Contributions to Game Theory and Management, VIII, 21–32

Bidding Games with Several Risky Assets and Random Walks of Stock Market Prices *

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Abstract We consider multistage bidding models where several types of risky assets (shares) are traded between two agents that have different information on the liquidation prices of traded assets. These random prices depend on "a state of nature", that is determined by the initial chance move according to a probability distribution that is known to both players. Player 1 (insider) is informed on the state of nature, but Player 2 is not. The bids may take any integer values. The n-stage model is reduced to a zero-sum repeated game with lack of information on one side of Player 2. We show that, if liquidation prices of shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration. We give the solutions for corresponding infinite games. Analogously to the case of two risky assets (see Domansky and Kreps (2013)) the optimal strategy of Player 1 generates a random walk of transaction prices. The symmetry of this random walk is broken at the final stages of the game.

Keywords: multistage bidding, asymmetric information, price fluctuation, random walk, repeated game, optimal strategy.

1. Introduction

Regular random fluctuations in stock market prices are usually explained by effects from multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley (2002) proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from the asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in the immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of an oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for risky assets (shares). The liquidation price of a share depends on a random "state of nature". Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation price of a share once and for all. Player 1 is informed on the "state of nature", but Player 2 is not. Both players know the probability of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

* This study was partially supported by the Russian Foundation for Basic Research, project 13-01-00462-a

In this model Player 2 should use the history of Player 1's moves to update his beliefs about the state of nature. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider a model where a share's liquidation price takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann and Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have the values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As n tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

The same result was demonstrated in De Meyer (2010) for models with perfectly general trading mechanisms. The thesis of Gensbittel (2010) contains analogous results for a model with two risky assets and with arbitrary bids.

It is more natural to assume that players may assign only discrete bids proportional to a minimal currency unit. De Meyer and Marino (2005), Domansky and Kreps (2005), Domansky (2007) analyze a bidding model with the same mechanism of the game as in the model of De Meyer and Moussa-Saley (2002), and where market makers have to post prices within a discrete grid. The n-stage games $G_n^m(p)$ are considered with two possible values of liquidation price, 1 with probability p and 0 with probability $1-p$, and with admissible bids being multiples of $1/m$.

The results of De Meyer and Marino (2005), Domansky and Kreps (2005), Domansky (2007) show that, unlike the model of De Meyer and Saley, the sequence of values $V_n^m(p)$ of the games $G_n^m(p)$ is bounded from above and converges as n tends to ∞. The authors calculate its limit H^m, that is a continuous, concave, and piecewise linear function with m domains of linearity $[k/m,(k+1)/m]$, $k=0,\dots,m-1$, and the values at peak points $H^m(k/m)=k(m-k)/2m$.

The proof in Domansky and Kreps (2005) differs in essential ways from the proof in De Meyer and Marino (2005). The first proof is more concise due to exploiting a "reasonable" strategy of Player 2. In fact, this is his optimal strategy for the game with infinite number of steps.

As the sequence $V_n^m(p)$ is bounded from above, it is reasonable to consider the games $G_\infty^m(p)$ with infinite number of steps. We do this in Domansky (2007). The games $G_\infty^m(p)$ are infinitely repeated, non-discounted games with non-averaged payoffs that differs from the classical model of Aumann and Maschler (1995).

We believe that the model is consistent and tractable with an endogenous random time for information disclosure that happens when a posterior probability takes the value 0 or 1. But the model with infinite number of steps does not allow to determine an exogenous time for information disclosure that is a base for the notion of liquidation value in the works of De Meyer. At time T, each player should be able to sell his shares of the risky asset at this liquidation price.

The infinite game may be reinterpreted in the following way, that allows us to conserve the exogenous time of disclosure T. The sequential stages $t_n, n=1,2,\dots$ of the game occur on the interval $[0,T)$ having an accumulation point at the point T. This means that transactions become more and more frequent as the disclosure of information approaches. For example, one can take $t_n = T(1-\alpha^n)$ for some $\alpha \in (0,1)$.

Unlike the case of $n < \infty$, the existence of a value for the games $G^m_\infty(p)$ has to be proved. We prove it by constructing explicitly the optimal strategies. We show that the value V^m_∞ is equal to H^m, that is the limit of the sequence of values $V^m_n(p)$.

We construct the optimal strategy of Player 1 that provides him the maximal possible expected gain $1/2m$ per step (the fastest optimal strategy). For this strategy the posterior probabilities perform a simple symmetric random walk over the admissible bids l/m, $l = 0, \dots, m$, with absorbing extreme points 0 and 1. The absorption of posterior probabilities means revealing of the true value of share by Player 2. For the initial probability k/m, the expected duration of this random walk before absorption is $k(m-k)$. The bidding terminates almost surely in a finite number of steps, and the expected number of steps is also finite. This random time of absorption is a time for disclosure of information. The game terminates naturally when the posterior expectation of liquidation price coincide with its real value.

