$, $h(u) = 0.1u^2$.

In fig.1 we can see the decreasing of the fraction of infected in a population under the influence of optimal control. For example, in uncontrolled case we have fraction of $I_{\max}(t) = 0.732$, but in controlled case the fraction of infected is $I_{\max}(t) = 0.581$, likewise we observe the decreasing of the sickness rate and aggregated system costs.

In fig.2 we show how parameters of the system are changed for different values of virus strength δ subject to the costs function $h(u)$ is concave. Common parameters of the SIR model are the same that in the previous case: $\sigma = 0.06667$, $\varepsilon = 0.1$, initial states are: $S(0) = 0.9$, $I(0) = 0.1$, $R(0) = 0$. Maximum of the intensity rate of vaccination is $u_{max} = 0.5$. Here we consider following costs functions: $f(I) = 10I(t)$, $l(R) = 3R(t)$, $h(u) = 0.1u$. Below we present the detailed description of graphs, presented in fig.2 and values for parameters of the system.

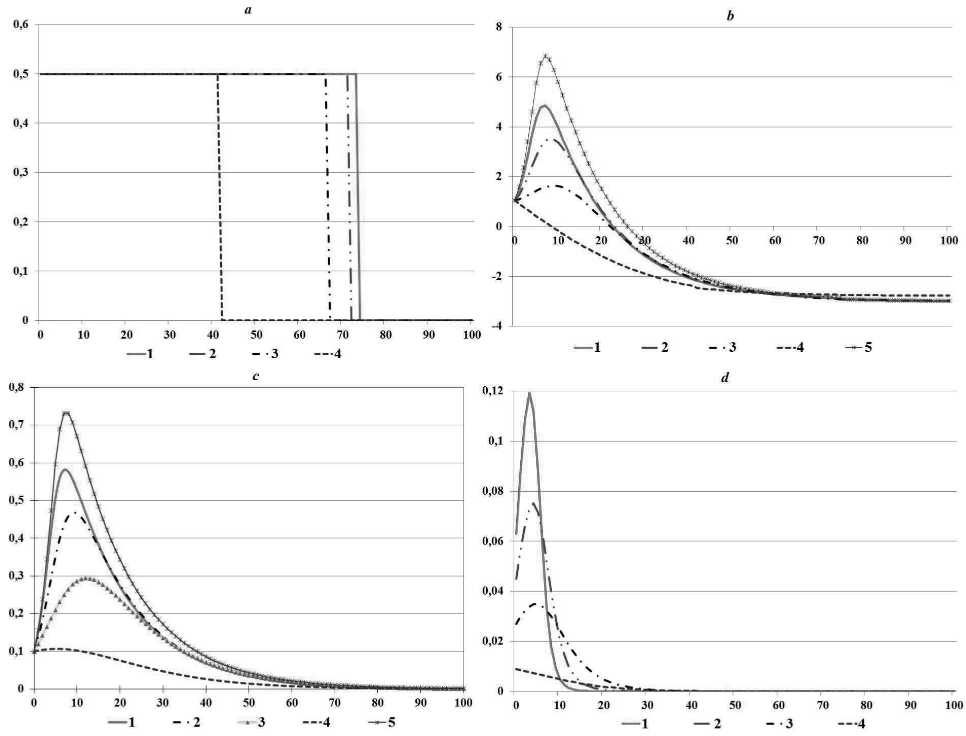

Fig. 2: a)Optimal vaccination policy. b) Aggregated costs. c) Dynamics of infected $I(t)$. d)Sickness rate $L(t)$.

- **a)** The optimal policy of vaccination u in case, when function $h(u)$ is concave for different values of the infection rate δ. (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$).
- **b)** Aggregated system costs (*line 1.* $J = -130.43$, $\delta = 0.7$, *line 2.* $J = -139.42$, $\delta = 0.5$, *line 3.* $J = -160,82$, $\delta = 0.3$, 4. $J = -201,64$, $\delta = 0.1$, *line 5.* uncontrolled case).
- **c)** Dynamics of changes in the subgroup of infected $I(t)$ (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$, 5. uncontrolled case).
- **d)** The sickness rate L for different values of parameter δ (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$).

We receive that the time moment t^*, when the control is off is shifted to the right, whereas an epidemic peak is shifted to the left, while value of δ is increased. For example if $\delta = 0.7$ then $t^* = 74$, and if $\delta = 0.1$ then $t^* = 42$. Aggregated system costs decrease as well as the sickness rate L.

In fig.3 we manipulate the infection rate δ in case of strictly convex costs function $h(u)$. Parameters of the dynamic system are are: $\sigma = 0.06667$, $\varepsilon = 0.5$. Initial states are: $S(0) = 0.9,\ I(0) = 0.1,\ R(0) = 0$. Maximum of the intensity rate of vaccination is $u_{\max} = 0.5$. Costs functions are: $f(I) = 10I$, $l(R) = 3R$, $h(u) = 0.1u^2$.

