Contributions to Game Theory and Management, IX, 266–275

A Design of Strategies in Alternative Pursuit Games

Igor Shevchenko[1,2] **and Dušan M. Stipanović**[3]

[1] *Information Technology Department,*
TINRO-Center,
Shevchenko Alley 4, Vladivostok, 690091, Russia
E-mail: igor@tinro.ru

[2] *Computer Science Department,*
Far East Federal University,
Sukhanova ul. 8, Vladivostok, 690091, Russia
E-mail: shevchenko.ii@dvfu.ru

[3] *Coordinated Science Laboratory,*
University of Illinois at Urbana-Champaign
E-mail: dusan@illinois.edu

Abstract In this work we consider the games where P can terminate pursuit at will on any of two terminal manifolds. If the optimal feedback strategies for every variant of termination are known, an obvious pursuit strategy assigns the control that corresponds to the alternative with less value at every state. On the manifold with equal alternative values, this strategy may become discontinuous even when the value functions themselves are smooth. We describe smooth approximations for the minimum functions that allow to construct smooth alternative strategies and to deal with generalized solutions for differential equations with discontinuous right-hand sides. However, as shown by an example, the state may stay on a equivalued manifold and the game never terminates.

Keywords: approximations of minimum and maximum functions, alternative pursuit, generalized solutions for differential equations with discontinuous right-hand sides.

1. Introduction

Differential games advanced far beyond the initial findings of their founders. However, finding solutions for concrete games still involves more art than craft. In this paper, we study a method for generating pursuit strategies and evaluation of their guaranteed results when the goal functions represent the minimum of two value functions.

2. Smooth approximations for minimum functions and their derivatives

Certain values between v_1 and v_2 that described, e.g., as

$$F_\alpha(v_1, v_2) = \alpha v_1 + (1-\alpha)v_2,\ 0 < \alpha < 1$$

may be considered as "rough" approximations for $\min(v_1, v_2)$ from above or for $\max(v_1, v_2)$ from below. In "more accurate" approximations, α depends on v_1 and v_2, and $\alpha(v_1, v_2)$ takes the value close to 1 for $\min(v_1, v_2)$ and to 0 for $\max(v_1, v_2)$ if $v_i < v_{3-i}$, $i = 1, 2$. Thus,

$$\overline{F}^{\xi}_{\lambda}(v_1, v_2) = \frac{\lambda_1 v_1^{\xi} v_2 + \lambda_2 v_1 v_2^{\xi}}{\lambda_1 v_1^{\xi} + \lambda_2 v_2^{\xi}},\ v_1, v_2, \xi \in \mathbb{R}^+, \tag{1}$$

and

$$\underline{F}_\lambda^\xi(v_1, v_2) = \frac{\lambda_1 v_1^{\xi+1} + \lambda_2 v_2^{\xi+1}}{\lambda_1 v_1^\xi + \lambda_2 v_2^\xi},\ v_1, v_2, \xi \in \mathbb{R}^+, \tag{2}$$

which correspond to F_α with

$$\alpha(v_1, v_2) = \frac{\lambda_2 v_2^\xi}{\lambda_1 v_1^\xi + \lambda_2 v_2^\xi}$$

and

$$\alpha(v_1, v_2) = \frac{\lambda_1 v_1^\xi}{\lambda_1 v_1^\xi + \lambda_2 v_2^\xi}$$

approximate $\min(v_1, v_2)$ from above and $\max(v_1, v_2)$ from below, $0 < \lambda_i < 1, \sum_{i=1}^2 \lambda_i = 1$, $i = 1, 2$, respectively; see, e.g., (Stipanović et al., 2009). Moreover, their partial derivatives approximate the corresponding partial derivatives of the minimum and maximum functions where they exist; see, e.g., (Shevchenko, 2009, 2012).

Since, e.g.,

$$\min(v_1, v_2, \ldots, v_n) = \min(v_1, \min(v_2, \ldots, v_n)),\ n > 2, \tag{3}$$

approximations for the arbitrary number of arguments minimum functions may be easily constructed with use of approximations for $\min(v_1, v_2)$ and $\max(v_1, v_2)$.

