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A Differential Game of Pollution Control with Participation of Developed and Developing Countries*

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Abstract In this paper, a 2-player non-cooperative differential game is considered. We assume that he first player is a developed country with linear-quadratic utility function while the second player is a developing country whose utility function explicitely depends on time. Furthermore, the duration of the game is considered to be a random variable which reflects the instability of the economical development of developing countries. In the paper the duration of the game is assumed to be exponentially distributed. The optimal strategies are sought in the class of open-loop strategies.

Keywords: pollution control, differential games, cooperative games, random time horizon

1. Introduction

This contribution is devoted to the important problem of the control of environmental pollution by a group of participating countries. This problem is formulated within a game-theoretic framework; the controls are the rates of pollution, the scalar state variable is the level of pollution.

In this paper, we considered the 2-player non-cooperative differential game proposed by Massoudi and Zaccour in (Masoudi and Zaccour, 2013). In this setup, the first player is a developed country with linear-quadratic utility function while the second player is a developing country whose utility function explicitely depends on time. In this work the described model was modified, namely we changed the payoff function of the second player. Furthermore, the duration of the game is considered to be a random variable which reflects the instability of the economical development of developing countries. We assumed that the duration of the game is exponentially distributed. The optimal strategies were sought in the class of open-loop strategies.

The paper is structured as follows: in Section 2 both cooperative and non-cooperative formulations were considered. We found a Nash equilibrium solution and a Pareto optimal solution (for the case of equal weights). Moreover, we computed a cooperative solution (Shapley value) on the base of three different characteristic functions: classical maxmin approach by Neumann-Morgenstern (Von Neumann and Morgenstern, 1944), max-approach with the use of Nash equilibrium solution for left out players (Petrosjan and Zaccour, 2003), and min-approach with the use of Pareto optimal solution for the players within the coalition (Petrosyan and Gromova, 2014). Finally, an analytical formulation for the components of the imputation distribution procedure guaranteeing time consistency (Petrosyan, 1977) of the chosed solution (Shapley value) were obtained. In Section 3 we consider the 3-player game in which the first player is the developed country and the remaining two players are the developing countries. All results are illustrated by simulation results.

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2. 2-person game model and main assumptions

As the basic model the 2-person differential game of pollution control proposed in (Masoudi and Zaccour, 2013) is considered. In the game $\Gamma(S_0)$, let the player 1 be a developed country, which is characterized by high level of industry development, and the player 2 be a developing country with evolving costs. The game evolves in continuous time on the time interval $t \in [0, \infty)$. Each country i controls the volume of pollution emission u_i, $i = 1, 2$, with upper critical value α_i ($u_i \in [0, \alpha_i]$).

Let $S(t)$ be the accumulated volume (stock) of pollution which evolves according to the following differential equation:

$$\dot{S}(t) = \mu(u_1(t) + u_2(t)) - \delta S(t), \quad S(0) = S_0, \tag{1}$$

where μ is a positive parameter, δ is the absorption coefficient, S_0 is the initial volume of the pollution.

The utility function for player i is described as a discounted difference between revenue $f_i(u_i)$ and damage cost $D_i(S)$, which corresponds to the expenses resulted from the environmental pollution. Suppose that functions $D_i(S)$ are continuously differentiable and convex.

Let the revenue functions $f_i(u_i)$ for both players have a similar form:

$$f_i(u_i) = \alpha_i u_i - \frac{1}{2}u_i^2, \quad i = 1, 2.$$

Furthermore, let the damage costs for the player 1 (developed country) have the following form: $D_1(S) = \beta_1 S$, i.e. the damage cost for developed country is proportional to the volume of pollution. Let ρ be the constant discount factor. Then the payoff function for player 1 is as follows:

$$J_1(u_1, u_2, S_0) = \int_0^\infty e^{-\rho t} \left(\alpha_1 u_1 - \frac{1}{2}u_1^2 - \beta_1 S\right) dt, \tag{2}$$

subject to (1).

We assume that T has the exponential distribution with parameter λ:

$$f(t) = \lambda e^{-\lambda(t-t_0)}, \qquad F(t) = 1 - e^{-\lambda(t-t_0)}, \qquad \frac{f(t)}{1-F(t)} = \frac{F'(t)}{1-F(t)} = \lambda(t) = \lambda.$$

Damage costs function in the model (Masoudi and Zaccour, 2013) is such that the damage cost increases with time:

$$D_2(S(t), t) = \beta_2 \frac{t}{T} S(t). \tag{3}$$

In our model we have T as a random value and hence instead of T in (3) we put its mathematical expectation $E(T) = \frac{1}{\lambda}$. Then we get:

$$D_2(S(t), t) = \beta_2 \lambda t S(t). \tag{4}$$

In this way the following form for expected integral payoff of player 2 is obtained:

$$J_2(u_1, u_2, S_0) = E\left(\int_0^T e^{-\rho\tau} \left(\alpha_2 u_2 - \frac{1}{2}u_2^2 - \beta_2 \lambda \tau S\right) d\tau dt\right) =$$

$$= \int_0^\infty \int_0^t e^{-\rho\tau} \left(\alpha_2 u_2 - \frac{1}{2} u_2^2 - \beta_2 \lambda \tau S \right) d\tau dF(t).$$

After simplification of the integral payoff (Kostyunin and Shevkoplyas, 2011), we get

$$J_2(u_1, u_2, S_0) = \int_0^\infty e^{-\rho\tau} \left(\alpha_2 u_2 - \frac{1}{2} u_2^2 - \beta_2 \lambda \tau S)(1 - F(\tau) \right) d\tau =$$

$$= \int_0^\infty e^{-(\rho+\lambda)\tau} \left(\alpha_2 u_2 - \frac{1}{2} u_2^2 - \beta_2 \lambda \tau S \right) d\tau. \quad (5)$$

It is easily seen that the integral payoff of player 2 is equivalent to the integral payoff of the respective player in the game with infinite time horizon and with discounting factor $\rho + \lambda$.

2.1. A cooperative 2-person game

Let us now consider the cooperative game $\Gamma_c(S_0)$ on the base of the game $\Gamma(S_0)$. It means that all players (developed country and developing country) act together to maximize their joint payoff

$$\max_{u_1, u_2} (J_1(u_1, u_2, S_0) + J_2(u_1, u_2, S_0)) =$$

$$\int_0^\infty e^{-\rho t} \left(\alpha_1 \overline{u_1} - \tfrac{1}{2} \overline{u_1^2} - \beta_1 S \right) dt + \int_0^\infty e^{-(\rho+\lambda)\tau} \left(\alpha_2 \overline{u_2} - \tfrac{1}{2} \overline{u_2^2} - \beta_2 \lambda \tau \bar{S} \right) d\tau,$$

$$\text{subject to } \dot{S}(t) = \mu \left(u_1(t) + u_2(t) \right) - \delta S(t), S(0) = S_0,$$

We will call $\overline{u_1}$, $\overline{u_2}$ optimal controls for players $1, 2$, $\bar{S}(t)$ – optimal (cooperative) trajectory. To find the optimal controls in the open-loop form, we apply Pontrygin's maximum principle (see Appendix 1). Then we get:

$$\overline{u}_1(t) = \alpha_1 - \mu \left(\frac{\beta_1}{\delta + \rho} + \frac{\beta_2 \lambda \mathrm{e}^{-\lambda t}}{(\delta + \lambda + \rho)^2} + \frac{\beta_2 \lambda t \mathrm{e}^{-\lambda t}}{\delta + \lambda + \rho} \right), \quad (6)$$

$$\overline{u}_2(t) = \alpha_2 - \mu \left(\mathrm{e}^{\lambda t} \frac{\beta_1}{\delta + \rho} + \frac{\beta_2 \lambda}{(\delta + \lambda + \rho)^2} + \frac{\beta_2 \lambda t}{\delta + \lambda + \rho} \right).$$

The controls of players must be taken from the compact sets $0 \le u_i \le \alpha_i$, $i = 1, 2$. Under the additional condition

$$\alpha_1 \ge \mu \frac{\beta_1}{\delta + \rho}, \quad (7)$$

the control $\bar{u}_1(t)$ is a nonnegative function and its upper value is equal to α_1. The economical sense of this condition could be investigated by economists.

