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A Differential Game Model for the Extraction of Non Renewable Resources with Random Initial Times*

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Abstract In this work we propose a model for the extraction of a non renewable resource in an economy where, initially, only one agent is enabled to perform extraction tasks. However, at certain *non predictable* (random) times, more companies receive the government's approval for extracting the country's resources. We provide a set up suitable for the use of standard dynamic programming results; we develop the corresponding HJB equations, prove a verification theorem, and give an example. Our framework is inspired by the trends that oil industries are experiencing in countries like Mexico and Russia.

Keywords: Conditional distribution, random start, HJB equations.

1. Introduction

To motivate the problem statemen analysed in this paper we consider the situation on the Mexican oil market. Currently, there is the only company named Petróleos Mexicanos (trademarked as Pemex), the Mexican state-owned petroleum company. Recently, there has been a serious concern regarding the annual production drop that has been taking place since year since 2004. It has become clear that the obsolete infrastructure and inefficient management of this nationalized company seriously hinder the development of the oil and gas industry. Since the last decade, there has beed a serious discussion whether this sector should be open up to private investment. Finally, a decision has been taken according which the oil market will be open up by 2018. In our view, this situation can be well reflected by the differential game model proposed in this paper.

We note that a – to some extent – similar approach has been presented in (Kostyunin et al., 2012), where two firms compete over time and their two terminal times of extraction are two different random variables. The winning firm will be the only one remaining in the game after the first one retires.

2. The game model and main assumptions

We begin this work by considering a degenerate game (i.e., with only one player). At certain random instants, more players can join the system, thus conforming a multi player game. For the sake of simplicity, we start by assuming the existence of only two players.

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Let $x(t)$ represent the stock of the resource at time $t \geq 0$, and Π^1, Π^2 denote the sets of actions for players 1 and 2, respectively. We characterize the dynamics of the stock by the following autonomous ordinary differential equation.

$$\begin{cases} \dot{x}(t) = G(u_1(t), u_2(t)), \\ x(t_0) = x. \end{cases} \tag{1}$$

Assumption 1 *The state dynamics* (1) *satisfies the following conditions:*

(a) *The coefficient G is a differentiable function and, moreover, $\frac{\partial G}{\partial u_k} < 0$ for $k = 1, 2$.*
(b) *The state dynamics can be written as an additive (separable) function, for instance, $G(u_1, u_2) := -u_1 - u_2$.*

The first player is supposed to be "stable". This player starts the resource extraction at a fixed time T_0, which, without loss of generality is assumed to be $T_0 = 0$; and continues to extract the resource *ad infinitum*. The performance index of the first firm is given by

$$K_1(x, T_0, u_1, u_2) = \int_{T_0}^{\infty} h_1(x(t), u_1(t), u_2(t)) e^{-\rho(t-T_0)} dt, \tag{2}$$

here, ρ stands for the force of interest, and h_1 is the utility function of such player.

The second firm is "unstable", in the sense that it starts the extraction at a random time $T_1 \in [T_0, \infty[$. The cumulative distribution function $F(t)$ of this random instant is known, and $t \in [T_0, \infty[$.

The utility function for player 2 is as follows:

$$\tilde{h}_2(x(t), u_1(t), u_2(t)) = \begin{cases} 0, & t \in [T_0, T_1[\\ h_2(x(t), u_1(t), u_2(t)), & t \in [T_1, \infty[. \end{cases} \tag{3}$$

Hence, the objective functional of the second firm is:

$$\begin{aligned} K_2(x, T_0, u_1, u_2) &= \mathbb{E}\left[\int_{T_0}^{\infty} \tilde{h}_2(x(t), u_1(t), u_2(t)) e^{-\rho(t-T_0)} dt\right] \\ &= \mathbb{E}\left[\int_{T_0}^{T_1} 0 dt + \int_{T_1}^{\infty} h_2(x(t), u_1(t), u_2(t)) e^{-\rho(t-T_0)} dt\right] \\ &= \mathbb{E}\left[\int_{T_1}^{\infty} h_2(x(t), u_1(t), u_2(t)) e^{-\rho(t-T_0)} dt\right]. \end{aligned}$$

The following hypothesis enables us to simplify K_2.