A more general approach is based on using monotonic functions; see, e.g., (Stipanović, 2012, 2014). Let g be a strictly decreasing non-negative differentiable function $\mathbb{R}^+ \to \mathbb{R}^+$ and $v_{i_0} = \min(v_1, \ldots, v_n)$. Then,

$$v_{i_0} \le v_i,$$

$$g(v_{i_0}) \ge g(v_i),$$

$$\lambda_i g(v_{i_0}) \ge \lambda_i g(v_i),\ 0 < \lambda_i < 1, \sum_{i=1}^n \lambda_i = 1,$$

$$\sum_{i=1}^n \lambda_i g(v_{i_0}) \ge \sum_{i=1}^n \lambda_i g(v_i),$$

$$g(v_{i_0}) \ge \sum_{i=1}^n \lambda_i g(v_i),$$

$$\sum_{i=1}^n \lambda_i g(v_{i_0}) \ge \sum_{i=1}^n \lambda_i g(v_i).$$

Since g is an invertible function,

$$v_{i_0} = g^{-1}(g(v_{i_0})) \le g^{-1}(\sum_{i=1}^n \lambda_i g(v_i)).$$

Let G_λ^g be the following symmetric function $(\mathbb{R}^+)^n \to \mathbb{R}^+$

$$G_\lambda^g(v_1, \ldots, v_n) = g^{-1}(\sum_{i=1}^n \lambda_i g(v_i)). \tag{4}$$

If $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$, (4) approximates $\min(v_1, \dots, v_n)$ from above since

$$\min(v_1, \dots, v_n) < G_\lambda^g(v_1, \dots, v_n), \text{ if } v_i \neq v_j,\ i \neq j, \tag{5}$$

In addition,

$$\min(v, \dots, v) = G_\lambda^g(v, \dots, v) = v. \tag{6}$$

Similarly,

$$g(v_{i_0}) \leq \sum_{i=1}^n \lambda_i g(v_i),\ \lambda_i \geq 1,$$

$$v_{i_0} = g^{-1}(g(v_{i_0})) \geq g^{-1}(\sum_{i=1}^n \lambda_i g(v_i)).$$

If $\lambda_i \geq 1$, (4) approximates $\min(v_1, \dots, v_n)$ from below since

$$\min(v_1, \dots, v_n) > G_\lambda^g(v_1, \dots, v_n), \text{ if } v_1 \neq v_2, \tag{7}$$

$$\min(v, \dots, v) > G_\lambda^g(v, \dots, v) = g^{-1}(\sum_{i=1}^n \lambda_i g(v)). \tag{8}$$

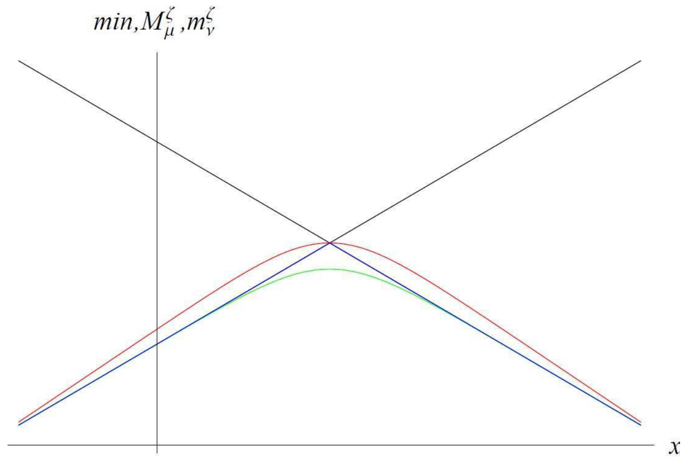


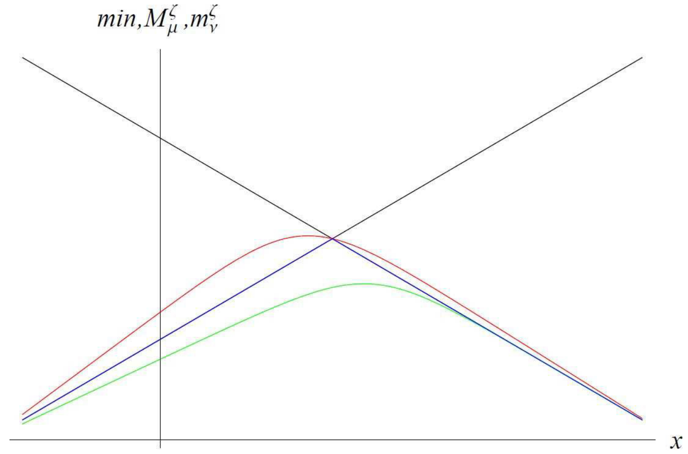
Fig. 1: Approximations for the minimum function; $\xi = 5$; $\mu_1 = 0.5, \mu_2 = 0.5$ (upper); $\nu_1 = 1, \nu_2 = 1$ (lower)

