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Minimax Estimation of Value-at-Risk under Hedging of an American Contingent Claim in a Discrete Financial Market*

Alexey I. Soloviev

Lomonosov Moscow State University,
Faculty of Computational Mathematics and Cybernetics,
Department of Operations Research,
Leninskie Gory, Moscow, 119991, Russia
E-mail: alex.solo.88@mail.ru

Abstract The game problems between seller and buyer of an American contingent claim relate to large scale problems because a number of buyer's strategies grows overexponentially. Therefore, decomposition of such games turns out to be a fundamental problem. In this paper we prove the existence of a minimax monotonous (in time) strategy of the seller in a loss minimization problem considering value-at-risk measure of loss. The given result allows to substantially decrease a number of constraints in the original problem and lets us turn to an equivalent mixed integer problem with admissible dimension.

Keywords: decision making under uncertainty, value-at-risk, scenario tree, stopping time, hedging.

1. Introduction

A seminal series of papers (Merton, 1973; Black and Scholes, 1973; Shiryaev, 1999) initiated an extensive number of studies on financial asset pricing and minimization of risk associated with failure of contingent claim hedging (building a portfolio of assets to exceed the claim value). The authors assumed that trades occur continuously in time. Consideration of discrete models of a financial market for solving investment problems allowed to apply new methods, particularly ones of mathematical programming and game theory. This is due to the fact that the number of market scenarios is finite.

The first discrete models of contingent claims valuation were examined in (Harrison and Kreps, 1979). This paper proposed a new concept of a discrete market applying stochastic programming approach. The novel idea was to describe a financial market with a scenario tree. They formulated the notions of arbitrage (market condition which permits investment strategies with a guaranteed profit), a self-financing strategy, hedging (implementing the contingent claim) basing on scenario tree framework. The fundamental theorem of asset pricing was presented as well. The problem of maximizing the expected value of terminal portfolio was formulated in (Pliska, 1997). The author derived analytically the amount of initial capital needed for perfect hedging of various contingent claims. In paper (King, 2002) the existence of arbitrage opportunities was analyzed using the duality theory. He stated the linear and nonlinear programming problems to determine optimal buyer

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and seller's strategies. In addition, the author proved a criterion for the existence of an optimal solution in the utility maximization problem.

SPAN system (Standard Portfolio Analysis of Risk) is a good example which illustrates practical use of the discrete models of a financial market. It was introduced at the Chicago Mercantile Exchange in 1988 (Chicago Mercantile Exchange, 1999). This is the portfolio risk assessment methodology which determines the minimum margin requirements to cover losses for one trading period. 16 market scenarios are simulated in the system representing possible ranges of percentage changes in price and volatility of the underlying asset.

The main feature of an American contingent claim is an uncertain moment of exercise (using the right to oblige a seller to make a transaction). So, American claims may be exercised by its buyer at any time $t = \{0, ..., T\}$ up to expiration date. Exercise time is usually considered as an uncertain factor in investment problems. As a result a zero-sum game between the seller and the buyer arises in this scope. Perfect hedging (with probability one) of an American contingent claim generally requires considerable initial endowment from the seller.

There are several common ways to assess the risk of imperfect hedging. The authors of (Föllmer and Leukert, 1999) suggested to use strategies of two types. The first one is quantile hedging. It allows to hedge the contingent claim with the highest probability. This approach does not take the investor's attitude towards the risk into account in contrast to the second type of strategies that minimize a linear function of losses associated with imperfect hedging. The authors proved existence of the optimal solutions for a continuous model of the market using Neyman-Pearson lemma. Perez-Hernandez formulated optimization problems of the described two types of imperfect hedging for financial markets with discrete time and an infinite number of states (Perez-Hernandez, 2007). In this paper he also stated new problems and proved the existence of their optimal solutions under minimizing the initial portfolio endowment and the fixed losses. The paper (Novikov, 1999) considers the analogous problem of minimizing the initial endowment. However, the constraints are more complicated to deal with. The probability of full hedging is bounded from below. There are two tradable assets: risky and riskless ones. It was assumed that the contingent claim can not be exercised until the specific time moment which is optimal for the buyer. Then, the optimal hedging strategies were found in (Lindberg, 2012) for a slightly more general model of the market but a set of exercise times was restricted in a foregoing way. The problem of imperfect hedging from the buyer's perspective was proposed in (Pinar, 2011). An alternative description of the decision making process connected with exercising the claim allowed to formulate the mixed-integer problem which is equivalent to the original one. The paper (Camci and Pinar, 2009) stated a theorem which leads to even more reduction and equivalently turns to finding the optimal solution to the relaxed problem. Pinar (2011) also provides numerical results using real data.