The set of all optimal strategies of Player 1 for $G^m_\infty(p)$ consists of the described fastest strategy obtained in Domansky (2007) and its slower modifications. In Sandomirskaia (2014b) it is shown that the constructed fastest optimal strategy of Player 1 for the infinitely repeated game $G^m_\infty(p)$ is an ε-optimal strategy of Player 1 for any finitely repeated game $G^m_n(p)$ of length n, where $\varepsilon = O(\cos^n \pi/m)$. This is not so for slower optimal strategies of Player 1.

The results of Domansky (2007) cannot be extended to a general transaction mechanism introduced by De Meyer (2010). As mentioned in the last paper, the discretized mechanism does not satisfy axioms of shift- and scale-invariance. Note that in practice a grid of possible bids is not shift- and scale-invariant simultaneously.

A more realistic model is studied in Sandomirskaia (2014a). It is analogous to the model considered in Domansky (2007), but equipped with a more general transaction mechanism. Namely, the agents fix different stakes for buying and selling a share.

In Domansky and Kreps (2009) we consider a model where the share liquidation price may take any integer values according to a probability distribution $\mathbf{p}$. Any integer bids are admissible. This n-stage model is reduced to a zero-sum repeated game $\bar{G}_n(p)$ with countable state and action spaces. The games considered in Domansky (2007) can be presented as particular cases of these games corresponding to probability distributions with two-point supports and with payoffs rescaling (the payoff for the game $G^m_n(p)$ is multplied by m).

We show that if the liquidation price of a share has a finite expectation, then the values of n-stage games exist. If its variance is finite, then, as n tends to ∞, the sequence of values is bounded from above and converges. The limit $\bar{H}$ is a continuous, concave, piecewise linear function with a countable number of domains of linearity. For distributions with integer mean values the function $\bar{H}$ is equal to the half of the liquidation price variance.

As the sequence of n-stage game values is bounded from above, it is reasonable to consider the games $\bar{G}_\infty(\mathbf{p})$ with an infinite number of steps. We show that the value $\bar{V}_\infty(\mathbf{p})$ is equal to $\bar{H}(\mathbf{p})$. We explicitly construct the optimal strategies for these games. To construct the optimal strategies of Player 1 we exploit symmetric representations of univariate probability distributions with given mean values as convex combinations of extreme points of corresponding sets, i.e. distributions with the same mean values and with supports containing at most two points.

The insider optimal strategy generates a symmetric random walk of posterior expectations over the one-dimensional integer lattice with absorption. For distributions with integer mean values the expected duration of this random walk is equal to the variance of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain 1/2 of informed Player 1.

In the paper Domansky and Kreps (2013) we consider multistage bidding models where two types of risky assets are traded. We show that, if expectations of share prices are finite, then the values $V_n(\mathbf{p})$ of n-stage bidding games $G_n(\mathbf{p})$ exist. The value of such a game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that the simultaneous bidding of two types of risky assets is at most so profitable for the insider as the separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if both share prices have finite variances, then the values of n-stage bidding games do not exceed the function $H(\mathbf{p})$ that is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

In the present paper we consider multistage bidding models where several types of risky assets (shares) are traded between two agents that have different information on the liquidation prices of traded assets. These random prices depend on "a state of nature", that is determined by the initial chance move according to a probability distribution that is known to both players. Player 1 (insider) is informed on the state of nature, but Player 2 is not. The bids may take any integer values. The n-stage model is reduced to a zero-sum repeated game with lack of information on one side of Player 2. We show that, if liquidation prices of shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration. We give the solutions for corresponding infinite games. Analogously to the case of two risky assets (see Domansky and Kreps (2013)) the optimal strategy of Player 1 generates a random walk of transaction prices. The symmetry of this random walk is broken at the final stages of the game.

In Russian the results are published in Domansky and Kreps (2014).

2. Repeated games modeling multistage bidding with several types of risky assets

We consider repeated games $G_n(\mathbf{p})$ with incomplete information on one side (see Aumann and Maschler (1995)) modeling the bidding with several types of risky assets.

Two players with opposite interests have money and shares of m types. The random liquidation price of a share of every type may take any integer values.

At stage 0 a chance move determines the "state of nature" and therefore the liquidation prices of shares $(z_1, z_2, \ldots, z_m)$ for the whole period of bidding according to the known to both Players probability distribution $\mathbf{p}$ over the m-dimensional integer lattice Z^m.