Certain upper (M_μ^ξ) and lower (m_ν^ξ) approximations for $\min(v_1, v_2)$ may be generated, e.g., with use of the family $\{v^{-\xi}\}_{\xi>0}$,

$$M_\mu^\xi(v_1, v_2) = \left(\mu_1 v_1^{-\xi} + \mu_2 v_2^{-\xi}\right)^{-\frac{1}{\xi}},\ 0 < \mu_i < 1, \sum_{i=1}^2 \mu_i = 1, \tag{9}$$

$$m_\nu^\xi(v_1, v_2) = \left(\nu_1 v_1^{-\xi} + \nu_2 v_2^{-\xi}\right)^{-\frac{1}{\xi}},\ \nu_i > 1, \tag{10}$$

see Figs. 1–2.



(a) (b)

Fig. 2: Approximations for the minimum function; $\xi = 5$; $\mu_1 = 0.3, \mu_2 = 0.7$ (upper); $\nu_1 = 3, \nu_2 = 1$ (lower)

The derivatives of G_λ^g are described as

$$\frac{\partial G_\lambda^g}{\partial v_j}(v_1, \ldots, v_n) = \lambda_j \frac{g'(v_j)}{g'\left(g^{-1}\left(\sum_{i=1}^n \lambda_i g(v_i)\right)\right)}, \tag{11}$$

and

$$\frac{\partial G_\lambda^g}{\partial v_j}(v, \ldots, v) = \lambda_j,$$

if $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$.

For example, the derivatives of $\min(v_1, v_2)$ (where they exist) are approximated by the derivatives of M_μ^ξ since

$$\lim_{\xi \to +\infty} \frac{\partial M_\mu^\xi}{\partial v_j}(v_1, v_2) = \begin{cases} 1 \text{ if } v_j < v_{3-j}, \\ 0 \text{ if } v_j > v_{3-j} \\ \mu_j \text{ if } v_j = v_{3-j}. \end{cases}$$

Now, let g be a strictly increasing non-negative differentiable function $\mathbb{R}^+ \to \mathbb{R}^+$. Obviously, if $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$, (4) approximates $\max(v_1, \ldots, v_n)$ from below and if $\lambda_i \geq 1$ from above. All other mentioned formerly results related to the use of (4) for approximations of the minimum function may be easily reproduced for approximations of the maximum function.

3. Alternative pursuit

Let $Z \subseteq \mathbb{R}^m$ be an open set, $\overline{Z}$ be a playing space and $\partial Z = \overline{Z} \backslash Z$ be its boundary. Let $z_P(t) \in \mathbb{R}^{n_P}$ and $z_E(t) \in \mathbb{R}^{n_E}$ meet

$$\dot{z}_P(t) = f_P(z_P(t), u_P(t)), \quad z_P(0) = z_P^0, \tag{12}$$

$$\dot{z}_E(t) = f_E(z_E(t), u_E(t)), \quad z_E(0) = z_E^0, \tag{13}$$

where $t \geq 0$, $u_P(t) \in \mathsf{U}_P \subset \mathbb{R}^{m_P}$, $u_E(t) \in \mathsf{U}_E \subset \mathbb{R}^{m_E}$, U_P and U_E are compact sets, $f_P : \mathbb{R}^{n_P} \times \mathsf{U}_P \to \mathbb{R}^{n_P}$ $f_E : R^{n_E} \times \mathsf{U}_E \to R^{n_E}$, $z_P^0 \in \mathbb{R}^{n_P}$ $z_E^0 \in R^{n_E}$ are initial states. Let $z(t) = (z_P(t), z_E(t)) \in Z \subseteq \mathbb{R}^n$, $n = n_P + n_E$,

$$\dot{z}(t) = f(z(t), u_P(t), u_E(t)),\ z(0) = z^0, \tag{14}$$

where $z(0) = z^0 = (z_P^0, z_E^0) \in Z$, $f(z, u_P, u_E) = (f_P(z_P, u_P), f_E(z_E, u_E))$. We assume that f is jointly continuous and locally Lipschitz with respect to z for all $u_P \in \mathsf{U}_P$ and $u_E \in \mathsf{U}_E$.