In a present paper we propose value-at-risk (VaR) as a risk measure to estimate the losses from imperfect hedging. It is equal to the minimum value such that the expected losses do not exceed it with a specified probability. In other words, VaR corresponds to the amount of uninsured risk which the seller can take, see first (Rockafellar and Uryasev, 2000) for the details. Nowadays VaR method meets the standards of banking regulation approved by the Basel Committee on Banking

Supervision. This measure is recommended primarily for monitoring market risks and effectiveness of hedging strategies.

VaR approach of risk estimation was widely studied in (Rockafellar and Uryasev, 2000). The authors proved that minimization of VaR, CVaR (conditional value-at-risk that is roughly interpreted as expected losses which exceed VaR value) and Markowitz problem have the same optimum under some conditions. The analytic formula of a CVaR value was obtained in (Rockafellar and Uryasev, 2002) for a discrete model of a financial market. The distribution of future losses was assumed to be known. The paper (Sarykalin et al., 2008) provided detailed comparison of VaR and CVaR. In short, advantages of VaR measure include the fact that it is not subject to errors in the measurement of the biggest losses, assessment of which is rather difficult. The disadvantage of VaR is its non-convexity (in contrast to CVaR) which complicates problem solving in practice.

The rest of the paper is organized as follows. We describe the discrete model of securities market and define the basic notions of subject area in Sect. 2. Sect. 3 formally defines a zero-sum game (strategies of players and a loss function) and introduces a problem of VaR minimization consisted in finding of minimax for the game. We state and prove the main result in Sect. 4. Then, we apply it showing how to substantially reduce a number of constraints in the original problem.

2. The Model of a Financial Market

The market consists of $d+1$ tradable securities, whose prices are denoted at each state n by a non-negative vector $S_n = (S_n^0, ..., S_n^d)$. We assume the security indexed by 0 to be riskless (a bank deposit or a bond), it has strictly positive prices at each state. We choose this asset to be the numeraire and introduce the discounts $1/S_n^0$. Let a vector $X_n = S_n/S_n^0$ denote the discounted security prices relative to the numeraire. Its zero entry X_n^0 equals 1 in any state n.

The set of states $\mathcal{N}$ of the market has a tree structure; see examples of it in Fig. 1 and in (Harrison and Kreps, 1979, p. 393; Pliska, 1997, p. 79). It is divided into pairwise disjoint subsets of states $\mathcal{N}_t$ which may occur at specific time moments $t = 0, ..., T$. The set $\mathcal{N}_0$ contains the only element – a root of the tree denoted by 0. Every node $n \in \mathcal{N}_t$, where $t = 1, ..., T$, has a unique parent $a(n) \in \mathcal{N}_{t-1}$. We put $a(0) = 0$, $a^0(n) = n$, $a^{s+1}(n) = a^s(a(n))$, $s = 1, ..., t$, for all $n \in \mathcal{N}_t$, $t = 1, ..., T$. Next, each node $n \in \mathcal{N}_t$, where $t = 0, ..., T-1$, has a set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. Let $\mathcal{D}(n)$ be a set of all the nodes which may occur after n, i.e. child nodes, their children and so on ($\mathcal{D}(0) = \mathcal{N}\backslash\{0\}$, $\mathcal{D}(n) = \mathcal{C}(n)$ for all $n \in \mathcal{N}_{T-1}$).