Assumption 2 *The cumulative distribution function of T_1 satisfies the following:*

(a) *it is absolutely continuous with respect to Lebesgue's measure, and $F'(t) = f(t)$;*
(b) $\lim_{t\to\infty}(1 - F(t)) \int_{t_0}^{t} h(x(t), u_1(t), u_2(t)) dt = 0$ *for all $h : X \times U_1 \times U_2 \to \mathbb{R}$.*

Assumption 2(a) yields

$$
\begin{aligned}
K_2(x,T_0,u_1,u_2) &= \int_{T_0}^{\infty}\left[\int_{T_1}^{\infty} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dt\right]dF(T_1)\\
&= \int_{T_0}^{\infty}\left[\int_{T_1}^{\infty} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dt\right]f(T_1)dT_1\\
&= \int_{T_0}^{\infty}\int_{T_0}^{\infty} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dtf(T_1)dT_1\\
&\quad -\int_{T_0}^{\infty}\int_{T_0}^{T_1} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dtf(T_1)dT_1\\
&= \int_{T_0}^{\infty} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dt\int_{T_0}^{\infty} f(T_1)dT_1\\
&\quad -\int_{T_0}^{\infty}\int_{T_0}^{T_1} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dtf(T_1)dT_1\\
&= \int_{T_0}^{\infty} h_2(t,x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dt\\
&\quad -e^{\rho T_0}\int_{T_0}^{\infty}\left[\int_{T_0}^{T_1} h_2(x(t),u_1(t),u_2(t))e^{-\rho t}dt\right]f(T_1)dT_1. \qquad (4)
\end{aligned}
$$

To simplify the last integral in (4) we use Fubini-Tonelli's Theorem and Assumption 2(b) to invoke Proposition 2 in (Kostyunin and Shevkoplyas, 2011), and thus assert:

$$
\begin{aligned}
&\int_{T_0}^{\infty}\left[\int_{T_0}^{T_1} h_2(x(t),u_1(t),u_2(t))e^{-\rho t}dt\right]f(T_1)dT_1\\
&= \int_{T_0}^{\infty}(1-F(t))h_2(x(t),u_1(t),u_2(t))e^{-\rho t}dt. \qquad (5)
\end{aligned}
$$

The details can be read in Section 2.1 in (Gromov and Gromova, 2014).

Plugging (5) into (4) yields:

$$
\begin{aligned}
K_2(x,T_0,u_1,u_2) &= \int_{T_0}^{\infty} h_2(x(t),u_1(t),u_2(t))e^{-\rho(t-T_0)}dt\\
&\quad -e^{\rho T_0}\int_{T_0}^{\infty}(1-F(t))h_2(x(t),u_1(t),u_2(t))e^{-\rho t}dt\\
&= \int_{T_0}^{\infty} F(t)h_2(x(t),u_1(t),u_2(t))e^{-\rho t}dt.
\end{aligned}
$$

Admissible strategies

Now we introduce the type of equilibria we are interested in.

Definition 1. We say that a pair of stationary strategies $(u_1^*,u_2^*)\in \Pi^1\times\Pi^2$ is a Nash (or noncooperative) equilibrium if

$$K_1(x,T_0,u_1^*,u_2^*)\geq K_1(x,T_0,u_1,u_2^*) \quad \text{for every} \quad u^1\in\Pi^1,$$

and

$$K_2(x,T_0,u_1^*,u_2^*)\geq K_2(x,T_0,u_1^*,u_2) \quad \text{for every} \quad u^2\in\Pi^2.$$