A strategy is a rule to determine the control depending on available information at any instant of the game. For a given strategy, the equation (14) is used to generate a pencil of all potential motions and evaluate the guaranteed payoff over all admissible countering actions.

For $z^0 \in Z$, $\Delta = \{t_0, t_1, \ldots, t_i, t_{i+1}, \ldots\}$ and a strategy S_P, let $Z_P(z^0, S_P, \Delta)$ be a pencil of piecewise constant solutions of the inclusion

$$\dot{z}(t) \in \mathrm{co}\{f(z(t_i), u_P(t_i), u_E) : u_E \in \mathsf{U}_E\}, \tag{15}$$

where $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$, $t_0 = 0$, $t_i \to_{i\to\infty} \infty$, $z : R^+ \to Z$ is a continuous function that has an absolutely continuous restriction to $[0, \theta]$ for any $\theta > 0$ and meets (15) for almost all $t \in [0, \theta]$.

A pursuit game is called alternative if

- from any internal state $z \in Z$, it can be terminated by P *at will* on any of two given terminal manifolds $M^a \subset \partial Z$ or $M^b \subset \partial Z$,
- for every alternative termination, the payoffs of Boltza type differ only in their terminal parts,
- for every alternative termination, the optimal feedback pursuit $(S_P^a(\cdot), S_P^b(\cdot))$ and evasion $(S_E^a(\cdot), S_E^b(\cdot))$ strategies and the value functions $(V^a(\cdot), V^b(\cdot))$ are known.

Among the games that may be considered as alternative are the obstacle tag (Isbell, 1967) and successive pursuit (Breakwell and Hagedorn, 1979) games.

For a given alternative terminal manifold M_l, let the payoff functional be defined as

$$\mathcal{P}_l^\varepsilon(z(\cdot)) = \begin{cases} \tau_l^\varepsilon + K_l(z(\tau_l^\varepsilon)),\ \text{if } \tau_l^\varepsilon = \tau_l^\varepsilon(z(\cdot)) < \infty, \\ \infty \text{ otherwise,} \end{cases} \tag{16}$$

where

$$\tau_l^\varepsilon(z(\cdot)) = \begin{cases} \min\{t_i \in \Delta : z(t_i) \in M_l^\varepsilon\},\ \text{if } \exists t_i \in \Delta : z(t_i) \in M_l^\varepsilon, \\ \infty \text{ otherwise,} \end{cases} \tag{17}$$

M_l^ε is the ε neighbourhood of M_l, $M_l^\varepsilon = \{z : z \in Z, \min_{z' \in M_l} ||z - z'|| \leq \varepsilon\}$, $K_l : Z \to \mathbb{R}^+$, $l \in L = \{a, b\}$. Then the guaranteed result may be evaluated as

$$\check{\mathcal{P}}_l(z^0) = \lim_{\varepsilon \to 0+} \mathcal{P}_l^\varepsilon(z^0), \tag{18}$$

where $\check{\mathcal{P}}_l^\varepsilon(z^0) = \inf_{S_P} \check{\mathcal{P}}_l^\varepsilon(z^0, S_P)$,

$$\check{\mathcal{P}}_l^\varepsilon(z^0, S_P) = \lim_{|\Delta| \to +0} \check{\mathcal{P}}_l^\varepsilon(z^0, S_P, \Delta),\ |\Delta| = \sup_{i \in \mathbb{N}}(t_{i+1} - t_i),$$

$$\check{\mathcal{P}}_l^\varepsilon(z^0, S_P, \Delta) = \sup_{z(\cdot) \in Z_P(z^0, S_P, \Delta)} \mathcal{P}_l^\varepsilon(z(\cdot)).$$

Let $\hat{\mathcal{P}}_l : Z \to \mathbb{R}$ be a similar index for E and $V^l(z^0) = \check{\mathcal{P}}_l(z^0) = \hat{\mathcal{P}}_l(z^0), \forall z^0 \in Z$. Then the value function $V^l : Z \to \mathbb{R}^+$ represents a joint guaranteed result for both players.