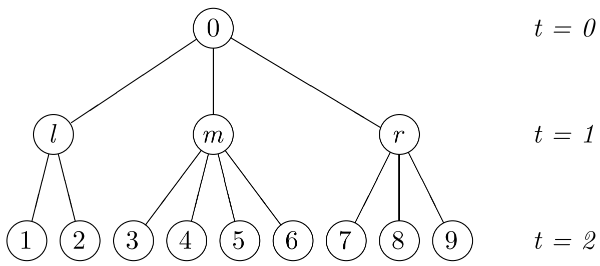

Fig. 1. A scenario tree ($T = 2$, $\mathcal{N}_1 = \mathcal{C}(0) = \{l, m, r\}$, $\mathcal{C}(l) = \{1, 2\} \subset \mathcal{N}_T = \{1, 2, ..., 9\}$).

A unique path $\omega = (n_0, n_1, ..., n_T)$ leads from the root to a leaf node $n \in \mathcal{N}_T$, where $n_0 = 0$, $n_{t-1} = a(n_t) \in \mathcal{N}_{t-1}$ for all $t = 1, ..., T$, $n_T = n$. These paths are interpreted as scenarios of market movement. They form atoms of probability space Ω. The set $\mathcal{N}_t$ partitions Ω into subsets (events). Each of them is defined by a node $n \in \mathcal{N}_t$ and consists of all the paths containing n. The partition generates an algebra $\mathcal{F}_t$ (algebra of events observed up to time moment t). Here, $\mathcal{F}_0 = \{\emptyset, \Omega\} \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_T$. The family of sets $\{\mathcal{F}_t\}$ is a filtration. Throughout the paper we will consider $\{\mathcal{F}_t\}$-adapted stochastic processes $b = \{b(t)\}$, where a random variable $b(t)$ takes values b_n, $n \in \mathcal{N}_t$, and, thus, is $\mathcal{F}_t$-measurable.

The probability measure $p = (p_n,\ n \in \mathcal{N})$ defined on Ω attaches values $p_n > 0$, $\sum_{n \in \mathcal{N}_T} p_n = 1$ to all the terminal states. The probabilities of other states can be defined consecutively: $p_n = \sum_{m \in \mathcal{C}(n)} p_m$ for all $n \in \mathcal{N}_t$, $t = T-1, ..., 0$. Note that $p_0 = 1$. Suppose that measure p defines true (statistical) probabilities of events. It can be uniquely determined by a probability distribution $p_T = (p_n,\ n \in \mathcal{N}_T)$. To define values of p it is convenient to set conditional measures at first:

$$p(\cdot|n) = (p(m|n) = p_m/p_n,\ m \in \mathcal{C}(n)).$$

They indicate the probabilities of turning from states $n \in \mathcal{N}_t$, $t = 0, ..., T-1$, to the next states $m \in \mathcal{C}(n)$. Then, values of p_n can be derived using the following formula:

$$p_n = \prod_{s=0}^{t-1} p(a^s(n)|\, a^{s+1}(n)).$$

3. Game Description

Let us consider a zero-sum game with two players: a seller of the contingent claim and its buyer. The seller is an investor in wide sense, he builds a trading strategy to hedge the American contingent claim. The buyer exercises the claim in some moment of time (i.e. obliges the seller to pay the claim value using his right specified in a contract). Next, we define strategies of players.

Seller's Strategy

We denote amount of security j held by the investor in state $n \in \mathcal{N}$ by θ_n^j. We will consider a portfolio process $\theta = \{\theta(t)\}$, where the portfolios $\theta_n = (\theta_n^0, ..., \theta_n^d)$, $n \in \mathcal{N}_t$, formed at stage t are the values of a random variable $\theta(t)$. So, the investor has an initial portfolio θ_0 at stage $t = 0$, then he forms a portfolio θ_n in state $n \in \mathcal{N}_1$ (buying some securities and selling others) and so on.