Remark 3 *In general, the set of deterministic control actions for a differential game is such that, except for a quite restricted class of games (such as scalar linear –separable- games, see (Bardi, 2012) and the references therein), one cannot assure the existence of a Nash equilibrium in the set of ordinary strategies for the players. Since the game under consideration is typically of this class (because the system* (1) *is linear and scalar; and the utility functions referred to in* (2) *and* (3) *are scalar and separable), we will use the framework of deterministic (pure) strategies for our developments.*

3. Dynamic programming equations

We begin by considering the non-cooperative case. To find a Nash equilibrium that depends on both, the system and the time, a valid option is to use the dynamic programming technique. With this in mind, we will need to solve a couple of Hamilton-Jacobi-Bellman (HJB) equations.

Suppose that $W_1(x,t)$ is the optimal value function for the first player, and that $W_2(x,t)$ is the corresponding optimal performance index for the second player. Thus,

$$W_1(x,t) = \int_t^\infty h_1(x(\tau), u_1^*(\tau), u_2^*(\tau))e^{-\rho(\tau-T_0)}d\tau.$$

The Bellman equation for this player is given by

$$\rho W_1 = \frac{\partial W_1}{\partial t} + \max_{u_1}\left\{\frac{\partial W_1}{\partial x}G + h_1\right\}. \tag{1}$$

On the other hand, the optimal value function for the second player should be given by

$$W_2(x,t) = \frac{1}{F(t)}\int_t^\infty F(\tau)h_2(x(\tau), u_1^*(\tau), u_2^*(\tau))e^{-\rho(\tau-T_0)}d\tau,$$

here, we divide by $F(t)$ to reflect the fact that the game started before time t.

Let us define the function

$$\tilde{W}_2(x,t) := \int_t^\infty F(\tau)h_2(x(\tau), u_1^*(\tau), u_2^*(\tau))e^{-\rho(\tau-T_0)}d\tau.$$

Obviously, $\tilde{W}_2(x,t) = F(t)W_2(x,t)$. Then,

$$\frac{\partial \tilde{W}_2}{\partial t} = \frac{\partial W_2}{\partial t}F(t) + W_2 f(t); \tag{2}$$

$$\frac{\partial \tilde{W}_2}{\partial x} = \frac{\partial W_2}{\partial x}F(t). \tag{3}$$

The HJB equation for $\tilde{W}_2$ then is:

$$\rho\tilde{W}_2 = \frac{\partial \tilde{W}_2}{\partial t} + \max_{u_2}\left\{\frac{\partial \tilde{W}_2}{\partial x}G + h_2F(t)\right\}. \tag{4}$$

Finally, the substitution of (2)-(3) into (4) yields:

$$\rho W_2 - W_2 \frac{f(t)}{F(t)} = \frac{\partial W_2}{\partial t} + \max_{u_2} \left\{ \frac{\partial W_2}{\partial x} G + h_2 \right\}. \tag{5}$$

We now state and prove a verification result that ensures that (1) and (5) solve effectively the game we are concerned with.

Theorem 4. *Let $(u_1^*, u_2^*) \in \Pi^1 \times \Pi^2$ be defined as follows:*

$$u_1^*(t) := \arg\max_{u_1} \left\{ \frac{\partial W_1}{\partial x}(x,t) G(u_1, u_2) + h_1(x, u_1, u_2) \right\}, \tag{6}$$

$$u_2^*(t) := \arg\max_{u_2} \left\{ \frac{\partial W_2}{\partial x}(x,t) G(u_1, u_2) + h_2(x, u_1, u_2) \right\}. \tag{7}$$

If Assumptions 1 and 2 hold, and there exist differentiable functions W_1 and W_2 that meet (1) and (5), respectively, and $e^{-\rho r} W_1(x(r), r) \to 0$, $e^{-\rho r} W_2(x(r), r) \to 0$ as $r \uparrow \infty$, then

(i) The pair (u_1^*, u_2^*) is a Nash equilibrium (see Definition 1).
(ii) The functions W_1 and W_2 are the optimal values of the game for each player, i.e., $W_1(x,t) = K_1(x, t, u_1^*, u_2^*)$, and $W_2(x,t) = K_2(x, t, u_1^*, u_2^*)$.