If $V^a(\cdot)$ and $V^b(\cdot)$ are continuous in $\overline{Z}$, satisfy the terminal conditions

$$V^l(z) = K_l(z),\, z \in M^l, \tag{19}$$

and are continuously differentiable in Z, then the Isaacs' main equation

$$H^l(z, DV^l(z)) + 1 = 0 \tag{20}$$

is satisfied, where

$$H^l(z, DV^l(z)) = \min_{u_P \in U_P} \max_{u_E \in U_E} \lambda^l f(z, u_P, u_E) = \max_{u_E \in U_E} \min_{u_P \in U_P} \lambda^l f(z, u_P, u_E)\, l \in L.$$

If there are bounded $u_P^* : Z \times \Lambda \to U_P$ and $u_E^* : Z \times \Lambda \to U_E$ such that

$$u_P^*(z^l, \lambda^l) \in Arg \min_{u_P \in U_P} \left(\max_{u_E \in U_E} \lambda^l f(z, u_P, u_E) \right), \tag{21}$$

$$u_E^*(z^l, \lambda^l) \in Arg \max_{u_E \in U_E} \left(\min_{u_P \in U_P} \lambda^l f(z, u_P, u_E) \right), \tag{22}$$

the optimal feedback strategies are designed as follows

$$S_P^l(z) = u_P^*(z, DV^l(z)), \quad S_E^l(z) = u_E^*(z, DV^l(z)). \tag{23}$$

When solving such kind of games, a standard problem is to combine S_P^a and S_P^b into a pursuit strategy that guarantees a result less or equal to $\min(V^a(z), V^b(z))$ for every state $z \in Z$ (Shevchenko, 2009).[1] An obvious candidate strategy for alternative pursuit is

$$S_P^{a|b}(z) = \begin{cases} S_P^a(z) \text{ if } V^a(z) < V^b(z), \\ S_P^b(z) \text{ if } V^b(z) < V^a(z), \\ u_P \in [S_P^a(z), S_P^b(z)] \text{ or} \\ u_P \in \{S_P^a(z), S_P^b(z)\} \text{ if } V^b(z) = V^a(z), \end{cases} \tag{24}$$

or

$$S_P^l(z) = \begin{cases} u_P^*(z, D\min(V^a(z), V^b(z))) \text{ if } V^b(z) \neq V^a(z) \\ u_P \in [u_P^*(z, DV^a(z)), u_P^*(z, DV^a(z))] \text{ or} \\ u_P \in \{u_P^*(z, DV^a(z)), u_P^*(z, DV^a(z))\} \text{ if } V^b(z) = V^a(z), \end{cases} \tag{25}$$

where $[v_1, v_2] = \{v : \kappa v_1 + (1-\kappa)v_2, \kappa \in [0,1]\}$. To evaluate the guaranteed payoff for $S_P^{a|b}$, one needs to determine pencils of solutions for differential equations with discontinuous right-hand sides. The generalized solutions (Krasovskii and Subbotin, 1988) include all possible absolutely continuous motions for all values of the control at a discontinuity point (as, e.g., the Filippov's solutions (Filippov, 1988)).

[1] A similar problem arises when optimal feedbacks are constructed with the use of a synthesis procedure based on the main equation and its smooth characteristics within the Isaacs' approach (Isaacs, 1967).

The constructive motions (Krasovskii and Subbotin, 1988) are absolutely continuous limits of motions along the Euler broken lines. The constructive motions are usually included into the generalized solutions and provide better guaranteed results. However, they are less stable (Krasovskii and Subbotin, 1988).

One way to avoid discontinuous controls at the regular states is to use smooth upper approximations of $\min(V^a(z), V^b(z))$ in (23) as, e.g., (9). Then with

$$S_P^l(z) = u_P^*(z, DM_\mu^\xi(z)),\ S_E^l(z) = u_E^*(z, DM_\mu^\xi(z)) \tag{26}$$

some subsets of the generalized motions are obtained.