Portfolio process θ is called an investor strategy if a self-financing condition is satisfied:

$$X_n \cdot \theta_n = X_n \cdot \theta_{a(n)},\ \forall\, n \in \mathcal{N}_t,\ t = 1, ..., T.$$

Self-financing means that an investor does not spend money and does not get any revenue from outside. Let

$$Y_n = X_n - X_{a(n)},\ \forall\, n \in \mathcal{N}_t,\ t = 1, ..., T,$$

be the vector of increments of securities prices. Then, $Y_n \cdot \theta_{a(n)}$ means a discounted profit of investor from portfolio $\theta_{a(n)}$ in state n.

Portfolio value process $V = \{V(t)\}$ corresponds to a trading strategy θ. A random variable $V(t)$ takes values V_n equal to scalar products of price and portfolio vectors:

$$V_n = X_n \cdot \theta_n = \sum_{j=0}^{d} X_n^j \theta_n^j, \ \forall n \in \mathcal{N}.$$

It is easy to see (Föllmer and Schied, 2011, Prop. 5.7; Pliska, 1997, Prop. 3.2) that $V(t)$ can be represented in the following form for any strategy θ

$$V(t) = V_0 + \sum_{s=1}^{t} Y(s) \cdot \theta(s-1), \ \forall t = 1, ..., T.$$

Since each child node of the scenario tree has a unique parent, we may specify all the preceding nodes for each $n \in \mathcal{N}_t$, $t = 1, ..., T$. They are $a^t(n)$, $a^{t-1}(n)$, ..., $a(n)$. Hence, the portfolio values V_n, $n \in \mathcal{N}_t$, are equal to

$$V_n = V_0 + \sum_{s=1}^{t} Y_{a^{s-1}(n)} \cdot \theta_{a^s(n)}, \ \forall n \in \mathcal{N}_t, \ t = 1, ..., T.$$

It also follows from (Föllmer and Schied, 2011, Prop. 5.7). We will use these equations later to describe the relationship between trading strategy and portfolio values. To make the dependance more convenient we denote the amount of portfolio value increment up to state n by

$$(Y\theta)_n = \sum_{s=1}^{t} Y_{a^{s-1}(n)} \cdot \theta_{a^s(n)}, \ \forall n \in \mathcal{N}_t, \ t = 1, ..., T,$$

and let $(Y\theta)_0 = 0$.

It is said that the market has an arbitrage opportunity if there is a trading strategy θ such that $V_0 \le 0$ and $V_n \ge 0$ for each $n \in \mathcal{N} \backslash \{0\}$ and at least one of these inequalities meets strictly. Following trading strategy θ, the investor loses nothing and yields a positive profit with a positive probability. Suppose further that there are no arbitrage opportunities in the market.

Strategy θ is called admissible if $V_n \ge 0$ for all $n \in \mathcal{N}$. We will consider only admissible trading strategies because they prevent the investor from ruin.

Remark 1. It is easy to show (King, 2002, p. 546) that strategy θ is admissible for arbitrage-free markets if $V_n \ge 0$ for all terminal states $n \in \mathcal{N}_T$. Indeed, otherwise suppose that portfolio value is negative in some state $m \in \mathcal{N} \backslash \mathcal{N}_T$ and $V_n \ge 0$ for all terminal states which follow m ($n \in \mathcal{N}_T \cap \mathcal{D}(m)$). Then, the investor may guarantee a positive profit for all future market scenarios.

Buyer's Strategy

Buyer's strategy is a moment of time when the contingent claim is exercised – a stopping time. Let us describe it with a random variable

$$\tau : \Omega \to \{0, ..., T\}$$

for which $\{\tau = t\} \in \mathcal{F}_t$. We use $\mathcal{T}$ to denote a finite set of all buyer's strategies.

Exercise time τ produces the only state $n_{\tau(\omega)} \in \mathcal{N}$, where stopping occurs for each simple event $\omega = (n_0, ..., n_T) \in \Omega$. Let us denote the set of such states as $\mathcal{N}_\tau$. It can be seen that a set $\tilde{\mathcal{N}} \subset \mathcal{N}$ conforms to some exercise time τ (in the sense that $\tilde{\mathcal{N}} = \mathcal{N}_\tau$) if there exists exactly one element of this set in each sequence of consecutive states $(n_0, ..., n_T)$.