Proof.

(i) The fact that $W_1(t,x) \geq K_1(t, x, u_1, u_2^*)$ for each $u_1 \in \Pi^1$ is quite standard (it can be consulted, for instance, in (Fleming and Soner, 2005 Theorem I.7.1)). We will include it here, however, for the sake of completeness. Consider any admissible pair of strategies $(u_1, u_2) \in \Pi^1 \times \Pi^2$. Using multivariate calculus and the dynamic programming equation (1) we obtain:

$$\begin{aligned} &W_1(x(r), r) e^{-\rho r} \\ &= W_1(x,t) e^{-\rho t} + \\ &\quad + \int_t^r e^{-\rho s} \left[\frac{\partial W_1}{\partial t}(x(s), s) - \rho W_1(x(s), s) + \frac{\partial W_1}{\partial x}(x(s), s) G(u_1(s), u_2(s)) \right] ds \\ &\leq W_1(x,t) e^{-\rho t} - \int_t^r e^{-\rho s} h_1(x(s), u_1(s), u_2(s)) ds \qquad \text{for } r \geq t. \end{aligned}$$

Now we let $r \uparrow \infty$ to obtain

$$W_1(t,x) \geq \int_t^\infty e^{-\rho(s-t)} h_1(x(s), u_1(s), u_2(s)) ds = K_1(x, t, u_1, u_2) \tag{8}$$

for every admissible pair $(u_1, u_2) \in \Pi^1 \times \Pi^2$. We can mimic this argument (with $F(t) W_2(x,t) e^{-\rho t}$ in lieu of $W_1(x,t) e^{-\rho t}$; and (5) instead of (1)) to get

$$W_2(t,x) \geq \frac{1}{F(t)} \int_t^\infty e^{-\rho(s-t)} F(s) h_2(x(s), u_1(s), u_2(s)) ds = K_2(x, t, u_1, u_2) \tag{9}$$

for every admissible pair $(u_1, u_2) \in \Pi^1 \times \Pi^2$.
Take u_2^* (as in (7)) in (8) and u_1^* (as in (6)) in (9) to see that the pair $(u_1^*, u_2^*) \in \Pi^1 \times \Pi^2$ is a Nash equilibrium.

(ii) To see that the functions W_1 and W_2 are the optimal values we are after, we let $(u_1^*, u_2^*) \in \Pi^1 \times \Pi^2$ be such that (6) and (7) hold. This yields $W_1(T_0, x) = K_1(x, T_0, u_1^*, u_2^*)$ and $W_2(T_0, x) = K_2(x, T_0, u_1^*, u_2^*)$. This concludes the proof.

□

References

Bardi, M. (2012). *Explicit solutions of some linear-quadratic mean field games.* Networks and heterogeneous media. American Institute of Mathematical Sciences, **7**, 1–19.

Fleming, W., Soner, H. M. (2005). *Controlled Markov Processes and Viscosity Solutions.* Springer. 2nd edition.

Gromov, D.; Gromova, E. V. (2014). *Differential games with random duration: a hybrid systems formulation*, Contributions to Game Theory and Management, **7**, 104–119.

Kostyunin, S.; Shevkoplyas, E. (2011). *On simplification of integral payoff in differential games with random duration*, Vestnik S. Petersburg Univ. Ser. 10. Prikl. Mat. Inform. Prots. Upr., **4**, 47–56.

Kostyunin, S.; Palestini, A.; Shevkoplyas, E. (2012). *A Differential Game-Based Approach to Extraction of Exhaustible Resource with Random Terminal Instants*, Contributions to Game Theory and Management, **5**, 147–155.