Player 1 is informed about the result of chance move, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \ldots, n$ both Players simultaneously propose their bids, meaning prices for one share of each type, $(i_1(t), \ldots, i_m(t)) \in Z^m$ for Player 1 and $(j_1(t), \ldots, j_m(t)) \in Z^m$ for Player 2. The bids are announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_r(t) > j_r(t)$, Player 1 gets one share of type r from Player 2 and Player 2 receives the sum of money $i_r(t)$ from Player 1. If $i_r(t) < j_r(t)$, Player 2 gets one share of type r from Player 1 and Player 1 receives the sum $j_r(t)$ from Player 2. If $i_r(t) = j_r(t)$, then no transaction of shares of type r occurs. Each player aims to maximize the value of his final portfolio (money plus the liquidation value of obtained shares).

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information for Player 2, with countable state space $S = Z^m$, and with countable action spaces $I = Z^m$, $J = Z^m$. The one-step gain $a(\mathbf{z}, \mathbf{i}, \mathbf{j})$ of Player 1 corresponding to the state $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ and the actions $\mathbf{i} = (i_1, i_2, \ldots, i_m)$, $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ is given with the sum $a(\mathbf{z}, \mathbf{i}, \mathbf{j} = \sum_{r=1}^{m} a_r(z_r, i_r, j_r)$, where

$$a_r(z_r, i_r, j_r) = \begin{cases} j_r - z_r, & \text{for } i_r < j_r; \\ 0, & \text{for } i_r = j_r; \\ -i_r + z_r, & \text{for } i_r > j_r. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a(\mathbf{z}, \mathbf{i}(t), \mathbf{j}(t)),$$

where $\mathbf{z}$ is the result of a chance move. This description is a common knowledge of both Players.

At the step t it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. A behavioral strategy σ for informed Player 1 depends on the result of a chance move. But a strategy τ for uninformed Player 2 does not depend. Formal description for randomized strategies of both players, for payoff functions $K_n(\mathbf{p}, \sigma, \tau)$ and for recursive structures of strategies and payoffs is given in Aumann and Maschler (1995) and in Domansky (2007).

Note that we consider non-discounted games with non-averaged payoffs that differs from the classical model of Aumann and Maschler (1995).

We also consider the infinite games $G_\infty(\mathbf{p})$. For certain pairs of strategies (σ, τ), the payoff function $K_\infty(\mathbf{p}, \sigma, \tau)$ may be indefinite. If we restrict the set of Player 1's admissible strategies to strategies with positive one-step gains against any action j of Player 2, then the payoff function of the game $G_\infty(\mathbf{p})$ becomes completely definite (may be infinite). Player 1 has many strategies, ensuring him a positive one-step gain against any action of Player 2. In fact, any reasonable strategy of Player 1 should possess this property.

For probability distributions $\mathbf{p}$ with finite supports, the games $G_n(\mathbf{p})$, being games with finite state and action spaces, have values $V_n(\mathbf{p})$. The functions V_n are continuous and concave in $\mathbf{p}$. Both players have optimal strategies $\sigma_n^*(\mathbf{p})$ and $\tau_n^*(\mathbf{p})$.

The value of such game does not exceed the sum

$$\sum_{r=1}^{m} V_n(\mathbf{p}^r)$$

of values of games modeling the bidding with one-type shares, where $\mathbf{p}^r$, $r = 1,\dots,m$, are the marginal distributions of the distribution $\mathbf{p}$. This follows from the fact that Player 2 can guarantee himself the loss that does not exceed this sum exploiting the direct combination of optimal strategies $\tau_n^*(\mathbf{p}^r)$ for the single asset games $G_n(\mathbf{p}^r)$ as a strategy for the game $G_n(\mathbf{p})$ with m risky assets.

Let $M^1(Z^m)$ be the set of probability distributions $\mathbf{p}$ over the m-dimension integer lattice Z^m with finite first moments $m^1[\mathbf{p}^r]$, $1 \le r \le m$. For $\mathbf{p} \in M^1(Z^m)$ the liquidation prices of all types of shares have finite expectations $\mathbf{E_p}[z_r] = m^1[\mathbf{p}^r]$.

The payoff of the game $G_n(\mathbf{p})$ with $\mathbf{p} \in M^1$ can be approximated using the payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support. The next theorem follows immediately from this fact.

Theorem 1. *If $\mathbf{p} \in M^1$, then the games $G_n(\mathbf{p})$ have values $V_n(\mathbf{p})$. The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps n increases.*

3. Upper bounds for values $V_n(\mathbf{p})$

Here we consider the set $M^2(Z^m)$ of probability distributions $\mathbf{p}$ over the m-dimension integer lattice Z^m with finite second moments $m^2[\mathbf{p}^r]$, $1 \le r \le m$. For $\mathbf{p} \in M^2(Z^m)$ the liquidation prices of all types of shares have finite variances $\mathbf{D_p}[z_r] = m^2[\mathbf{p}^r]$.