A set of buyer's strategies grows very quickly while a number of trading periods T increases. A number of exercise times can be determined recursively. Let K_n denote it for the subtree with node n and other nodes $\mathcal{D}(n)$. Then

$$K_n = 1,\ \forall\, n \in \mathcal{N}_T,\ \ K_n = 1 + \prod_{m \in \mathcal{C}(n)} K_m,\ \forall\, n \in \mathcal{N}_t,\ t = T-1, ..., 0.$$

There is an exact formula for the value of $\mathcal{T}$ for specific cases when a number of child nodes $\mathcal{C}(n)$ is constant for all $n \in \mathcal{N} \backslash \mathcal{N}_T$ and equals 2 or 3. It is the following:

$$|\,\mathcal{T}| = \left[k^{c^T}\right], \tag{1}$$

where $[x]$ is an integral part of x, $k \approx 1{,}5028$ when $c = |\,\mathcal{C}(n)| = 2$ (Aho and Sloane, 1973), and $k \approx 1{,}2766$ when $c = |\,\mathcal{C}(n)| = 3$ (McGarvey, 2007). A number of buyer's strategies (exercise times) is shown in Table 1 for different values of T.

Table 1: A number of exercise times under different numbers of trading periods T.

T	0	1	2	3	4	5
$\lvert\mathcal{T}\rvert$, where $c = 2$	1	2	5	26	677	458330
$\lvert\mathcal{T}\rvert$, where $c = 3$	1	2	9	730	389017001	$\approx 5{,}9 \times 10^{25}$

American contingent claim

We describe an American contingent claim with a non-negative stochastic process $F = \{F(t)\}$, where a random variable $F(t)$ takes discounted values F_n with probability p_n, $n \in \mathcal{N}_t$, $t = 0, ..., T$. The simple example of a contingent claim is an option payment. Portfolio strategy θ hedges an American contingent claim F exercised in time τ if the corresponding portfolio value process V satisfies $V_n \geq F_n$ for all $n \in \mathcal{N}_\tau$.

Suppose that the seller does not have a necessary sum for perfect hedging and decides to manage with less initial endowment taking the risk of future losses. So, if the claim is exercised in state $n \in \mathcal{N}$ of the market, then seller's losses are equal to $(F_n - V_n)^+ = \max\{F_n - V_n;\ 0\}$. Let us evaluate seller's losses in exercise time τ using the value-at-risk function:

$$\mathrm{VaR}_\alpha\big((F(\tau) - V(\tau))^+\big) = \min\{B \in \mathbb{R} \,|\, \mathbb{P}\big((F(\tau) - V(\tau))^+ \leq B\big) \geq \alpha\},$$

where α is a preset level of significance which is usually not less than 95%.

Therefore, we defined a zero-sum game between a seller of the claim and its buyer. Let us state the optimization problem from the seller's side to find an optimal investment strategy (V, θ) which imperfectly hedges contingent claim F and minimizes a loss function VaR_α under uncertain exercise time τ. The given problem consists in finding a minimax value of the game and can be formulated in the following way:

$$\min_{(V,\,\theta)}\max_{\tau\in\mathcal{T}}\mathrm{VaR}_\alpha((F(\tau)-V(\tau))^+) \\ V_n = v+(Y\theta)_n \ge 0,\ \forall\, n\in\mathcal{N}. \tag{2}$$

4. Conversion of an Original Problem

Let us introduce an auxiliary variable u to bound the maximum of (2) from above. Then, we may rewrite the problem (2)

$$\min_{(V,\,\theta,\,u)} u$$
$$\begin{cases} u \ge \mathrm{VaR}_\alpha((F(\tau)-V(\tau))^+)\,, & \forall\,\tau\in\mathcal{T} \\ V_n = v+(Y\theta)_n \ge 0, & \forall\, n\in\mathcal{N}. \end{cases}$$