For the player 2 the inequality $\bar{u}_2 \le \alpha_2$ is satisfied. But it may happen that at some time instant t^* the function $\bar{u}_2(t)$ becomes negative. Direct calculations give t^*:

$$t^* = -\frac{\gamma \mathrm{e}^{\gamma}}{\lambda} - \frac{1}{(\delta + \lambda + \rho)},$$

where

$$\gamma = \frac{(\delta+\lambda+\rho)(\beta_1\mu - \alpha_2(\delta+\rho))}{(\delta+\rho)\mu\beta_2} \mathrm{e}^{-\frac{\lambda}{\delta+\lambda+\rho}}.$$

If $t \geq t^*$ then the control for player 2 leaves the compact set $[0, \alpha_2]$. Hence after time instant t^* the optimal control takes its value on the boundary of the set of admissible control values. Finally the optimal control for player 2 has the following form:

$$\overline{u}_2 = \begin{cases} \alpha_2 - \mu \left(\mathrm{e}^{\lambda t} \frac{\beta_1}{\delta+\rho} + \frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2} + \frac{\beta_2\lambda t}{\delta+\lambda+\rho} \right), & t > t^*, \\ 0, & t \leqslant t^*. \end{cases} \quad (8)$$

Then we get the joint maximal payoff for all players:

$$J(\overline{u}_1, \overline{u}_2, S_0, R_0) = \int_0^{t^*} e^{-\rho\tau} \left(\alpha_1 \overline{u}_1 - \frac{1}{2}\overline{u_1}^2 - \beta_1 S \right) d\tau +$$
$$+ \int_{t^*}^{\infty} e^{-\rho\tau} \left(\alpha_1 \overline{u}_1 - \frac{1}{2}\overline{u_1}^2 - \beta_1 R \right) d\tau + \int_0^{t^*} e^{-(\rho+\lambda)\tau} \left(\alpha_2 \overline{u}_2 - \frac{1}{2}\overline{u_2}^2 - \beta_2\lambda\tau S \right) d\tau +$$
$$+ \int_{t^*}^{\infty} -e^{-(\rho+\lambda)\tau} \beta_2 \lambda\tau R d\tau, \quad (9)$$

where $R(t)$ is the cooperative trajectory after t^*, i.e. the solution of the differential equation

$$\dot{R}(t) = \mu u_1 - \delta R, \qquad R(t^*) = S(t^*). \quad (10)$$

We find the optimal (cooperative) trajectory before time instant t^*

$$\overline{S(t)} = S_0 e^{-\delta t} + \mu\left(\alpha_1 + \alpha_2\right)\varphi_1(0) - \frac{\mu^2\beta_1}{\delta+\rho}\left(\varphi_1(0) + \varphi_1(\lambda)\right) -$$
$$- \frac{\mu^2\beta_2\lambda}{(\delta+\lambda+\rho)^2}\left(\varphi_1(0) + \varphi_1(-\lambda)\right) - \frac{\mu^2\beta_2\lambda t}{\delta+\lambda+\rho}\left(\varphi_2(0) + \varphi_2(-\lambda)\right),$$

and after time instant t^*

$$\overline{R(t)} = R_0 e^{-\delta t} + \mu\left(\alpha_1 - \frac{\mu^2\beta_1}{\delta+\rho}\right)\varphi_1(0) - \frac{\mu^2\beta_2\lambda}{(\delta+\lambda+\rho)^2}\varphi_1(-\lambda) - \frac{\mu^2\beta_2\lambda t}{\delta+\lambda+\rho}\varphi_2(-\lambda),$$

where

$$\varphi_1(x) = \frac{\mathrm{e}^{xt} - \mathrm{e}^{-\delta t}}{\delta + x}, \qquad \varphi_2(x) = \frac{\mathrm{e}^{xt}(\delta+x)t - \mathrm{e}^{xt} - \mathrm{e}^{-\delta t}}{(\delta+x)^2}.$$

Let us denote

$$a = \alpha_1 + \alpha_2 - \frac{\beta_1}{\delta+\rho} - \frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2},$$

$$b = -\frac{\beta_1}{\delta+\rho}, \; c = -\frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2}, \; d = \alpha_1 - \frac{\beta_1}{\delta+\rho}, \; f = -\frac{\beta_2\lambda}{\delta+\lambda+\rho}.$$

Then we get an analytic formula for the joint maximal payoff:

$$J(\overline{u}_1,\overline{u}_2,S_0,R_0)=\int\limits_0^{t^*}e^{-\rho\tau}\left(\alpha_1\overline{u}_1-\frac{1}{2}\overline{u}_1^2-\beta_1\overline{S}\right)d\tau+$$

$$+\int\limits_{t^*}^{\infty}e^{-\rho\tau}\left(\alpha_1\overline{u}_1-\frac{1}{2}\overline{u}_1^2-\beta_1\overline{R}\right)d\tau+\int\limits_0^{t^*}e^{-(\rho+\lambda)\tau}\left(\alpha_2\overline{u}_2-\frac{1}{2}\overline{u}_2^2-\beta_2\lambda\tau\overline{S}\right)d\tau+$$

$$+\int\limits_{t^*}^{\infty}-e^{-(\rho+\lambda)\tau}\beta_2\lambda\tau\overline{R}d\tau=\frac{\left(\rho^2\,(d+c)+((c+2\,d)\,\lambda+f)\,\rho+d\lambda^2\right)\alpha_1}{\rho\,(\rho+\lambda)^2}+$$

$$+\frac{-(\rho+2\,\lambda)^3\,d^2-\rho\,\left(4\,c^2\lambda^2+\left(4\,\rho\,c^2+4\,fc\right)\lambda+c^2\rho^2+2\,\rho\,cf+2\,f^2\right)}{2\rho\,(\rho+2\,\lambda)^3}+$$

$$+\frac{-2\,(\rho+2\,\lambda)^3\,(\rho\,c+f+\lambda\,c)\,d}{2\,(\rho+\lambda)^2}+\frac{a\beta\,\mu\,\left(\delta\,\mathrm{e}^{\rho\,t^*}-\rho-\delta+\mathrm{e}^{-\delta\,t^*}\rho\right)\mathrm{e}^{-\rho\,t^*}}{\delta\,\rho\,(\rho+\delta)}-$$

$$-\frac{b\left((-\rho-\delta)\,\mathrm{e}^{-t^*\,(\rho-\lambda)}+(\rho-\lambda)\,\mathrm{e}^{-(\rho+\delta)t^*}+\delta+\lambda\right)\beta\,\mu}{(\delta+\lambda)\,(\rho-\lambda)\,(\rho+\delta)}+\frac{\beta\,\mu\,c}{(\rho+\lambda)\,(\rho+\delta)}+$$