Practically in all games solved with use of the Isaacs' approach, the value functions are not smooth globally (Isaacs, 1967). Call a state $z \in Z$ regular with respect to known value functions V^a and V^b if S_P^l and S_E^l meet (20)–(23) with $V^l, H^l \in C^2, u_P^* \in C$, $l \in L$, in some neighbourhood of z. A set of such states is also called regular. Let $\mathcal{E}^{a|b} = \{z \in Z : V^a(z) = V^b(z)\}$ be a two-sided regular smooth hypersurface separating two regular sets. Moving along any direction η from $z \in \mathcal{E}^{a|b}$ for a small enough time, the state shifts to a regular state z' on $\mathcal{E}^{a|b}$ $(V^a(z') = V^b(z'))$ or in Z $(V^a(z') < V^b(z'))$ or in Z $(V^a(z') > V^b(z'))$.

If

$$\frac{\partial}{\partial z}(V^a(z) - V^b(z))f(z, S_P^a(z), S_E^a(z)) > 0, \tag{27}$$

$$\frac{\partial}{\partial z}(V^b(z) - V^a(z))f(z, S_P^b(z), S_E^b(z)) > 0, \tag{28}$$

on $\mathcal{E}^{a|b}$ and P uses (25) or (26) in its close neighbourhood, the state may stay there for some time (Shevchenko, 2014).

4. Constructing strategies in a simple alternative pursuit game

Three points P, E_1, and E_2, $E = (E_1, E_2)$, with bounded velocities move on the plane as

$$\dot{z} = (u_P, u_e),\ z = z^0,$$
$$z_P, z_1, z_2 \in \mathbb{R}^2,\ z_e = (z_1, z_2) \in \mathbb{R}^4,\ z = (z_P, z_e) \in \mathbb{R}^6,$$
$$z^0 = (z_P^0, z_e^0),\ z_e^0 = (z_1^0, z_2^0),$$
$$u_P \in U_P,\ u_e = (u_1, u_2) \in U_e,$$
$$U_P = \{u_P : ||u_P|| \le 1\},\ U_e = \{u_e : ||u_1|| \le \beta_1 < 1, ||u_2|| \le \beta_2 < 1\}$$

It's required to determine the minimal guaranteed time $\tau^{1|2}$ for P to approach one of E's by a given distance $r > 0$ and the corresponding strategy.

The value of the game at the initial state $z^0 \in Z$ may be evaluated as

$$V(z^0) = \min(V^1(z^0), V^2(z^0)), \tag{29}$$

where

$$V^i(z^0) = \frac{||z_i^0 - z_P^0|| - r}{1 - \beta_i}, \tag{30}$$

the optimal feedback pursuit strategy is

$$S_P^{i(z^0)}(z) = -\frac{\partial V^{i(z^0)}}{\partial z_{i(z^0)}}(z)/||\frac{\partial V^{i(z^0)}}{\partial z_{i(z^0)}}(z)||,\ z, z^0 \in Z, \tag{31}$$

and $i(z^0)(= 1 \vee 2)$ satisfies the condition

$$V^{i(z^0)}(z^0) = \min(V^1(z^0), V^2(z^0)). \tag{32}$$

It was noted in (Krasovskii and Subbotin, 1988) that in this game with $\beta_1 = \beta_2$ and $r = 0$ there is a dispersal line where both pursuers are equidistant and it may become a singular line with the payoff equal to $+\infty$ due to the measurement errors.

Look at an alternative version of the game with the pursuit strategy

$$S_P^{1|2}(z) = -\frac{\partial V^{i^*(z)}}{\partial z_{i^*(z)}}(z)/||\frac{\partial V^{i^*(z)}}{\partial z_{i^*(z)}}(z)||,\ z \in Z, \tag{33}$$

where $i^*(z)(= 1 \vee 2)$ meets the condition

$$V^{i^*(z)}(z) = V^{1|2}(z) = \min(V^1(z), V^2(z)). \tag{34}$$

This strategy is discontinuous in z on

$$\mathcal{E}^{1|2} = \{z : V^1(z) = V^2(z), z \in Z\}.$$

To construct smooth in z approximations for $S_P^{1|2}$, M_μ^ξ may be used as follows

$$S_P^\xi(z) = -\frac{\partial M_\mu^\xi}{\partial z_P}(V^1(z), V^2(z))/||\frac{\partial M_\mu^\xi}{\partial z_P}(V^1(z), V^2(z))||,\ z \in Z, \tag{35}$$

where

$$\frac{\partial M_\mu^\xi}{\partial z_P}(V^1(z), V^2(z)) = \sum_{i=1,2} \frac{\partial M_\mu^\xi}{\partial v_i}(V^1(z), V^2(z))\frac{\partial V^i}{\partial z_P}(z),$$

$$\frac{\partial V^i}{\partial z_P}(z) = -\frac{1}{1-\beta_i}\frac{z_i - z_P}{||z_i - z_P||},\ z \in Z, i = 1, 2.$$