Next, we use the definition of VaR and introduce variables B_τ for all $\tau\in\mathcal{T}$. Hence:

$$\min_{(V,\,\theta,\,u)} u \\ \begin{cases} u \ge \min\limits_{B_\tau\in X(V,\tau)} B_\tau, & \forall\,\tau\in\mathcal{T} \\ V_n = v+(Y\theta)_n \ge 0, & \forall\, n\in\mathcal{N}, \end{cases} \tag{3}$$

where $X(V,\tau)=\{B_\tau\in\mathbb{R}\,|\,\mathbb{P}((F(\tau)-V(\tau))^+\le B_\tau)\ge\alpha\}$ for any fixed V and τ. Now we show that problem (3) can be equivalently reduced to the following one:

$$\min_{(V,\,\theta,\,u)} u \\ \begin{cases} \mathbb{P}((F(\tau)-V(\tau))^+\le u)\ge\alpha, & \forall\,\tau\in\mathcal{T} \\ V_n = v+(Y\theta)_n \ge 0, & \forall\, n\in\mathcal{N}. \end{cases} \tag{4}$$

Indeed, for any $B_\tau\in X(V,\tau)$ and $u\ge B_\tau$

$$\alpha \le \mathbb{P}\big((F(\tau)-V(\tau))^+\le B_\tau\big)\le \mathbb{P}\big((F(\tau)-V(\tau))^+\le u\big)\,.$$

Conversely, for optimal solution (V^*,θ^*,u^*) of (4) we may put

$$B^*_\tau = \mathrm{Arg}\min_{B_\tau\in X(V^*,\tau)} B_\tau = \max_{n\in\mathcal{N}_\tau}(F_n-V^*_n)^+\le u^*.$$

The first group of constraints in (4) shows that losses do not exceed u with probability not less than α for all the exercise times. So, when the seller determines his investment strategy, he separates all the states of the market into two groups whether planned losses exceed u or not. Let us incorporate binary variables $x_n\in\{0,1\}$ for all $n\in\mathcal{N}$ which represent the seller's choice of states. Then, the problem (4) has the following reformulation:

$$\min_{(x,\,V,\,\theta,\,u)} u \\ \begin{cases} \sum\limits_{n\in\mathcal{N}_\tau} p_n x_n \ge \alpha, & \forall\,\tau\in\mathcal{T} \\ V_n \ge x_n F_n - u, & \forall\, n\in\mathcal{N} \\ V_n = v+(Y\theta)_n \ge 0, & \forall\, n\in\mathcal{N} \\ u\ge 0, & \\ x_n\in\{0,1\}\,, & \forall\, n\in\mathcal{N}. \end{cases} \tag{5}$$

Indeed, $u \geq (F_n - V_n)^+$ if $x_n = 1$. The constraint $V_n \geq x_n F_n - u$ becomes redundant if $x_n = 0$.

Direct solving of (5) is complicated by a huge number of coupling constraints which correspond to all possible exercise times $\tau \in \mathcal{T}$. A lot of binary variables remains an issue to deal with as well. Next theorem proves the main outcome of this study – the existence of optimal solution for problem (5) such that x^* has a monotonic nature over time. Namely, we will show that

$$x_n^* \geq x_m^*,\ \forall\, m \in \mathcal{C}(n),\ n \in \mathcal{N} \backslash \mathcal{N}_T. \tag{6}$$

It can be interpreted in the following way. For each scenario $\omega = (n_0, ..., n_T)$, i.e. for each sequence of consecutive nodes of the scenario tree, leading from the root to a leaf node, the following is true: if $x_{n_t}^* = 0$, then $x_{n_s}^* = 0$ for each $s = t+1, ..., T$, hence we only want the portfolio value to be non-negative from a state n_t up to the terminal moment of time.

Theorem 1. *There will always be an optimal solution* $(x^*, V^*, \theta^*, u^*)$ *of* (5) *such that* x^* *satisfies the monotone condition* (6).