The main result of this section is that, for $\mathbf{p} \in M^2(Z^m)$ the sequence $V_n(\mathbf{p})$ of values remains bounded as $n \to \infty$. To prove this, we define the set of infinite strategies of Player 2, suitable for the games $G_n(\mathbf{p})$ with arbitrary n.

Let k_r be the integer part of expectation $\mathbf{E_p}[z_r]$ of liquidation price of a share of type r, $r = 1,\dots,m$. Define the set of Player 2' strategies $\tau^{(k_1,\dots,k_m)}$, $(k_1,\dots,k_m) \in Z^m$, by the following way.

The first move $\tau_1^{((k_1,\dots,k_m))}$ is the action $(k_1,\dots,k_m)$. For $t > 1$, the r-th component of the move $\tau_t^{(k_1,\dots,k_m)}$, $r = 1,\dots,m$, depends on the last observed pair of r-th components of actions (i_{t-1}^r, j_{t-1}^r) for both players:

$$j_t^r = \begin{cases} j_{t-1}^r - 1, & \text{if } i_{t-1}^r < j_{t-1}^r\ ; \\ j_{t-1}^r, & \text{if } i_{t-1}^r = j_{t-1}^r; \\ j_{t-1}^r + 1, & \text{if } i_{t-1}^r > j_{t-1}^r. \end{cases}$$

Thus, for each asset r, $r = 1,\dots,m$, strategy $\tau^{(k_1,\dots,k_m)}$ independently reproduces the optimal strategy of Player 2 τ^{k_r} for single type bidding game of infinite duration (see Domansky and Kreps (2009)).

Proposition 1. *If all share prices have finite variances $D_{\mathbf{p}}[z_r] < \infty$, $r = 1,\dots,m$, then for the game $G_n(\mathbf{p})$ strategy $\tau^{(k_1,\dots,k_m)}$ ensures the Player 1' gain, not exceeding $H(\mathbf{p})$ that is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of all share prices*

$$H(\mathbf{p}) = \frac{1}{2}\sum_{1}^{m} D_{\mathbf{p}}[z_r],$$

for

$$\Theta(k_1,\ldots,k_m) = \{\mathbf{p} : E_{\mathbf{p}}[z_r] = k_r, r = 1,\ldots,m\}.$$

The proof is analogous to the case $m = 2$ (see Domansky and Kreps (2013)).

Thus if all share prices have finite variances, then the values $Val_n(\mathbf{p})$ of n-stage bidding games do not exceed the function $H(\mathbf{p})$. This makes it reasonable to consider games $G_\infty(\mathbf{p})$ of infinite duration.

Below we formulate the main result.

Theorem 2. *If all share prices have finite variances $D_{\mathbf{p}}[z_r] < \infty$, $r = 1,\ldots,m$, then*
a) the game $G_\infty(\mathbf{p})$ has the value

$$Val_\infty(\mathbf{p}) = H(\mathbf{p});$$

b) the strategy of Player 2 $\tau^{(k_1,\ldots,k_m)}$, where k_r is the integer part of price expectation of r-th asset, $r = 1,\ldots,m$, is Player 2' optimal strategy;
c) there exists Player 1' strategy $\sigma^*(\mathbf{p})$ that ensures him the gain $H(\mathbf{p})$.

4. Insider's optimal strategies for elementary games.

In Domansky and Kreps (2009) it is shown that the Player 2' strategy τ^{k_r} is his optimal strategy for the infinite bidding game with one risky asset of type r. For distribution with integer expectation the strategy ensures him the loss equal to one half of the variance of random price of asset of type r.

As the Player 2' strategy $\tau^*(\mathbf{p}) = \tau^{(k_1,\ldots,k_m)}$ is the independent combination of strategies τ^{k_r}, $r = 1,\ldots,m$, we get that for distribution with integer expectations the strategy $\tau^*(\mathbf{p})$ ensures him the loss equal to one half of the sum of variances of random prices of assets of all types.

To prove theorem 2 for arbitrary distribution $\mathbf{p}$ with integer variances we should construct a Player 1' strategy $\sigma^*(\mathbf{p})$ that guarantees him the gain $H(\mathbf{p})$.

It is sufficient to show that for distribution with integer expectations the strategy $\sigma^*(\mathbf{p})$ guarantees him the payoff equal to one half of the sum of variances of random prices of assets of all types.

It follows that the strategies $\sigma^*(\mathbf{p})$ and $\tau^*(\mathbf{p})$ are optimal strategies for Player 1 and Player 2.