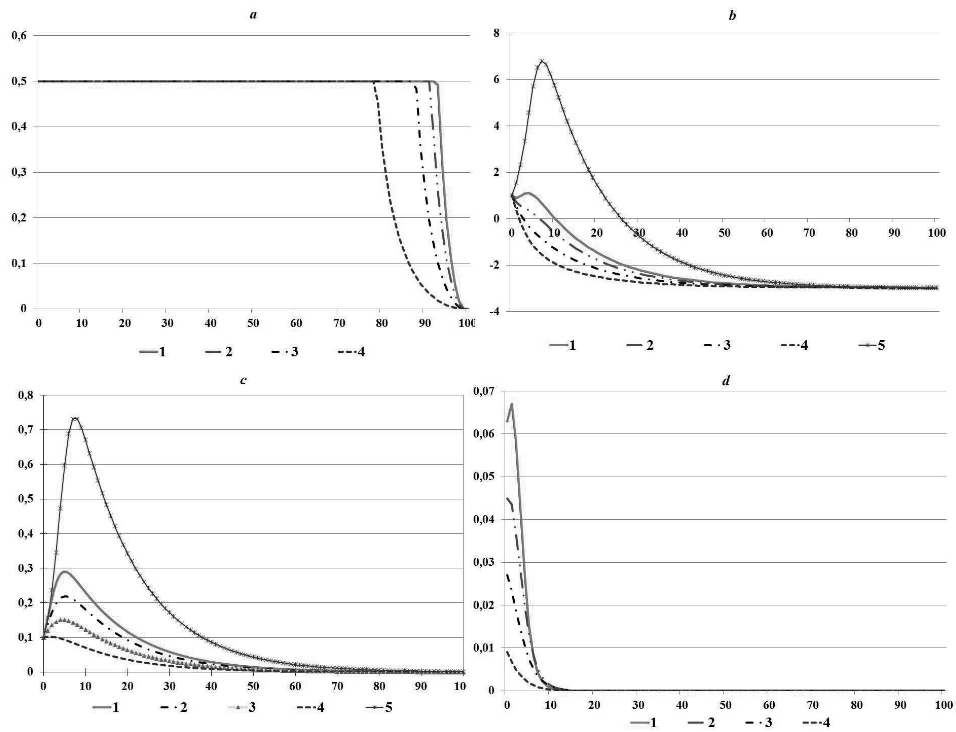

Fig. 3: a)Optimal vaccination policy. b) Aggregated costs. c) Dynamics of infected $I(t)$. d)Sickness rate $L(t)$.

Below we point the detailed definition of graphs a), b), c), d) and variations of parameters of the system.

a) The optimal policy of vaccination u in case, when function $h(u)$ – strictly convex for different values of the infection rate δ. (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$).

b) Aggregated system costs (*line 1.* $J = -217.38$, $\delta = 0.7$, *line 2.* $J = -232.29$, $\delta = 0.5$, *line 3.* $J = -249.03$, $\delta = 0.3$, *line 4.* $J = -264.58$, $\delta = 0.1$, *line 5.* uncontrolled case).

c) Dynamics of changes in the subgroup of infected $I(t)$ (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$, *line 5.* uncontrolled case).

d) The sickness rate L for different values of parameter δ (*line 1.* $\delta = 0.7$, *line 2.* $\delta = 0.5$, *line 3.* $\delta = 0.3$, *line 4.* $\delta = 0.1$).

In this case it have been shown that optimal vaccination policy influence on a number of infected and the aggregated system costs decrease. The control is switched off early if the rate of parameter δ decreases.

We also illustrate a behavior of agents in a population if each agent can choose an action, using information from its surroundings, according to the algorithm presented in section 3.1. The common parameters for the SIR model are: $\delta = 0.7$, $\sigma = 0.06667$. Initial states are: $S(0) = 0.9$, $I(0) = 0.1$, $R(0) = 0$. Maximum of intensity rate of vaccination is $u_{\max} = 0.5$, function $h(u) = 0.1u^2$ is convex, here the value of the threshold level is $L^* = 0.02$.