Proof. Let us fix an optimal solution $(x^*, V^*, \theta^*, u^*)$ and define a process $\tilde{x} = \{\tilde{x}(t)\}$ recursively for $t = T, ..., 0$:

$$\tilde{x}_n = x_n^*,\ \forall\, n \in \mathcal{N}_T,\ \tilde{x}_n = \min\left\{ x_n^*,\ \sum_{m \in \mathcal{C}(n)} p(m|n)\tilde{x}_m \right\},\ \forall\, n \in \mathcal{N}_t,\ t = T-1, ..., 0.$$

A stochastic process $\tilde{x}$ is analogous to Snell envelope (Föllmer and Schied, 2011, p. 285) and turns out to be a submartingale, i.e.

$$\tilde{x}_n \leq \sum_{m \in \mathcal{C}(n)} p(m|n)\tilde{x}_m,\ \forall\, n \in \mathcal{N}_t,\ t = 0, ..., T-1.$$

Besides, we conclude the following from the definition of process $\tilde{x}$:

$$\min_{\tau \in \mathcal{T}} \sum_{n \in \mathcal{N}_\tau} p_n x_n^* = \min_{\tau \in \mathcal{T}} \sum_{n \in \mathcal{N}_\tau} p_n \tilde{x}_n = \tilde{x}_0.$$

Next, we define the stopping rule $\tilde{\tau} \in \mathcal{T}$ for each scenario $\omega = (n_0, ..., n_T) \in \Omega$, where $n_t \in \mathcal{N}_t$:

$$\tilde{\tau}(\omega) = \begin{cases} T, \text{ if } x_{n_t}^* \geq \displaystyle\sum_{m \in \mathcal{C}(n_t)} p(m|n_t)\tilde{x}_m,\ \forall\, n_t \in \omega,\ t = 0, ..., T-1, \\ \min\left\{ t \,\middle|\, \tilde{x}_{n_t} = x_{n_t}^* < \displaystyle\sum_{m \in \mathcal{C}(n_t)} p(m|n_t)\tilde{x}_m \right\},\ \text{otherwise.} \end{cases} \tag{7}$$

According to definition of $\tilde{\tau}$ the buyer stops and exercises the contingent claim until T if $\tilde{x}_{n_t} = x_{n_t}^* = 0$. Therefore, (7) is equivalent to

$$\tilde{\tau}(\omega) = \begin{cases} T, & \text{if } x_{n_t}^* = 1,\ \forall\, n_t \in \omega,\ t = 0, ..., T-1, \\ \min\left\{ t | \, x_{n_t}^* = 0 \right\}, & \text{otherwise.} \end{cases}$$

In other words, we stop in the first possible state where $x_n^* = 0$ or in the final state if there was no stopping before that. Let us note that

$$\tilde{\tau} \in \operatorname*{Argmin}_{\tau \in \mathcal{T}} \sum_{n \in \mathcal{N}_\tau} p_n x_n^*.$$

For all states $m \in \mathcal{D}(n)$ which occur after $n \in \mathcal{N}_{\tilde{\tau}}$ we may put $x_m^* = 0$. Indeed, $V_m^* \geq F_m x_m^* - u^*$ and a value $\min_{\tau \in \mathcal{T}} \sum_{n \in \mathcal{N}_\tau} p_n x_n^*$ does not change. Minimum value of the objective function u remains unchanged too. Therefore, the optimal process x^* satisfies the monotone condition (6). □

We use this theorem to reduce the problem (5) excluding the exercise times:

$$\min_{(x,\, V,\, \theta,\, u)} u$$

$$\begin{cases} \sum\limits_{n \in \mathcal{N}_T} p_n x_n \geq \alpha, & \\ x_n \geq x_m, & \forall\, m \in \mathcal{C}(n),\, n \in \mathcal{N} \backslash \mathcal{N}_T \\ V_n \geq x_n F_n - u, & \forall\, n \in \mathcal{N} \\ V_n = v + (Y\theta)_n \geq 0, & \forall\, n \in \mathcal{N} \\ u \geq 0, & \\ x_n \in \{0, 1\}, & \forall\, n \in \mathcal{N}_T. \end{cases} \tag{8}$$