For $m > 2$, analogously to the two-dimensional case ($m = 2$), we begin with constructing Player 1's optimal strategies for "elementary" games that is games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p}$ having $k+1$-point supports $\mathbf{z}^1,\ldots,\mathbf{z}^{k+1} \in R^m$, where $k \leq m$. The support points belong to a k-dimensional hyperplane $Hyp(\mathbf{z}^1,\ldots,\mathbf{z}^{k+1})$. Such a hyperplane is determined by the set of $m-k$ unit vectors $\{\mathbf{e}^1,\ldots,\mathbf{e}^{m-k}\}$, that are pairwise orthogonal and orthogonal to the hyperplane.

At this hyperplane the points $\mathbf{z}^1,\ldots,\mathbf{z}^{k+1}$ determine a simplex $\triangle(\mathbf{z}^1,\ldots,\mathbf{z}^{k+1})$. Its points $\mathbf{w} \in \triangle(\mathbf{z}^1,\ldots,\mathbf{z}^{k+1})$ correspond to distributions over $(\mathbf{z}^1,\ldots,\mathbf{z}^{k+1})$, namely, vector $\mathbf{w}$ gives expectations of random prices of assets

$$\mathbf{w} = \sum_{u=1}^{k+1} P(\mathbf{z}^u|\mathbf{w}) \cdot \mathbf{z}^u,$$

where $P(\mathbf{z}^u|\mathbf{w})$ are corresponding probabilities of the points of the simplex.

On the other hand for point $\mathbf{w} \in \triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1})$ the probability distribution $P(\cdot|\mathbf{w})$ is given by the following equalities

$$P(\mathbf{z}^u|\mathbf{w}) = \frac{\det[\mathbf{z}^1, \ldots, \mathbf{z}^{u-1}, \mathbf{z}^{u+1}, \ldots, \mathbf{z}^{k+1}, e^1, \ldots, e^{m-k}]}{\sum_{v=1}^{k+1} \det[\mathbf{z}^1, \ldots, \mathbf{z}^{v-1}, \mathbf{z}^{v+1}, \ldots, \mathbf{z}^{k+1}, \mathbf{e}^1, \ldots, \mathbf{e}^{m-k}]},$$

where $\det[\cdot]$ is determinant of matrix $[\cdot]$.

For any k-dimensional hyperplane Hyp^k of R^m its points with at least k integer coordinates form a discrete lattice Lat^k. Define a lattice $Lat(\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1}))$ over the simplex $\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1})$ by the following way. For the interior of the simplex $\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1})$ it is the lattice Lat^k. For its boundary, that is for any simplex of dimension less than k it is an analogous lattice of corresponding dimension.

Proposition 2. *Let* $\mathbf{w} = (w_1, \ldots, w_m)$ *be an interior point of the lattice* $Lat(\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1}))$. *For the initial expectation vector* $\mathbf{w}$ *Player 1 has optimal strategy* $\sigma^*(\mathbf{w})$ *generating a random walk of posterior probabilities over the points of the lattice* $Lat(\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1}))$.

Proof. By definition the point $\mathbf{w}$ has at least k integer coordinates. Without loss of generality we assume that its first k coordinates $w_1, \ldots, w_k$ are integer.

To proof the proposition we use the following lemma.

Lemma 1. *The point* $\mathbf{w} = (w_1, \ldots, w_m)$ *is corresponded by* $(k+1)$*-tuple of points* $\{\mathbf{w}^1, \ldots, \mathbf{w}^l\}$, $l \leq k+1$ *of the lattice* $Lat(\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{k+1}))$ *with the following properties:*

1. The point $\mathbf{w}$ *is the convex hull of points* $\{\mathbf{w}^1, \ldots, \mathbf{w}^l\}$, $l \leq k+1$*:*

$$\mathbf{w} = \sum_{i=1}^{l} q_i \mathbf{w}^i,$$

where coefficients $q_i > 0$ *and* $\sum_{i=1}^{l} q_i = 1$.

2. If a coordinate w_j *of the vector* $\mathbf{w}$ *is integer (in particular, if* $j < k+1$*), then for all* $1 \leq i \leq k+1$ *the coordinate* w_j^i *of vector* $\mathbf{w}^i$ *belongs to the interval* $[w_j - 1, w_j + 1]$. *If a coordinate* w_j *of the vector* $\mathbf{w}$ *is not integer, that is* $w_j = k_j + \alpha_j$, *where* k_j *is the integer part of* w_j *and* $\alpha_j > 0$, *then* w_j^i *belongs to the interval* $[k_j, k_j + 1]$.

3. For integer coordinate w_j *of vector* $\mathbf{w}$ *put* $\sigma_j^i = w_j$ *if* $w_j^i > w_j$, *put* $\sigma_j^i = w_j - 1$ *if* $w_j^i < w_j$ *and put* $\sigma_j^i = w_j$ *or* $\sigma_j^i = w_j - 1$ *if* $w_j^i = w_j$. *For non-integer coordinate* $w_j = k_j + \alpha_j$ *put* $\sigma_j^i = k_j$.