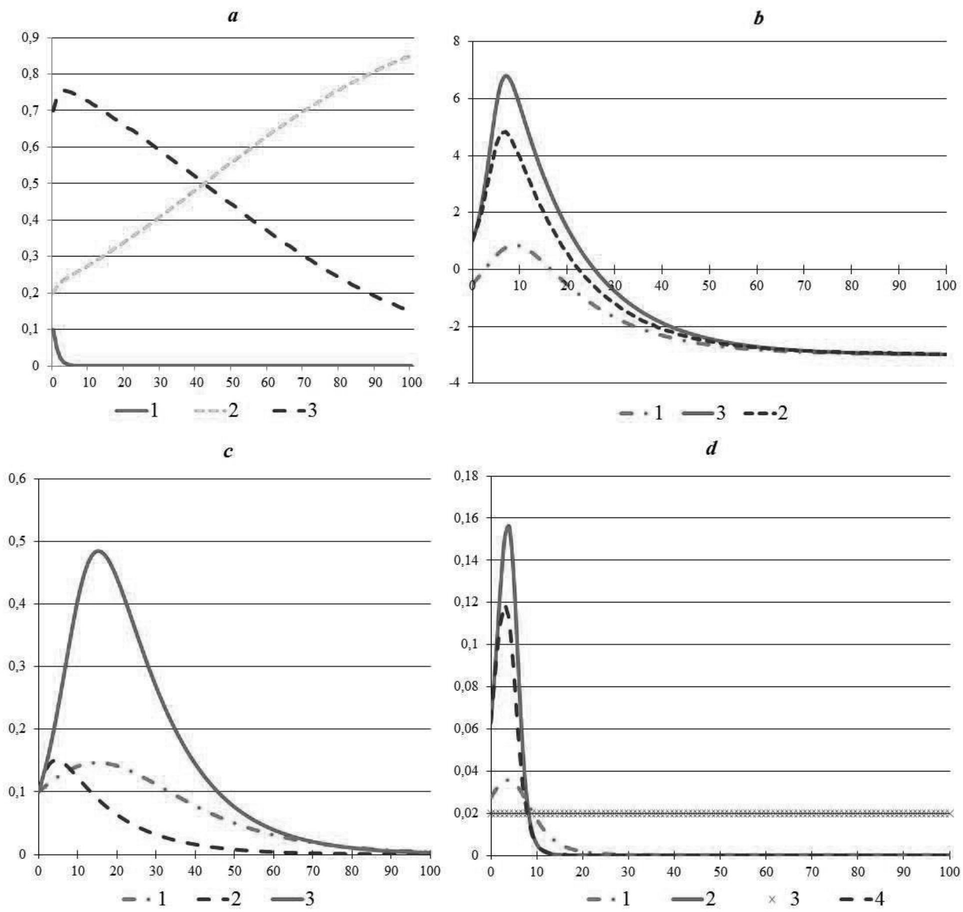

Fig. 4: a)Optimal vaccination policy. b) Aggregated costs. c) Dynamics of infected $I(t)$. d)Sickness rate $L(t)$.

The behavior of the curves $I(t)$, $J(t)$ and $L(t)$ demonstrate different scenarios of using vaccination in the system is shown in fig. 4 (b, c, d) and corresponding level of sickness rates.

a) The dynamics, which shows the changes in subgroups of agents, using different actions k_i $c(k_1) = 1$, $c(k_2) = 0.7$, $c(k_3) = 0.715$, $c(k_3) = 0.715$, benefits are

$b(k_i) = 3$, $p(k_1) = 2$, $p(k_2) = 2.3$, $p(k_3) = 2.285$. Initial states are: $x(k_1) = 0.1$, $x(k_2) = 0.2$, $x(k_3) = 0.7$. (*line 1.* $x(k_1)$, *line 2.* $x(k_2)$, line *line 3.* $x(k_3)$).

b) Aggregated system costs (*line 1.* Agents choose their behavior, $J = -197.57$, *line 2.* Controlled case, vaccination policy is chosen by the medical organization, $J = -131.73$, *line 3.* uncontrolled case, $J = -95.45$).

c) Dynamics of changes in the subgroup of infected $I(t)$ (*line 1.* Agents choose their behavior, *line 2.* Controlled case, vaccination policy is chosen by the medical organization, *line 3.* uncontrolled case).

d) The sickness rate L for different values of parameter δ (*line 1.* Agents choose their behavior, *line 2.* Controlled case, vaccination policy is chosen by the medical organization, *line 3.* threshold level L^*. *line 4.*uncontrolled case).

The experiments show that the different methods of applying vaccination program as a control in the system lead to significant cost savings, as well as to reduce the sickness rate. For example, in the case of $u_{max} = 0,5$ functional is $J = -131.73$, and if $u_v = 0.51$ (at time moment $t^* = 43$), then the value of aggregated system costs is $J = -197.57$. In the uncontrolled case the value of functional is J =-95.45.

5. Conclusion

In this paper, we have studied an epidemic model that takes into account the agent motivation of participation in the vaccination program. We incorporate procedure of making decision to the simple Susceptible-Infected-Recovered model and have formulated this model in special case. Using Pontryagin's maximum principle, we have shown the structure of optimal control for ideal situation. We supported our results with numerical simulations, observing different cases of epidemic process in entire urban population. In future work we would extend this model including different structure of population, it means that human decision may depend on his social group, not only his costs and to modify the model, using number of contacts as a function of the time.