Corollary 1. *Any optimal solution of* (8) *is optimal for* (5) *too.*

Proof. A set of feasible solutions of (5) can be reduced by Theorem 1 incorporating the monotone condition. First, we show that a minimum value of the objective function does not change if we remove the constraint $x_n \in \{0, 1\}$ for all $n \in \mathcal{N} \backslash \mathcal{N}_T$. Non-negativity $x_n \geq 0$ for all $n \in \mathcal{N}$ follows from this system of inequalities

$$x_{a^T(n)} \geq \ldots \geq x_{a(n)} \geq x_n \geq 0, \ \forall\, n \in \mathcal{N}_T.$$

If there is a state $\tilde{n} \in \mathcal{N} \backslash \mathcal{N}_T$ such that $x_{\tilde{n}} > \max_{m \in \mathcal{C}(\tilde{n})} x_m$, then we may put $x_{\tilde{n}} = \max_{m \in \mathcal{C}(\tilde{n})} x_m$ decreasing $x_{\tilde{n}}$. Indeed,

$$V_{\tilde{n}} \geq F_{\tilde{n}} x_{\tilde{n}} - u > F_{\tilde{n}} \max_{m \in \mathcal{C}(\tilde{n})} x_m - u, \ \ x_{a(\tilde{n})} \geq x_{\tilde{n}} > \max_{m \in \mathcal{C}(\tilde{n})} x_m.$$

So, $x_n = \max_{m \in \mathcal{C}(n)} x_m$ for each $n \in \mathcal{N} \backslash \mathcal{N}_T$, that is why $x_n \in \{0, 1\}$ for all $n \in \mathcal{N}$. Decreasing of x does not change the values of x_T. Using monotone condition (6), we conclude that

$$\sum_{n \in \mathcal{N}_\tau} p_n x_n \geq \sum_{n \in \mathcal{N}_T} p_n x_n$$

for all exercise times $\tau \in \mathcal{T}$. Thus,

$$\sum_{n \in \mathcal{N}_T} p_n x_n = \min_{\tau \in \mathcal{T}} \sum_{n \in \mathcal{N}_\tau} p_n x_n \geq \alpha.$$

Therefore, turning to (8) is equivalent in the sense that the objectives achieve the same minimum values and any optimal solution of (8) remains optimal for (5). □

Equivalent reduction to problem (8) allows to change $|\mathcal{T}|$ linear coupling constraints into $|\mathcal{N}|$ monotone conditions and one constraint which couples terminal values x_T. We remark here that a number of nodes $|\mathcal{N}|$ in the scenario tree grows exponentially with increasing number of trading periods T. For a constant number of child nodes ($|\mathcal{C}(n)| = c$ for all $n \in \mathcal{N}\backslash\{0\}$) it equals a sum of geometric series:

$$|\mathcal{N}| = \frac{c^{T+1} - 1}{c - 1}. \tag{9}$$

To estimate a number of monotone conditions if a number of child nodes is non-constant and does not exceed c we may bound $|\mathcal{N}|$ above with the fraction of (9). One should compare this formula with (1) to see clearly the effect of Theorem 1.

5. Conclusion

In this study we suppose that security trading in financial market occurs in deterministic time moments and a market has a finite number of scenarios. If there are a lot of trading periods and market scenarios, we deal with large-scale problems. There are no transaction costs during the trading and the market has no arbitrage opportunities.

Here for the first time we state the problem of finding the investment strategy which produces the minimal losses associated with imperfect hedging of American contingent claim using VaR measure. The main problem has a game form and consists in finding a minimax value of a specific zero-sum game. The main result of this study states that the seller always has a minimax strategy which is monotonous over time. It allows us to not only reduce the dimension of the original optimization problem, but also actually exclude the uncertainty associated with the time of exercising the contingent claim. The outcome can be used to create the software systems for financial institutions which deal with valuation and hedging of contingent claims, building trading strategies and so on.

In future research we are planning to investigate the formulations of the main game problem in which the seller may use mixed strategies and the buyer may use behavioral ones.

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