$$+\frac{\left(\mathrm{e}^{-\delta\,t^*}\rho^2-\delta^2\mathrm{e}^{\rho\,t^*}+((\delta\,t^*-1)\,\rho+\delta)\,(\rho+\delta)\right)f\beta\,\mathrm{e}^{-\rho\,t^*}\mu}{\delta^2\rho^2\,(\rho+\delta)}+\frac{f\beta\,\mu}{(\rho+\delta)\,(\rho+\lambda)^2}+$$

$$+2\,\frac{\beta\,S_0\,\left(-1+\mathrm{e}^{-(\rho+\delta)t^*}\right)}{\rho+\delta}+\frac{\mathrm{e}^{-(\rho+\delta)t^*}\beta\,R_0}{\rho+\delta}-\frac{\beta\,\mu\,d\left(-\rho-\delta+\mathrm{e}^{-\delta\,t^*}\rho\right)\mathrm{e}^{-\rho\,t^*}}{\delta\,\rho\,(\rho+\delta)}+$$

$$+\frac{\alpha_2\,\mathrm{e}^{-\rho\,t^*}\,((\alpha_2+c+t^*\,f)\,\rho+(\alpha_2+c+t^*\,f)\,\lambda+f)\,\mathrm{e}^{-\lambda\,t^*}}{(\rho+\lambda)^2}+$$

$$+\frac{\alpha_2\left((\alpha_2+c+b)\,\rho^2+((2\,b+c+\alpha_2)\,\lambda+f)\,\rho+b\lambda^2\right)}{\rho\,(\rho+\lambda)^2}-\frac{\alpha_2\,\mathrm{e}^{-\rho\,t^*}b}{\rho}-$$

$$-\frac{e^{-\rho t^*}}{2\,(\rho-\lambda)\,\rho^2\,(\rho+\lambda)^3}(-\left(\rho^2b\mathrm{e}^{t^*\,\lambda}+2\,(\rho-\lambda)\,((\alpha_2+c+ft^*)\,\rho+f)\right)b\,(\rho+\lambda)^3)-$$

$$-\frac{(c+\alpha_2+b)^2\,\rho^5+3\,((b+1/3\,\alpha_2+1/3\,c)\,\lambda+2/3\,f)\,(c+\alpha_2+b)\,\rho^4}{2\,(\rho-\lambda)\,\rho^2\,(\rho+\lambda)^3}-$$

$$-\frac{\left(\left(3\,b^2-(\alpha_2+c)^2\right)\lambda^2+4\,\lambda\,bf+2\,f^2\right)\rho^3}{2\,(\rho-\lambda)\,\rho^2\,(\rho+\lambda)^3}-$$

$$-\frac{\left(\left(b^2+(-4\,\alpha_2-4\,c)\,b-(\alpha_2+c)^2\right)\lambda^2-2\,f\,(\alpha_2+c)\,\lambda-2\,f^2\right)\lambda\,\rho^2}{2\,(\rho-\lambda)\,\rho^2\,(\rho+\lambda)^3}-$$

$$-\frac{-2\,b\,(\lambda\,(\alpha_2+c)+2\,f)\,\lambda^3\rho-2\,bf\lambda^4}{2\,(\rho-\lambda)\,\rho^2\,(\rho+\lambda)^3}+$$

$$+\frac{\beta_2\,\lambda\,\mu\,a\left(-(\rho\,t^*+\lambda\,t^*+\delta\,t^*+1)\,\mathrm{e}^{-(\rho+\lambda+\delta)t^*}\right)}{\delta\,(\rho+\lambda+\delta)^2}+\frac{\beta_2\,\lambda\,\mu\,a\left((\rho\,t^*+\lambda\,t^*+1)\,\mathrm{e}^{-(\rho+\lambda)t^*}\right)}{\delta\,(\rho+\lambda)^2}-$$

$$-\frac{\beta_2\,\lambda\,\mu\,a\,(-2\,(1/2\,\delta+\lambda+\rho))}{(\rho+\lambda)^2\,(\rho+\lambda+\delta)^2}-$$

$$-\frac{\beta_2\,\lambda\,\mu\,b\mathrm{e}^{-\rho\,t^*}\left(\rho^2\,(\rho\,t^*+\lambda\,t^*+\delta\,t^*+1)\,\mathrm{e}^{-t^*\,(\delta+\lambda)}+2\,(\delta+\lambda)\,(1/2\,\delta+\rho+1/2\,\lambda)\,\mathrm{e}^{\rho\,t^*}\right)}{(\delta+\lambda)\,\rho^2\,(\rho+\lambda+\delta)^2}+$$
$$+\frac{\beta_2\,\lambda\,\mu\,(\rho\,t^*+1)}{(\delta+\lambda)\,\rho^2}+\frac{(\delta+2\,\rho+3\,\lambda)\,(\beta)\,\lambda\,\mu\,c}{(\rho+2\,\lambda)^2\,(\rho+\lambda+\delta)^2}+$$
$$+\frac{\lambda\,\mu\,\left((\delta\,t^*+1+(\rho+\lambda)\,t^*)\,(\rho+\lambda)^3\,\mathrm{e}^{-(\rho+\lambda+\delta)t^*}+\left((2+(\rho+\lambda)^2\,t^{*2}+(2\,\rho+2\,\lambda)\,t^*)\,\delta\right)\right)\beta_2\,f}{\delta^2\,(\rho+\lambda)^3\,(\rho+\lambda+\delta)^2}-$$
$$-\frac{\lambda\,\mu\,\left((-\,(1+(\rho+\lambda)\,t^*)\,(\rho+\lambda))\,(\rho+\lambda+\delta)^2\,\mathrm{e}^{-(\rho+\lambda)t^*}-3\,\delta^2\,(\rho+\lambda+2/3\,\delta)\right)\beta_2\,f}{\delta^2\,(\rho+\lambda)^3\,(\rho+\lambda+\delta)^2}-$$
$$-\frac{(2\,\delta+3\,\rho+4\lambda)\,\beta_1\,\lambda\,\mu\,f}{(\rho+2\,\lambda)^3\,(\rho+\lambda+\delta)^2}+\frac{R_0\,\left(1+(\rho+\delta)\,t^*\right)\lambda\,(\beta)\,\mathrm{e}^{-(\rho+\delta)t^*}}{(\rho+\delta)^2}-$$
$$-\frac{d\lambda\,\left(-\,(\rho+\lambda)^2\,(\rho\,t^*+\lambda\,t^*+\delta\,t^*+1)\,\mathrm{e}^{-(\rho+\lambda+\delta)t^*}+(\rho+\lambda+\delta)^2\,(\rho\,t^*+\lambda\,t^*+1)\,\mathrm{e}^{-(\rho+\lambda)t^*}\right)\beta_2}{\delta\,(\rho+\lambda)^2\,(\rho+\lambda+\delta)^2}.$$