Note that the time derivatives of $V^{1|2}$ and M_μ^ξ along a trajectory that corresponds to the strategies S_P, S_1, S_2 are described by the following expressions (where exist)

$$\frac{dV^{1|2}}{dt} = \sum_{i=1,2} \frac{\partial \min}{\partial v_i}(V^1(z), V^2(z))\frac{\partial V^i}{\partial z}(z)\frac{dz}{dt},\ \frac{\partial \min}{\partial v_i} \in \{0, 1\},$$

$$\frac{dM_\mu^\xi}{dt} = \sum_{i=1,2} \frac{\partial M_\mu^\xi}{\partial v_i}(V^1(z), V^2(z))\frac{\partial V^i}{\partial z}(z)\frac{dz}{dt},\ \frac{\partial M_\mu^\xi}{\partial v_i} \in [0, 1]$$

where

$$\frac{dz}{dt} = (S_P, (S_1, S_2)),\ z^0 = (z_P^0, (z_1^0, z_2^0)).$$

A locally optimizing strategy corresponds to the case when instant controls minimize/maximize the time derivative of a goal function for any $z \in Z$. Obviously, $S_P^{1|2}$ (see (33)) is locally optimizing for $V^{1|2}$ as well as

$$S_i^{\max} = \beta_i \frac{z_i - z_P}{||z_i - z_P||},\ z \in Z, i = 1, 2, \tag{36}$$

for V^i. S_P^ξ, $\xi > 0$, (see (35)) may be considered as an approximation for $S_P^{1|2}$.

Let E_i use $S_i^{\max}(z) = \beta_i e(\psi_i^{\max})$ and P use $S_P^{\mathcal{E}} = e(\varphi^{\mathcal{E}})$. The state moves along $\mathcal{E}^{1|2}$ if $\varphi^{\mathcal{E}}$ meets the condition

$$\frac{\beta_1}{1-\beta_1} - \frac{\beta_2}{1-\beta_2} - \left(\frac{e(\psi_1^{\max})}{1-\beta_1} - \frac{e(\psi_2^{\max})}{1-\beta_2}\right) e(\varphi^{\mathcal{E}}) = 0. \tag{37}$$

If E_i uses $S_i^{\max}$, $i = 1, 2$, and $\beta_1 = \beta_2 = \beta$, then $S_P^\xi = e(\varphi^\xi)$ keeps the state on $\mathcal{E}^{1|2}$ with the corresponding payoff

$$\tau^{1|2}(z, S_P^\xi, (S_1^{\max}, S_2^{\max})) = +\infty,\ z \in \mathcal{E}^{1|2}, \xi > 0,$$

see Fig. 3.