It is possible to determine vectors σ^i *such that* $\sigma^i \neq \sigma^{i'}$ *if* $i \neq i'$.

For vector $\mathbf{w}$ of posterior share price expectations

$$\mathbf{w} = \sum_{u=1}^{k+1} P(\mathbf{z}^u|\mathbf{w}) \cdot \mathbf{z}^u,$$

where $P(\mathbf{z}^u|\mathbf{w})$ are corresponding probabilities of the points of the simplex. For the first step Player 1 chooses the actions σ^i, $1 \leq i \leq l$ with total probabilities q_i such that action σ^i generates posterior expectations $(\cdot|\sigma^i) = \mathbf{w}^i$,

$$\mathbf{w}^i = \sum_{u=1}^{k+1} P(\mathbf{z}^u|\mathbf{w}^i) \cdot \mathbf{z}^u,$$

where $P(\mathbf{z}^u|w^i)$ are corresponding probabilities of the points of the simplex.

Thus for the state $\mathbf{z}^u$ Player 1 chooses his action σ^i with probability

$$P(\sigma^i|\mathbf{z}^u) = \frac{P(\sigma^i \cap \mathbf{z}^u)}{P(\mathbf{z}^u|\mathbf{w})} = \frac{P(\mathbf{z}^u|\mathbf{w}^i)q_i}{P(\mathbf{z}^u|\mathbf{w})}.$$

Analogously to the two-dimensional case ($m = 2$) $m = 2$ (see Domansky and Kreps (2013)), we get that the martingale of posterior expectations generated by optimal strategy of Player 1 $\sigma^*(\mathbf{w})$ for game with $(m+1)$-point distribution represents symmetric random walks over points of integer lattice lying within the simplex spanned across the support points of distribution. The symmetry is broken at the moment when the walk hits the simplex boundary. From this moment the game turns into one of games with distributions having not more m-point supports.

5. Symmetric representation of probability distributions p over Z^m

In this section we construct symmetric representation of probability distributions $\mathbf{p}$ over Z^m with given mean values as probability mixtures of distributions with the same mean values and with supports containing at most $m+1$ points ("elementary" distributions).

This representation allows us for general distribution to construct Player's optimal strategy $\sigma^*(\mathbf{p})$ for infinite game $G_\infty(\mathbf{p})$ as convex combinations of his optimal strategies for "elementary" games.

Here we use an other approach for constructing representation of multivariate probability distributions than the approach elaborated for the case $m = 2$ in Domansky (2013).

Without loss of generality we assume that $\mathbf{p} \in \Theta^m(0)$, where $\Theta^m(0)$ is the set of probability distributions $\mathbf{p}$ over Z^m with zero mean values (centered probability distributions). Denote

$$\Delta^m(0) = \{(\mathbf{z}^1, \ldots, \mathbf{z}^{m+1}) : 0 \in \triangle(\mathbf{z}^1, \ldots, \mathbf{z}^{m+1})\},$$

where $\triangle(\mathbf{z}^1, \ldots, \mathbf{z}^k)$ is the convex hull of points $\mathbf{z}^1, \ldots, \mathbf{z}^k$ belonging to Z^m.

The centered distribution $\mathbf{p}^0_{\mathbf{z}^1,\ldots,\mathbf{z}^{m+1}}$ with support $(\mathbf{z}^1, \ldots, \mathbf{z}^{m+1}) \in \Delta^m(0)$ is given by the formula

$$\mathbf{p}^0_{\mathbf{z}^1,\ldots,\mathbf{z}^{m+1}} = \frac{\sum_{i=1}^{m+1} \det[\mathbf{z}^{i+1}, \ldots, \mathbf{z}^{i+m}] \cdot \delta^{\mathbf{z}^i}}{\sum_{j=1}^{m+1} \det[\mathbf{z}^{j+1}, \ldots, \mathbf{z}^{j+m}]},$$

where $\delta^{\mathbf{x}}$ is the degenerate distribution with the point x in its support and $\det[\mathbf{z}^{i+1}, \ldots, \mathbf{z}^{i+m}]$ is determinant of square coordinate matrix. All arithmetical operations with subscripts are fulfilled in modulo $m+1$.

Note that if $\det[\mathbf{z}^{i+1}, \ldots, \mathbf{z}^{i+m}] = 0$, then $0 \in \triangle \det[\mathbf{z}^{i+1}, \ldots, \mathbf{z}^{i+m}]$.

We say $\mathbf{p} \in \Theta^m(0)$ *does not contain (free of) k-point distributions*, if there is no k-point set $(\mathbf{z}^1, \ldots, \mathbf{z}^k)$ with $\mathbf{p}(\mathbf{z}^i) > 0$ and $0 \in \triangle(\mathbf{z}^1, \ldots, \mathbf{z}^k)$.