2.2. Nash equilibrium for 2-player game

Now we find a pair of strategies $(u_1{}^{NE}, u_2{}^{NE})$ which form the Nash equilibrium solution. The optimization problems for players $1, 2$ are formulated as follows

$$\max_{u_1} J_1(u_1, u_2^{NE}, S_0) = \int_0^\infty e^{-\rho t}\left(\alpha_1 u_1{}^{NE} - \frac{1}{2}\left(u_1^{NE}\right)^2 - \beta_1 S^{NE}\right)dt,$$

$$\max_{u_2} J_1(u_1^{NE}, u_2, S_0) = \int_0^\infty e^{-(\rho+\lambda)t}\left(\alpha_2 u_2{}^{NE} - \frac{1}{2}\left(u_2^{NE}\right)^2 - \beta_2\lambda t S^{NE}\right)dt,$$

$$\dot{S}(t) = \mu\left(u_1^{NE}(t) + u_2^{NE}(t)\right) - \delta S(t), S(0) = S_0.$$

Using Pontryagin's maximum principle we get

$$u_1{}^{NE} = \alpha_1 - \mu\frac{\beta_1}{\delta+\rho}, \tag{11}$$
$$u_2{}^{NE} = \alpha_2 - \mu\left(\frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2} + \frac{\beta_2\lambda t}{\delta+\lambda+\rho}\right).$$

From condition (7) it follows that control $u_1{}^{NE}$ for player 1 will not leave the admissible set $[0, \alpha_1]$. It is also clear that control $u_2{}^{NE}$ will be bounded from above by α_2. But as for cooperative case, there is a time instant t^{**} at which $u_2{}^{NE}(t)$ reaches zero (and then leave the compact set). In the same way as before we find

$$t^{**} = \frac{\alpha_2(\delta+\lambda+\rho)}{\mu\beta_2\lambda} - \frac{1}{(\delta+\lambda+\rho)}, \tag{12}$$

and

$$u_2{}^{NE} = \begin{cases} \alpha_2 - \mu\left(\dfrac{\beta_2\lambda}{(\delta+\lambda+\rho)^2} + \dfrac{\beta_2\lambda t}{\delta+\lambda+\rho}\right), & t > t^{**}, \\ 0, & t \leqslant t^{**}. \end{cases} \tag{13}$$

We also rewrite the payoff of the player 2:

$$J_2(u_1,u_2,S_0,R_0)=\int\limits_0^{t^{**}} e^{-(\rho+\lambda)\tau}\left(\alpha_2 u_2-\frac{1}{2}u_2^2-\beta_2\lambda\tau S\right)d\tau+ \\ +\int\limits_{t^{**}}^{\infty} -e^{-(\rho+\lambda)\tau}\beta_2\lambda\tau R d\tau,$$

where $R(t)$ is the trajectory after time instant t^*,

$$\dot{R}(t)=\mu u_1-\delta R,\ R(t^{**})=S(t^{**}).$$

Finally we get the Nash trajectory before t^{**} in the following form:

$$S(t)=S_0e^{-\delta t}+\frac{(1-e^{-\delta t})\mu}{\delta^2}\left(\delta\left(\alpha_1-\frac{\mu\beta_1}{\delta+\rho}+\alpha_2-\frac{\mu\beta_2\lambda}{(\delta+\lambda+\rho)^2}\right)+ \\ +\frac{\mu\beta_2\lambda}{(\delta+\lambda+\rho)}\right)-\frac{\mu\beta_2\lambda t}{\delta(\delta+\lambda+\rho)},$$

and after time instant t^{**} we get

$$R(t)=R_0e^{-\delta t}+\frac{(1-e^{-\delta t})\mu}{\delta}\left(\delta\left(\alpha_1-\frac{\mu\beta_1}{\delta+\rho}\right)\right).$$

The payoff of the player 2 is not shown here because of the bulkiness.

2.3. Analysis of results

In the preceding section we presented results for cooperative and non-cooperative 2-person games in analytic form. Now we give a numerical illustration of these results for the following parameters of the model:

$$\alpha_2 = 20; \alpha_1 = 30; \beta_2 = 0.03; \beta_1 := 0.06; \lambda = 0.5; \rho = 0.1; \delta = 0.5; \mu = 0.8.$$

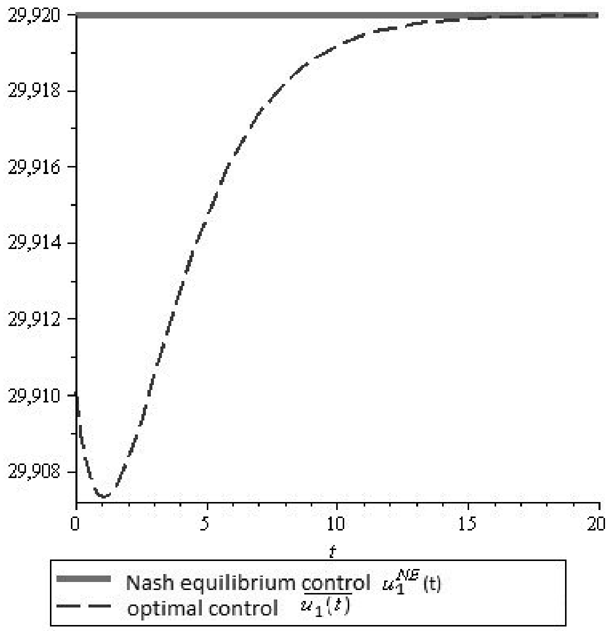

Fig. 1: Volume of emissions for developed country

As shown in Fig. 1, in cooperative case the volume of emissions for the developed country is lower than in the non-cooperative case.

Figure 2 demonstrates that in cooperative case the volume of emissions for developing country is lower than in the non-cooperative case. Finally, Figure 3 shows that in the cooperative case of the game the accumulated volume of the pollutions is lower than in the non-cooperative case of the game.

Thus the central conclusion that the cooperative behavior of the players (countries) is beneficial both for all participants of the game and also for the environment.

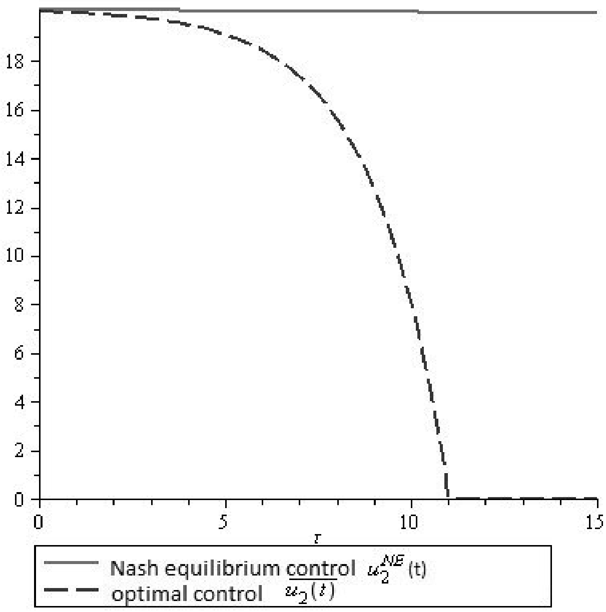

Fig. 2: Volume of emissions for developing country

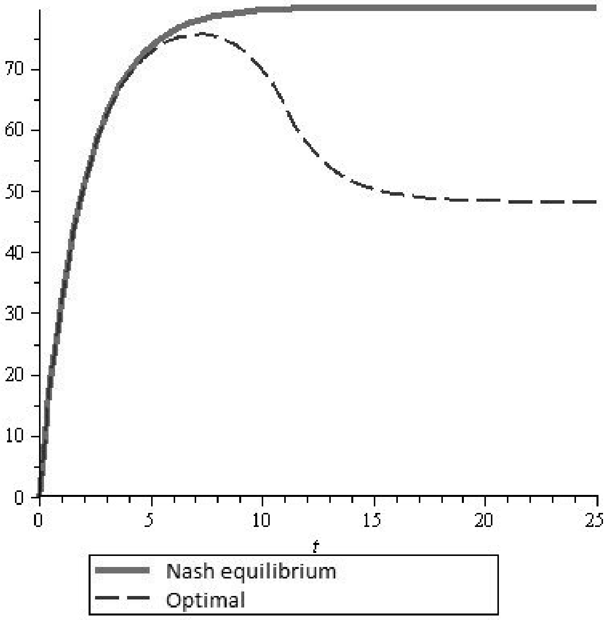

Fig. 3: Stock of the pollution

2.4. Imputations in cooperative game

In cooperative game players have to share the total maximal payoff according to some optimality principle. In this paper we use the Shapley Value as a rule for the allocation of this amount:

$$Sh_i(S_0, 0) = \sum_{K \ni i} \frac{(n-k)!\,(k-1)!}{n!} \left[V\left(K; S_0, 0\right) - V\left(K \setminus \{i\}; S_0, 0\right)\right], i \in N,$$

where k is number of players in coalition K, $V(K, S_0, t_0)$, $K \subseteq N$ is the characteristic function which shows the power of the coalition K in cooperative game $\Gamma_c(S_0)$, constructed by any appropriate approach (see (Von Neumann and Morgenstern, 1944; Petrosjan and Zaccour, 2003; Reddy and Zaccour, 2014; Petrosyan and Gromova, 2014)).

For game with 2 players we get:

$$Sh_1(S_0, 0) = \frac{1}{2}[V(N, S_0, 0) - V(\{2\}, S_0, 0) + V(\{1\}, S_0, 0)];$$
$$Sh_2(S_0, 0) = \frac{1}{2}[V(N, S_0, 0) - V(\{1\}, S_0, 0) + V(\{2\}, S_0, 0)].$$

We construct the characteristic function in three ways. The first way was proposed in (Von Neumann and Morgenstern, 1944) and now it is a classical approach for construction of a superaddtivie characteristic function:

$$V^{\alpha}(K, S_0, t_0) = \begin{cases} 0, & K = \{\emptyset\} \\ \max\limits_{u_i, i \in K} \min\limits_{u_j, j \in N \setminus K} J_i(u_1, u_2, t_0, S_0), & K = \{i\} \\ \max\limits_{u_1, u_2}(J(u_1, u_2, t_0, S_0)), & K = N \end{cases}. \quad (14)$$

By using Pontryadin's maximum principle it is easy to obtain controls, trajectory and payoffs for this maxmin approach. We do not show the analytic formulas for the reason of their bulkiness.

The second way of the characteristic function construction was proposed in the work (Petrosjan and Zaccour, 2003). It is based on the two-step procedure: first we find the Nash equilibrium for all players and then we take the Nash strategies for left-out players from $N \setminus K$ while players from K maximize their joint payoff:

$$V^{\delta}(K, S_0, t_0) = \begin{cases} 0, & K = \{\emptyset\} \\ \max\limits_{\substack{u_i, j \in K \\ u_j = u_j{}^{NE}, j \in N \setminus K}} J_i(u_1, u_2, t_0, S_0), & K = \{i\} \\ \max\limits_{u_1, u_2}(J(u_1, u_2, t_0, S_0)), & K = N \end{cases}. \quad (15)$$

In general it is not a superadditive characteristic function. The values of the characteristic function for the case of 2-player were obtained above, because here we only need to calculate payoffs in Nash equilibrium and cooperative joint payoff.

The third approach was proposed in (Petrosyan and Gromova, 2014). The characteristic function $V(K, S_0)$ is constructed by the following way: first we find the optimal controls for all players, then freeze these controls for players from coalition K and then find controls of left-out players from $N \setminus K$ which give the minimum to the payoff of players from coalition K. The advantages of this approach in addition to the simplification of the calculation is that the constructed characteristic function is superadditive in general. For the case of 2-player game we have the simple formula for characteristic function:

$$V(K,S_0,t_0)=\begin{cases} 0, & K=\{\emptyset\} \\ \min\limits_{\substack{u_j, j\in N\setminus K\\ u_i=\overline{u_i}, i\in K}} J_i(u_1,u_2,t_0,S_0), & K=\{i\} \\ \max\limits_{u_1,u_2}(J(u_1,u_2,t_0,S_0)), & K=N \end{cases}. \tag{16}$$

From direct calculations we get

$$V(\{1\},S_0)=\frac{\alpha_1\left((d+c)\,\rho^2+((c+2\,d)\,\lambda+f)\,\rho+d\lambda^2\right)}{\rho\,(\rho+\lambda)^2}+$$
$$+1/2\,\frac{-(\rho+\lambda)^2(\rho+2\,\lambda)^3d^2-2\,\rho\,(\rho+2\,\lambda)^3(\rho\,c+\lambda\,c+f)\,d}{\rho\,(\rho+\lambda)^2(\rho+2\,\lambda)^3}+$$
$$++1/2\,\frac{-\rho\,(\rho+\lambda)^2\left(4\,c^2\lambda^2+\left(4\,fc+4\,\rho\,c^2\right)\lambda+c^2\rho^2+2\,f^2+2\,\rho\,cf\right)}{\rho\,(\rho+\lambda)^2(\rho+2\,\lambda)^3}+$$
$$+\frac{(\alpha_2+b+\alpha_1)\,\beta_1\,\mu}{\rho\,(\delta+\rho)}+\frac{\beta_1\,\mu\,b}{(\rho-\lambda)\,(\delta+\rho)}+\frac{\mu\,\beta_1\,c}{(\rho+\lambda)\,(\delta+\rho)}+\frac{\mu\,\beta_1\,f}{\rho^2\,(\delta+\rho)}+$$
$$+\frac{\mu\,\beta_1\,f}{(\rho+\lambda)^2(\delta+\rho)}+\frac{\beta_1\,S_0}{\delta+\rho},$$

and $V(\{2\},S_0)$.

The Shapley Value was calculated for all three methods of the characteristic function, but these are not shown here for the reasin of the very big volume of formulas. The analysis of the results will be shown below.

2.5. Imputation distribution procedure

It is important to establish the dynamically stable (time-consistent) cooperative agreement which is the Shapley Value in the framework of this paper. Time-consistency involves the property that, as the cooperation develops cooperating partners are guided by the same optimality principle at each instant of time and hence do not possess incentives to deviate from the previously adopted cooperative behavior. The approach how to avoid the problem of time-inconsistency of the solution by the special imputation distribution procedure was proposed in (Petrosyan, 1977).

We use the same ideology as in (Petrosyan, 1977). Then we suggest to distribute the imputations from Shapley Value by the inputation distribution procedure $B_1(t)$, $B_2(t)$ such that

$$Sh_1\left(\overline{S}_0\right)=\int_0^\infty e^{-\rho t}B_1(t)\,dt,\quad Sh_2\left(\overline{S}_0\right)=\int_0^\infty e^{-(\rho+\lambda)t}B_2(t)\,dt,$$
$$Sh_1\left(\overline{S}_\tau\right)=\int_\tau^\infty e^{-\rho(t-\tau)}B_1(t)\,dt,$$
$$Sh_2\left(\overline{S}_\tau\right)=\int_\tau^\infty e^{-(\rho+\lambda)(t-\tau)}B_2(t)\,dt,\quad \tau\in[0,\infty),$$

where $\{Sh_i\left(\overline{S}_\tau\right)\}_{i=1,2}$ is the Shapley value in subgames $\Gamma_c(\bar{S}(t))$ occuring along the optimal cooperative trajectory $\bar{S}(t)$.