6. Appendix

We will proof this statement base on the following properties (Khouzani et al., 2011, Gubar et al., 2013):

Property 1. Let $w(t)$ be a continuous and piecewise differential function of t. Let $w(t_0) = L$ and $w(t) < L$ for all $t \in (t_0, \dots, t_1]$. Then $\dot{w}(t_0^+) \leq 0$, where $w(t_1^+) = \lim_{x \to x_0} w(x)$.

Property 2. For any convex and differentiable function $y(x)$, which is 0 at $x = 0$, $y'(x)x - y(x) \geq 0$ for all $x \geq 0$.

Proof. (of lemma2)

Step I. Consider instant time moment $t = T$, from transversality conditions (6) we have $\lambda_S(T) - \lambda_I(T) = 0$, and $\dot{\lambda}_S(T) - \dot{\lambda}_I(T) = -\sum_i^n f'(I(T)) < 0$, $\dot{\lambda}_I(T) = \sum_i^n f'(I(T)) > 0$, therefore function λ_I is increasing on the interval $[0, T]$.

Step II.(Proof by contradiction).

Let $0 \leq t^* < T$ be the last time moment at which one of the inequality constraints are active:

$$\lambda_I(t) < 0,\ \lambda_S(t) < 0,\ \lambda_I(t) - \lambda_S(t) < 0 \text{ for } t^* < t < T,$$
$$\text{or}$$
$$\lambda_I(t^*) = 0,\ \lambda_S(t^*) = 0,\ \lambda_S(t^*) - \lambda_I(t^*) = 0.$$

Case **A.**

$\lambda_S(t^*) = 0,\ \ \lambda_I(t^{*+}) - \lambda_S(t^{*+}) < 0$ for $t^* < t < T$. Then

$$\dot{\lambda}_S(t^{*+}) = \delta I(\lambda_S - \lambda_I); \tag{24}$$

Since $\lambda_I(t) - \lambda_S(t) < 0$ for $t^* < t < T$, values δ and I are nonnegative then $\dot{\lambda}_S(t^{*+}) > 0$ which contradicts with property 1. Then case A. is impossible.

Case **B.**

$\lambda_S(t^*) = 0$, $\lambda_S(t^*) < 0$ for $t^* < t < T$ and $\lambda_S(t^*) - \lambda_I(t^*) = 0$, $\lambda_S(t^{*+}) - \lambda_I(t^{*+}) < 0$.

Now let consider $\dot{\lambda}_S(t^{*+}) - \dot{\lambda}_I(t^{+*})$.

The system ODE is autonomous, i.e., Hamiltonian and the constraints on the control u do not have an explicit dependency on the independent variable t. Then at time $t = T$ Hamiltonian is:

$$H = H(T) = -(f(I(T)) - l(R(T)) + h(u(T)). \tag{25}$$

costs functions follow the next conditions $f(I(T)) \geq 0$, $l(R(T)) \geq 0$, $h((u(T)) \geq 0$ and transversality conditions (6) at time moment T are justified then $H \leq 0$.

Hence as far as functions f, l, h are non-decreasing we have:

$$H + l(R(t)) \leq -(f(I(T)) + h(u(T))) \leq 0 \tag{26}$$

Then

$$\begin{aligned} \dot{\lambda}_S(t^{*+}) - \dot{\lambda}_I(t^{+*}) = \delta I(t)(\lambda_S - \lambda_I) - \psi\varepsilon - (f'(I) + \delta S(\lambda_S - \lambda_I) + \lambda_I \sigma) = \\ \delta I(\lambda_S - \lambda_I) - f'(I) - (\tfrac{H-(l(R)+h(u)+f(I))}{I} - \tfrac{\psi\varepsilon}{I}) = \\ \delta I(\lambda_S - \lambda_I) - [f(I) - If'(I)] - (\tfrac{H+l}{I}) + (\tfrac{h(u)+\varepsilon\psi}{I}) \end{aligned} \tag{27}$$

For any admissible control u and according to (7) for all $t \in [0, T]$

$$h(u^*(t) + \varepsilon\psi) \geq h(u(t) + \varepsilon\psi) \geq 0.$$

By property 2 $(f(I(t) - f'(I(t))I)) \leq 0$ is negative, from the case assumptions we have $(\lambda_S - \lambda_I) > 0$ and together with (26) we received that $\dot{\lambda}_S(t^{*+}) - \dot{\lambda}_I(t^{+*}) > 0$, then $\frac{d}{dt}(\lambda_S(t^{*+}) - \lambda_I(t^{*+})) > 0$, which contradicts property 1 then case B. is impossible.

Thus, non of the cases **A** and **B** can occur, which contradicts the lemmas' statement then a time moment t^{*+} does not exist. Hence the lemma follows.

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