Theorem 3. *Let distribution* $\mathbf{p} \in \Theta^m(0)$ *does not contain k-point distributions with* $k < m+1$. *Then*

$$\mathbf{p} = \sum_{\Delta^m(0)} \frac{V(\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^{m+1})\mathbf{p}(\mathbf{z}^1)\mathbf{p}(\mathbf{z}^2)\ldots\mathbf{p}(\mathbf{z}^{m+1})}{\sum_{\Delta^m(0)} V(\mathbf{t}^1, \mathbf{t}^2, \ldots, \mathbf{t}^{m+1})\mathbf{p}(\mathbf{t}^1)\mathbf{p}(\mathbf{t}^2)\ldots\mathbf{p}(\mathbf{t}^{m+1})} \mathbf{p}^0_{\mathbf{z}^1,\mathbf{z}^2,\ldots,\mathbf{z}^{m+1}}, \quad (1)$$

where

$$V(\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^{m+1}) = \frac{\sum_{j=1}^{m+1} \det[\mathbf{z}^{j+1}, \ldots, \mathbf{z}^{j+m}]}{m!} \tag{1}$$

is m-dimensional volume of corresponding simplex.

Proof. Clarify the structure of denominator in formula 1.

With each $\mathbf{z} \in Z^m$ we associate the set of unordered collections $(\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m)$:

$$\Delta^m(0, \mathbf{z}) = \{(\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m), \mathbf{z}^i \neq (0) : (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m, \mathbf{z}) \in \Delta^m(0)\}.$$

We accept that points $\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m$ are indexed so that

$$\det[\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m] > 0.$$

Put

$$\Phi(\mathbf{p}, \mathbf{z}) = \sum_{\Delta^m(0,\mathbf{z})} \det[\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^m]\mathbf{p}(\mathbf{z}^1)\mathbf{p}(\mathbf{z}^2)\ldots\mathbf{p}(\mathbf{z}^{m+1}) =$$

$$= \frac{1}{m!} \sum_{\mathbf{z} \in \mathrm{Supp}\ \mathbf{p}} \Phi(\mathbf{p}, \mathbf{z})\mathbf{p}(\mathbf{z}).$$

The next theorem provides a base for constructing symmetric representation of centered probability distributions $\mathbf{p}$ over Z^m.

Theorem 4. *Let distribution $\mathbf{p} \in \Theta^m(0)$ does not contain k-point distributions with $k < m+1$. Then $\Phi(\mathbf{p}, \mathbf{z})$ does not depend on z, i.e. $\Phi(\mathbf{p})$ is an invariant of distribution $\mathbf{p}$.*

Remark 1. This result is m-dimensional analog of the equality

$$\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t) = \sum_{t=1}^{\infty} t \cdot \mathbf{p}(-t),$$

holding for centered probability distributions $\mathbf{p}$ over Z^1.

Corollary 1. *For any centered probability distributions $\mathbf{p}$ over Z^m not containing k-point distributions with $k < m+1$, $\Phi(\mathbf{p})$ has the following invariant representation:*

$$\Phi(\mathbf{p}) = \sum_{\Delta^m(0)} \sum_{j=0}^{m} \det[\mathbf{z}^{j+1}, \ldots, \mathbf{z}^{j+m}]\mathbf{p}(\mathbf{z}^1)\ldots\mathbf{p}(\mathbf{z}^{m+1}) =$$

$$m! \sum_{\Delta^m(0)} V(\mathbf{z}^1, \ldots, \mathbf{z}^{m+1})\mathbf{p}(\mathbf{z}^1)\ldots\mathbf{p}(\mathbf{z}^{m+1}).$$

For constructing decomposition of arbitrary centered probability distribution $\mathbf{p}$ over Z^m we elaborate a procedure that allows us step by step to eliminate distributions with supports containing less than $m+1$ points.

We start with deleting the atom at the point 0. Denote the rest distribution by $\mathbf{p}^1$. There exists not more than countable number of one-dimensional subspaces R_l^1 with a rest positive measure (we enumerate these subspaces by an index l). Of each obtained subspace we delete a centered probability distribution $\mathbf{p}_l^1$ such that the rest distribution is concentrated on a half-line. We denote $\mathbf{p}^2$ the final rest distribution that is a centered distribution over Z^m. It does not contain any two-point centered distribution.

There exists not more than countable number of two-dimensional subspaces R_l^1 with positive measure $\mathbf{p}^2$ concentrated on more than a half-line. Of each such subspace we delete a centered probability distribution $\mathbf{p}_l^2$ such that the rest distribution is concentrated on a half-plane. We denote $\mathbf{p}^3$ the final rest distribution that is a centered distribution over Z^m. It does not contain any three-point centered distribution. And so on ...