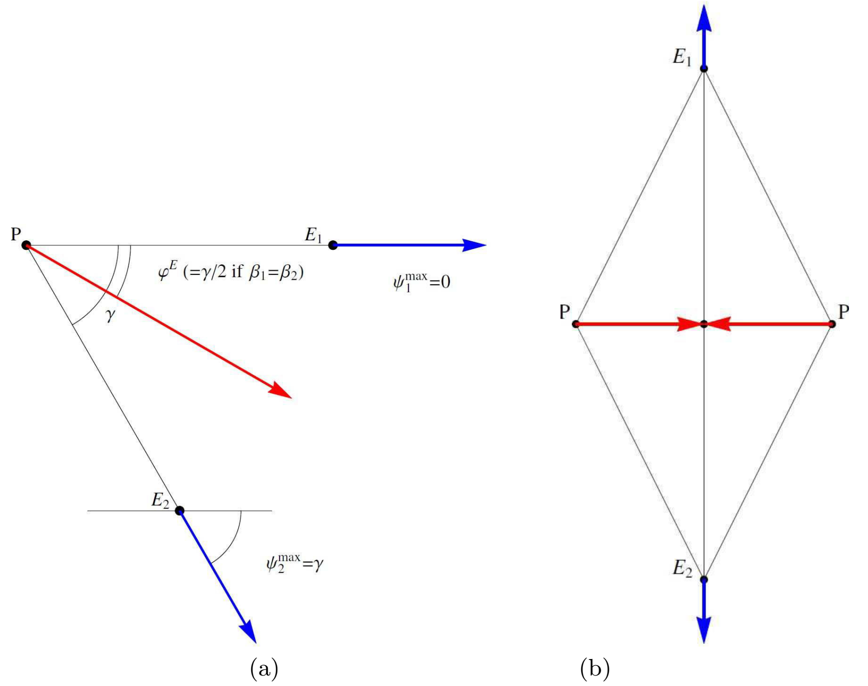


Fig. 3: Instant velocities on $\mathcal{E}^{1|2}$ (a) and trajectories in vicinity of a point of attraction (b)

The inequality

$$\frac{dM_\mu^\xi}{dt} < 0, \tag{38}$$

holds on $\mathcal{E}^{1|2}$ only if $0 \le \gamma < 2\arccos\beta$. Whereas, for $S_P^{i(z^0)}$, $S_1^{\max}$ and $S_2^{\max}$,

$$\frac{dV^{i(z^0)}}{dt} = -1,\ z \in Z. \tag{39}$$

5. Conclusions

In this paper, we provide an analysis on smooth approximations of the minimum function thus allowing construction of the corresponding strategies for a pursuer in an alternative pursuit game with two evaders. As a drawback of the procedure we show through an example how the state may never leave a manifold where two value functions are equal and thus the game never ends if the pursuit strategy is applied.

References

Bernhard, P. (1977). *Singular Surfaces in Differential Games: An Introduction.* In Differential Games and Applications. Springer Lecture Notes in Information and Control Sciences (Hargedorn P., H.W. Knobloch and G.H. Olsder, eds), Berlin: Springer, Vol. 3, 1–33.

Breakwell, J. V. and P. Hagedorn (1979). *Point Capture of Two Evaders in Succession.* JOTA, **27 (1)**, 90–97.

Filippov, A. F. (1988). *Differential Equations with Discontinuous Righthand Sides.* Kluwer: Dordrecht.

Isaacs, R. (1967). *Differential Games.* John Wiley: New York.

Isbell, J. R. (1967). *Pursuit Around a Hole.* Naval Research Quarterly, Vol. 14, 569–571.

Krasovskii, N. N. and A. I. Subbotin (1988). *Game-Theoretical Control Problems.* Springer-Verlag: New York.

Shevchenko, I. (2009). Strategies for Alternative Pursuit Games. Advances in Dynamic Games Theory and Their Applications: Analytical and Numerical Developments. (Bernhard P., V. Gaitsgory and O. Pourtallier eds.) (Annals of the International Society of Dynamic Games, Vol. 10. Birkhäuser, 121 – 131.

Shevchenko, I. (2012). *Locally Optimizing Strategies for Approaching the Furthest Evader*, *Contributions to Game Theory & Management*, Vol. 5, 293-303.

Shevchenko, I. (2014). *Guaranteed strategies with memory for alternative pursuit*, Automation and Remote Control, **75(10)**, 1861–1868.

Stipanović, D. M., A. Melikyan and N. Hovakimyan (2009). *Some Sufficient Conditions for Multi-Player Pursuit-Evasion Games with Continuous and Discrete Observations*, Annals of the International Society of Dynamic Games, Vol. 10, 1-13.

Stipanović, D. M., A. Melikyan and N. Hovakimyan (2010). *Guaranteed strategies for nonlinear multi-player pursuit-evasion games*, International Game Theory Review, **12(1)**, 1–17.

Stipanović, D. M., C. J. Tomlin and G. Leitmann (2012). *Monotone Approximations of Minimum and Maximum Functions and Multi-Objective Problems*, Applied Mathematics & Optimization, Vol. 66, 455–473.

Stipanović, D. M., C. Valicka and A. E. Abbas (2014). *Control Strategies For Players In Pursuit-Evasion Games Based On Their Preferences.* International Game Theory Review, **16(02)**, 1440008.