Then we get

$$Sh_1\left(\overline{S}_0\right) = \int_0^\tau e^{-\rho t} B_1\left(t\right) dt + e^{-\rho\tau} \int_\tau^\infty e^{-\rho(t-\tau)} B_1\left(t\right) dt,$$

$$Sh_2\left(\overline{S}_0\right) = \int_0^\tau e^{-(\rho+\lambda)t} B_2\left(t\right) dt + e^{-(\rho+\lambda)\tau} \int_\tau^\infty e^{-(\rho+\lambda)(t-\tau)} B_2\left(t\right) dt, \quad \tau \in [0,\infty).$$

It means that we can represent the Shapley Value in the game $\Gamma_c(S_0)$ in the followin form:

$$Sh_1\left(\overline{S}_0\right) = \int_0^\tau e^{-\rho t} B_1\left(t\right) dt + e^{-\rho\tau} Sh_1\left(\overline{S}_\tau\right),$$

$$Sh_2\left(\overline{S}_0\right) = \int_0^\tau e^{-(\rho+\lambda)t} B_2\left(t\right) dt + e^{-(\rho+\lambda)\tau} Sh_2\left(\overline{S}_\tau\right).$$

Thus at the each time instant players are oriented to the same optimality principle (Shapley Value).

By taking the first derivative we obtain the formulas for imputation distribution procedure which are very close to formulas in (Petrosjan and Zaccour, 2003):

$$B_1\left(\tau\right) = \rho Sh_1\left(\overline{S}_\tau\right) - \frac{d}{d\tau} Sh_1\left(\overline{S}_\tau\right), \tag{17}$$
$$B_2\left(\tau\right) = (\rho+\lambda) Sh_2\left(\overline{S}_\tau\right) - \frac{d}{d\tau} Sh_2\left(\overline{S}_\tau\right).$$

We calculate the imputation distribution procedure for all 3 forms of the characteristic functions but do not put it here for their bulkiness. The results will be analyzed below.

2.6. Analysis of the cooperative solution

Below, a number of plots illustrated the obtained results will be presented. We will compare different approaches to the definition of the characteristic function and consider both cooperative and non-cooperative behaviour of the players.

We see from Figure 4 that the functions have very similar values especially for small values of t. In the Fig. 5 we consider these plots for $t > 150$.

As shown in Fig. 6, the maximal payoff for the developed country is obtained when using the characteristic function found according to the formula(15) while the minimal payoff corresponds to the value function given by (16).

We see that the functions have similar values. In Figure 7 the plots are zoomed in.

The plots presented in Fig. 8 and Fig. 9 show that both for the developed and developing country the maximal payoff is obtained for the case of characteristic function defined according to (15) and the minimal payoff corresponds to(16).

It tuns out that that the cooperative behaviour is beneficial both for developed and developing countries.

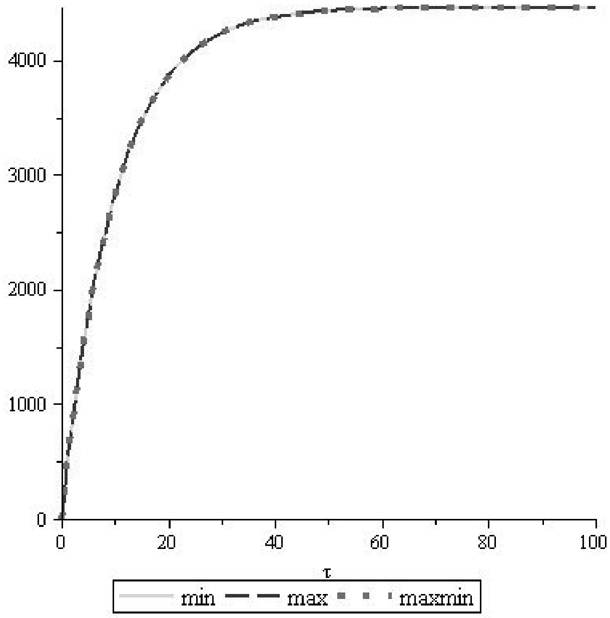

Fig. 4: Payoff of the developed country when considering non-cooperative solution for different types of characteristic functions

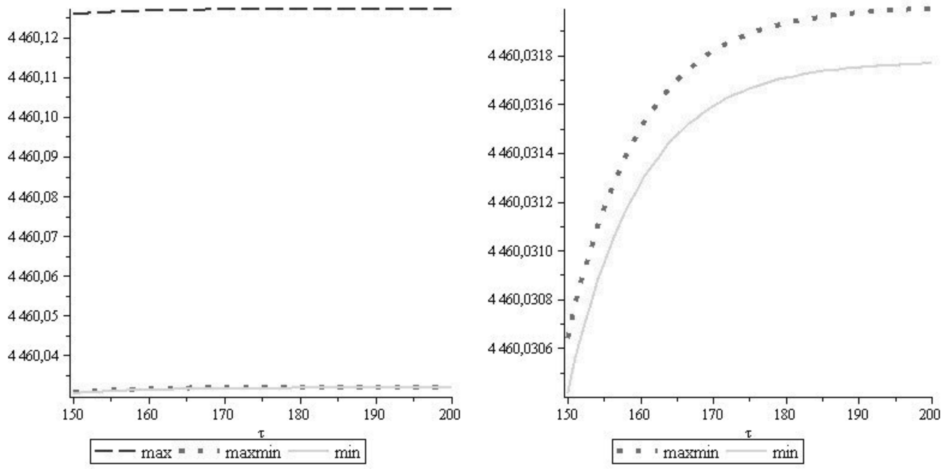

Fig. 5: Payoff of the developed country when considering non-cooperative solution for different types of characteristic functions ($t > 150$)

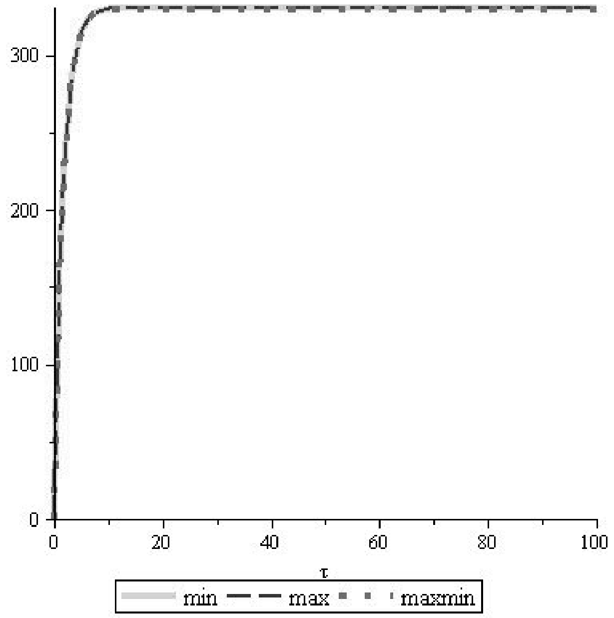

Fig. 6: Payoff of the developing country when considering non-cooperative solution for different types of characteristic functions