Theorem 5. *Any centered distribution* $\mathbf{p}$ *over* Z^m *may be represented as a probability mixture*

$$\mathbf{p} = \mathbf{p}(0)\delta^0 + \sum_{k=1}^{m-1}\sum_{l=1}^{\infty} \alpha_l^k \mathbf{p}_l^k + \alpha^m \mathbf{p}^m, \tag{2}$$

where $\mathbf{p}_l^k$ $(0 < k < m)$ *is centered probability distributions with integer supports over* k*-dimensional subspaces* R_l^k *of* R^m*, not containing* r*-point distributions with* $r < k+1$*. The last distribution* $\mathbf{p}^m$ *is a centered probability distribution over* Z^m *and it contains no* k*-point distributions with* $k < m+1$*.*

Remark 2. Any distribution $\mathbf{p}_l^k$ may be represented as a convex combination of $(k+1)$-point distributions. Distribution $\mathbf{p}^m$ may be represented as a convex combination of $(m+1)$-point distributions.

6. Constructing Player 1' optimal strategy for arbitrary distribution p

In this section we terminate the proof of theorem 2. Namely with help of the obtained in section 5 decomposition of distribution $\mathbf{p}$ we construct Player 1' optimal strategy $\sigma^*(\mathbf{p})$, that ensures him the gain $\sum_{r=1}^{m} D_p[z_r]/2$ for the game $G_\infty(\mathbf{p})$. It proves the paragraph c) of theorem 2.

We construct the strategy $\sigma^*(\mathbf{p})$ by the following algorithm. As before without loss of generality we consider centered probability distributions $\mathbf{p}$.

a) If the state chosen by chance move coincides with the zero vector of expected prices, then Player 1 stops the game. In this case he cannot get any profit from his informational advantage.

b) Let the state chosen by chance move $\mathbf{z} \neq 0$. Then Player 1 makes a choice among distributions $\mathbf{p}_l^k$ and $\mathbf{p}^m$ by means of a lottery with the probabilities

$$\frac{\alpha_l^k \mathbf{p}_l^k(\mathbf{z})}{\mathbf{p}(\mathbf{z})}, \quad \frac{\alpha^m \mathbf{p}^m(\mathbf{z})}{\mathbf{p}(\mathbf{z})}.$$

Here $\mathbf{p}_l^k$ and $\mathbf{p}_l^k$ are elementary distributions of decomposition 2 and α_l^k, α^m are corresponding coefficients.

c) If distribution $\mathbf{p}_l^k$ is chosen, then Player 1 chooses k points $\mathbf{z}^1, \ldots, \mathbf{z}^k$ by means of a lottery with the probabilities

$$\frac{\det[\mathbf{z}^1, \ldots, \mathbf{z}^k] \cdot \mathbf{p}_l^k(z_1) \ldots \mathbf{p}_l^k(z_k)}{\Phi(\mathbf{p}_l^k)}$$

and he plays his optimal strategy $\sigma^*(\cdot|z)$ for in $(k+1)$-point game $G(\mathbf{p}^0_{\mathbf{z},\mathbf{z}^1,\ldots,\mathbf{z}^k})$ the state $\mathbf{z}$.

d) If distribution $\mathbf{p}^m$ is chosen, $\mathbf{p}^m$, then Player 1 chooses m points $\mathbf{z}^1, \ldots, \mathbf{z}^m$ by means of a lottery with the probabilities

$$\frac{\det[\mathbf{z}^1, \ldots, \mathbf{z}^m] \cdot \mathbf{p}^m(\mathbf{z}^1) \ldots \mathbf{p}^m(\mathbf{z}^m)}{\Phi(\mathbf{p}^m)}$$

and he plays his optimal strategy $\sigma^*(\cdot|\mathbf{z})$ in $(m+1)$-point game $G(\mathbf{p}^0_{\mathbf{z},\mathbf{z}^1,\ldots,\mathbf{z}^m})$ the state $\mathbf{z}$.

It is sufficient to prove paragraph c) of theorem 2 for distributions $\mathbf{p} \in \Theta(k_1, \ldots, k_m)$.

In Section 4 for k-point game with $\mathbf{p} \in \Theta(0)$ we demonstrated that the strategy σ^* ensures for Player 1 the payoff equal to one half of the sum of variance components $\sum_{r=1}^m D_p[z_r]/2$.

Note that the sum of variance components is a linear function over $\Theta(0) \cap M^2$, where M^2 is the set of distributions with finite second moments. Thus with help the decomposition of section 5 we get that the depicted above compound strategy of Player 1 ensures him the gain $\sum_{r=1}^m D_p[z_r]/2$ in game $G_\infty(\mathbf{p})$ for any distribution $\mathbf{p} \in \Theta(0) \cap M^2$.

It terminates the proof of theorem 2.

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