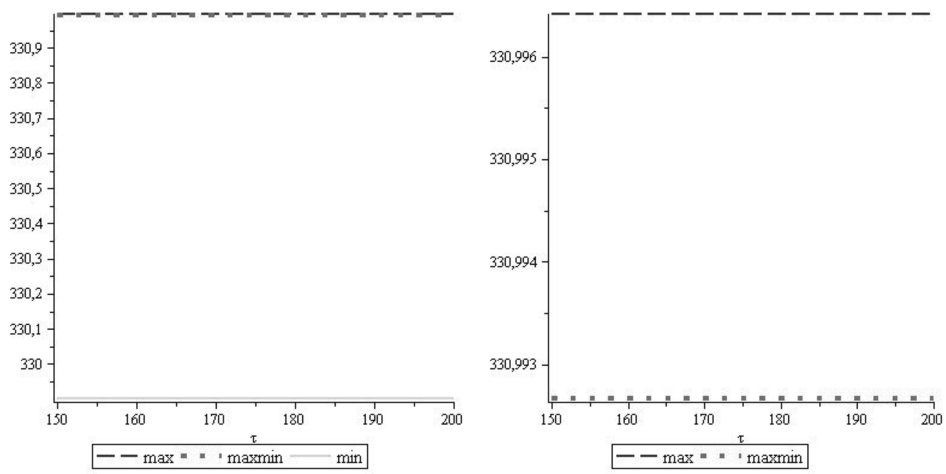

Fig. 7: Payoff of the developing country when considering non-cooperative solution for different types of characteristic functions (zoomed in)

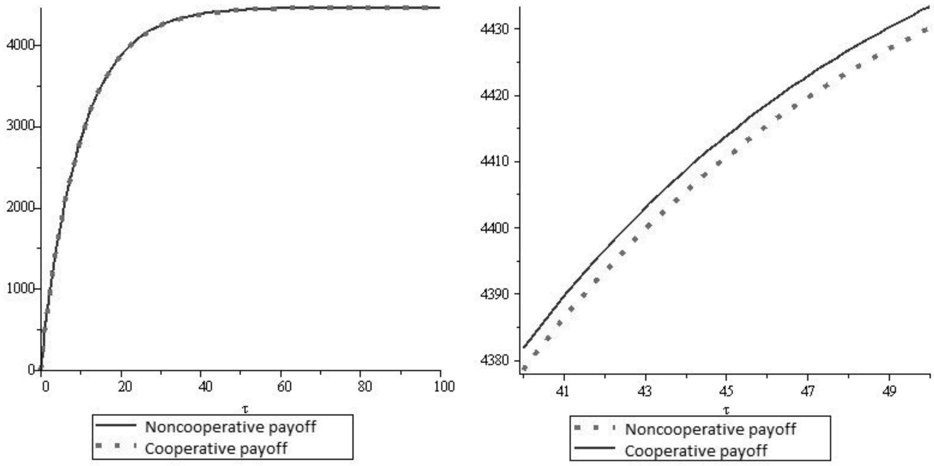

Fig. 8: Payoff of the developed country for the characteristic function (16) in the cooperative and non-cooperative cases.

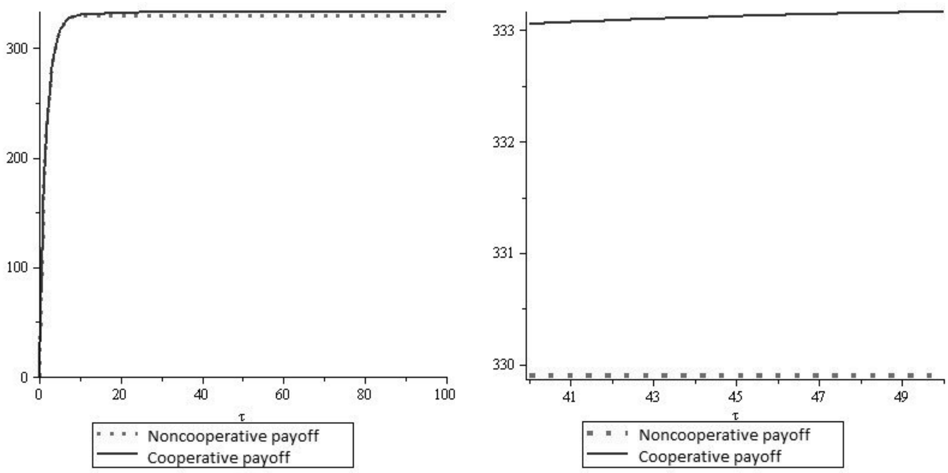

Fig. 9: Payoff functions of a developing country for (16)

3. 3-player game

Consider 3-player game, organized by the following way: the first player (player 1) is the same developed country as before, the second player (player 2) is the same developing country as before and the third player (player 3) is develping country identical to player 2. Then we get the following payoffs

$$J_1(u_1, u_2, u_3, S_0) = \int_0^\infty e^{-\rho t} \left(\alpha_1 u_1 - \frac{1}{2} u_1^2 - \beta_1 S \right) dt,$$

$$J_2(u_1, u_2, u_3, S_0) = \int_0^\infty e^{-(\rho+\lambda)\tau} \left(\alpha_2 u_2 - \frac{1}{2} u_2^2 - \beta_2 \lambda \tau S \right) d\tau,$$

$$J_3(u_1, u_2, u_3, S_0) = \int_0^\infty e^{-(\rho+\lambda)\tau} \left(\alpha_2 u_3 - \frac{1}{2} u_3^2 - \beta_2 \lambda \tau S \right) d\tau,$$

and dynamics of the pollution

$$\dot{S}(t) = \mu \left(u_1(t) + u_2(t) + u_3(t) \right) - \delta S(t), S(0) = S_0.$$

We do not put here details how to calculate optimal controls, cooperative trajectory, payoffs, characteristic function, Shapley value and imputation distribution procedure for cooperative case of the game. We also calculate the Nash equilibrium for non-cooperative game and compare cooperative and non-cooperative behaviour of players by similar but more complicated methods as for 2-player game. Below, we present some numerical results.

According to Fig. 10, for the developed country it is more advantageous to choose the characteristic function (14).

Figure 11 shows that for the developing country the characteristic function (14) gives a bigger payoff either.

Next, as Fig. 12 shows, for the developed country it is more advantageous to choose the characteristic function (16).

For the developing country it is better to choose the characteristic function (15) (see Fig. 13).

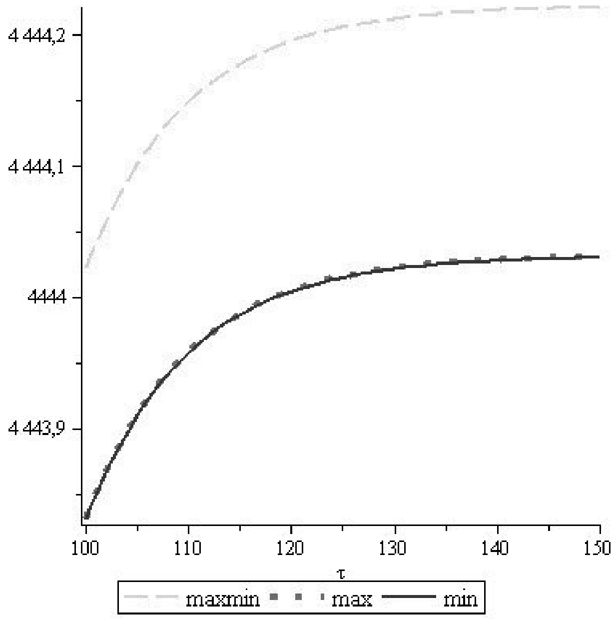

Fig. 10: Comparison of the payoff functions for the non-cooperative case and different types of characteristic functions: Developed country

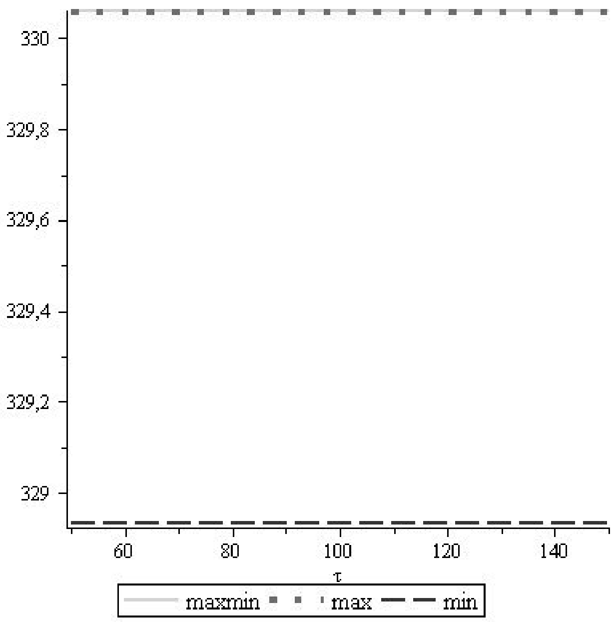

Fig. 11: Comparison of the payoff functions for the non-cooperative case and different types of characteristic functions: Developing country

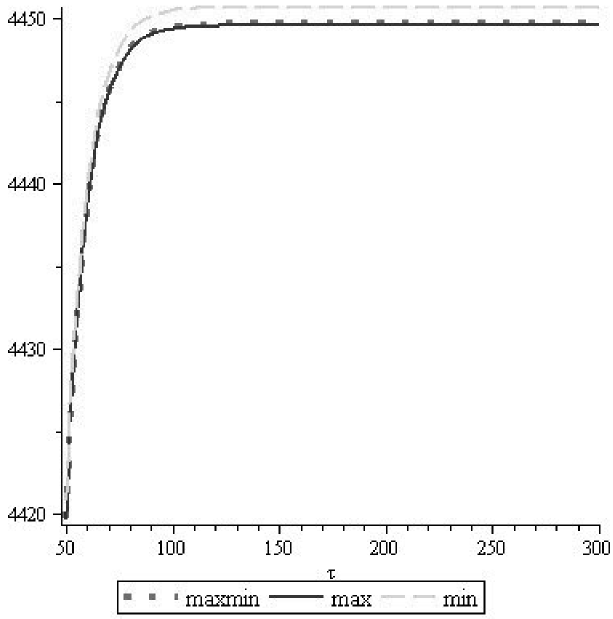

Fig. 12: Comparison of payoff functions for different types of characteristic function and the cooperative behaviour: Developed country

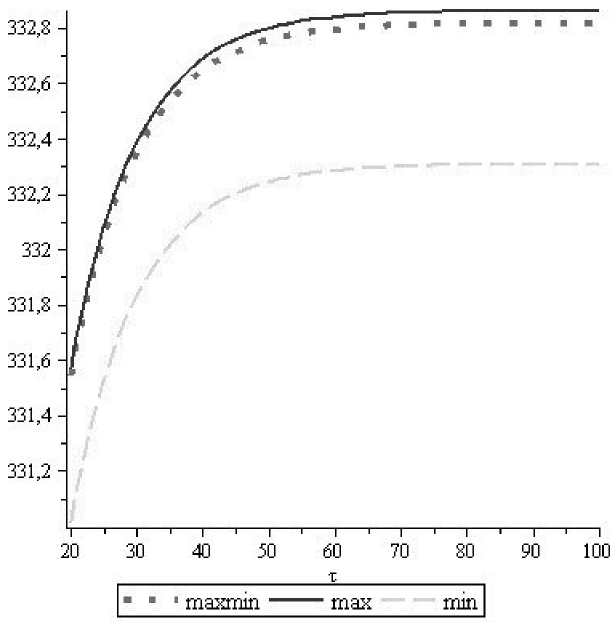

Fig. 13: Comparison of payoff functions for different types of characteristic function and the cooperative behaviour: Developing country

4. Appendix 1

We use Pontryagin's maximum principle to find the optimal controls for 2-player cooperative game. The Hamiltonian has a form

$$H = \left[e^{-\rho t}\left(\alpha_1 u_1 - \frac{1}{2}u_1^2 - \beta_1 S\right)\right] + \left[e^{-(\rho+\lambda)t}\left(\alpha_2 u_2 - \frac{1}{2}u_2^2 - \beta_2 \lambda t S\right)\right] + \\ + \psi\Big[\mu(u_1+u_2) - \delta S\Big], \quad (18)$$

where $\psi(t)$ is adjoint variable. From the first-order optimality condition

$$\frac{\partial H}{\partial u_1} = (\alpha_1 - u_1)e^{-\rho t} + \psi\mu = 0,$$

$$\frac{\partial H}{\partial u_2} = (\alpha_2 - u_2)e^{-(\rho+\lambda)t} + \psi\mu = 0,$$

we get the following formulas for optimal controls:

$$\overline{u_1} = \alpha_1 + e^{\rho t}\psi\mu,$$

$$\overline{u_2} = \alpha_2 + e^{(\rho+\lambda)t}\psi\mu.$$

Adjoint variable $\psi(t)$ can be found from differential equation $\dot{\psi(t)} = -\frac{\partial H}{\partial S}$, $\lim\limits_{t\to\infty}\psi(t) = 0$. Then

$$\dot{\psi(t)} = e^{-\rho t}\beta_1 + e^{-(\rho+\lambda)t}\beta_2\lambda t + \delta\psi.$$

Finally we get

$$\psi(t) = e^{\delta t}\left(\psi_0 + \frac{\beta_1}{\delta+\rho} + \frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2}\right) - \frac{\beta_1 e^{-\rho t}}{\delta+\rho} - \frac{\beta_2\lambda e^{-(\rho+\lambda)t}}{(\delta+\lambda+\rho)^2} - \frac{\beta_2\lambda t e^{-(\rho+\lambda)t}}{\delta+\lambda+\rho},$$

where ψ_0 is calculated as follows:

$$\psi_0 = -\frac{\beta_1}{\delta+\rho} - \frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2} - \frac{\beta_2\lambda t_0}{\delta+\lambda+\rho}.$$

Optimal controls have the following form:

$$\overline{u}_1(t) = \alpha_1 - \mu\left(\frac{\beta_1}{\delta+\rho} + \frac{\beta_2\lambda \mathrm{e}^{-\lambda t}}{(\delta+\lambda+\rho)^2} + \frac{\beta_2\lambda t \mathrm{e}^{-\lambda t}}{\delta+\lambda+\rho}\right),$$

$$\overline{u}_2(t) = \alpha_2 - \mu\left(\mathrm{e}^{\lambda t}\frac{\beta_1}{\delta+\rho} + \frac{\beta_2\lambda}{(\delta+\lambda+\rho)^2} + \frac{\beta_2\lambda t}{\delta+\lambda+\rho}\right).$$

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